Advanced Engineering Mathematics

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Lecture No. # 05

System of Linear Equations, Eigenvalues and Eigenvectors

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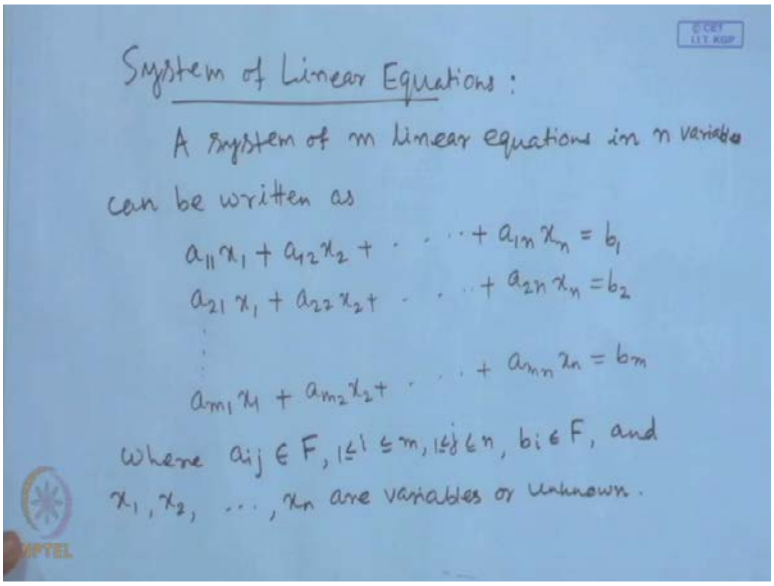

So, today we shall see this system of linear equations, system system of linear equations. This is also an important concept in linear algebra. Here, we shall discuss about consistency of a system of linear equations; and also we shall discuss, how to find solution of this. A system of a system of m linear equations in n variables can be written as a 1 1 x 1 plus a 1 2 x 2 plus a 1 n x n is equal to b 1, a 2 1 x 1 plus a 2 2 x 2 plus a 2 n x n is equal to b 2. Like this m equations a m 1 x 1 plus a m 2 x 2 plus a m n x n is equal to b m.

Here the coefficients, where this a i j that belong to a field; i from 1 to m, j from 1 to n, b i also belong to this filed, and x 1, x 2 to x n are variables or unknown that we have to find out, or in other words solving the system of linear equations means; we have to find unknowns x 1, x 2 to x n, so that this equations will be true. Here we are considering this coefficients a i j are from a field F, and also the elements b 1, b 2 to b m are also elements of this field F. Here in fact, we say that the system of linear equations is a system over this field F. In place of this field F, one can also consider the real field or complex field also.

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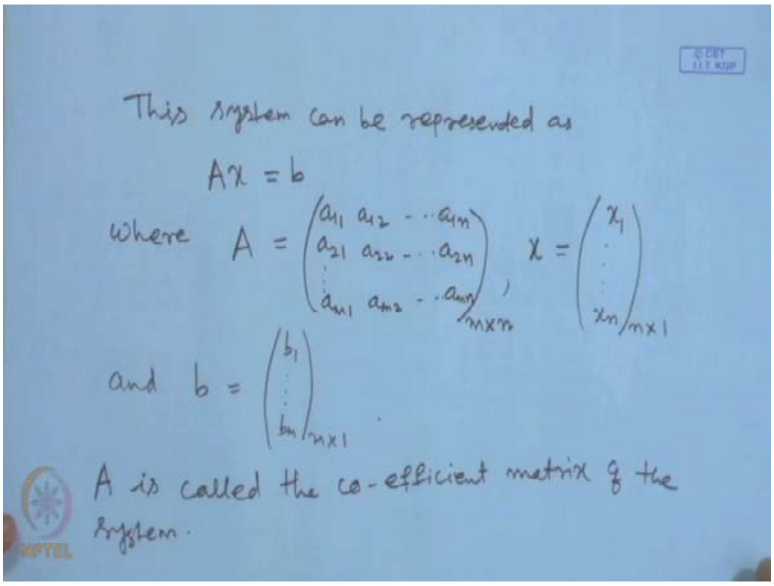

So, this system of m linear equations in n variables can also be written as; this system can be represented as Ax is equal to b, where A is this matrix that a 1 1 a 1 2 a 1 n a 2 1 a 2 2 a 2 n a m 1 a m 2 a m n, and this x is.. So, this A is m by n matrix and x is this column vector of this matrix x 1, x 2 to x n; this is of size n by 1, and b is also a column vector, or this matrix b 1 up to b m; this is m by 1 matrix. So, this matrix is called the co-efficient matrix of the system, A is called the co-efficient matrix of the system.

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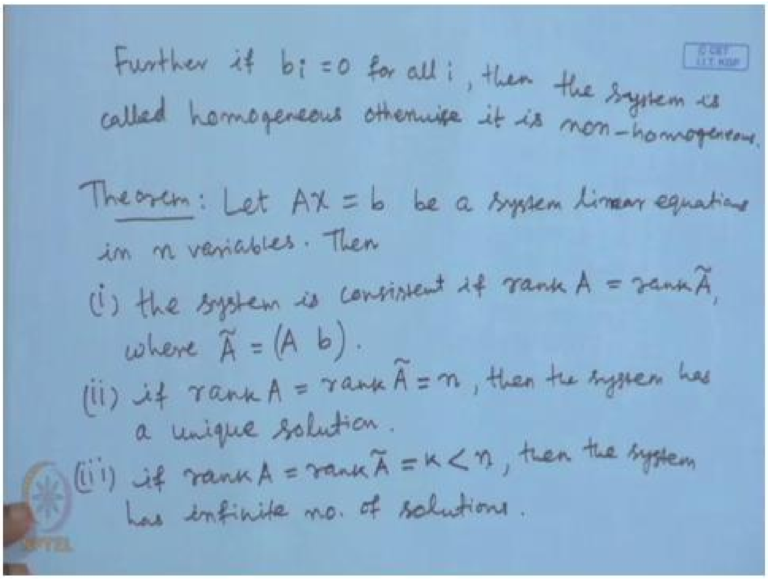

Further, if further if b i is equal to 0 for all i, then the system is called homogeneous. Called homogeneous Otherwise, it is non-homogeneous Otherwise, it is not homogeneous. So, here we shall discuss about consistency of the system; that means, when solutions exit for the system, and also we shall see how to find solutions. So, the following theorem gives the condition for consistency of the system as well as existence of number of solutions of the system. So, let Ax equal to b be a system of linear equations be a system of be a system of linear equations in n variables, then we have the following results: The system is consistent, if rank of the co-efficient matrix A is equal to rank of the augmented matrix A tilta, where A tilta is this augmented matrix A b.

Second condition says that if rank of A is equal to rank of A tilta; that means, the system is consistent, and if this rank is equal to n, then the system has a unique solution. So, third result here is that: If rank of A is equal to rank of A tilta; and this is equal to k, and that is strictly n less than n that is, number of variables variables strictly less than the number of variables, then the system has infinite number of solutions infinite number of solutions. So, we are not proving this theorem, but will use it and solve problems.

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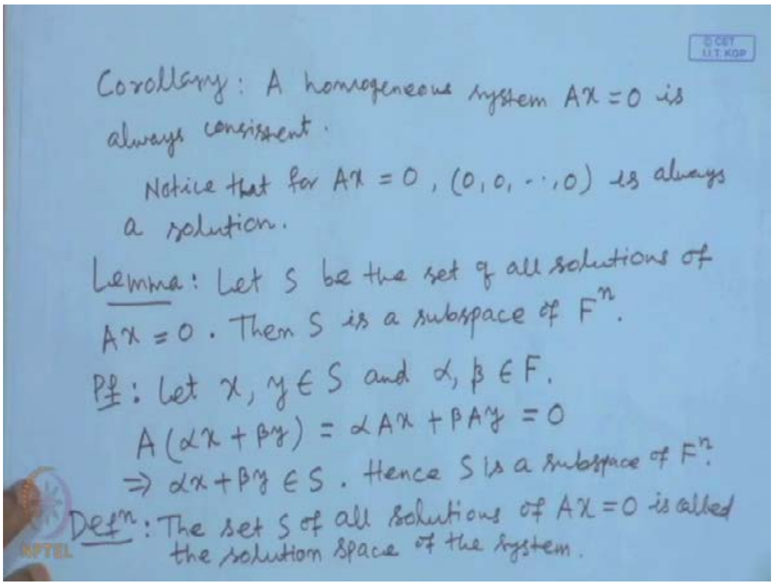

One immediate corollary is that; So, In case of a homogeneous system of equations, we get that the rank of the co-efficient matrix is equal to rank of the augmented matrix, because all b i are equal to 0. Therefore, a homogeneous system a homogeneous system Ax equal to 0 is always consistent, or always a solution exists for the system. Notice that for Ax equal to 0, this 0 0 0; that means, all x i(s) are 0 is always a solution. So, we will have in fact, the following lemma that tells that the collection of all solution of a homogeneous system forms a subspace. Let S be the set of all solutions of this homogeneous system Ax is equal to 0, then S is a subspace of this vector space F n. This is not difficult to proof.

We can take any two solutions in a S. Let x, y belongs to S that means; x and y be 2 solutions of this system Ax equal to 0, and alpha, beta be any two scalar or elements of this field F. Then we shall see whether this alpha x plus beta y is a solution for the system or not. So, this can be written as alpha Ax plus beta Ay, and this is equal to zero. Since Ax equal to 0 and Ay equal to zero we get this combination which also zero. This implies that alpha x plus beta y belongs to x. Hence S is a subspace of this F n. And therefore, this set of all solution of the system Ax equal to 0 is called the solution space of this homogeneous system. So, we have the following definition; the set S of all solutions of Ax equal to 0. This homogeneous system is called the solution space of the system is called the solution space of the system.

So next, we shall see some examples of system of linear equations; and we shall apply the results in theorem one, and check whether that system consistent; and if the system is consistent, then we shall find all solutions of that system.

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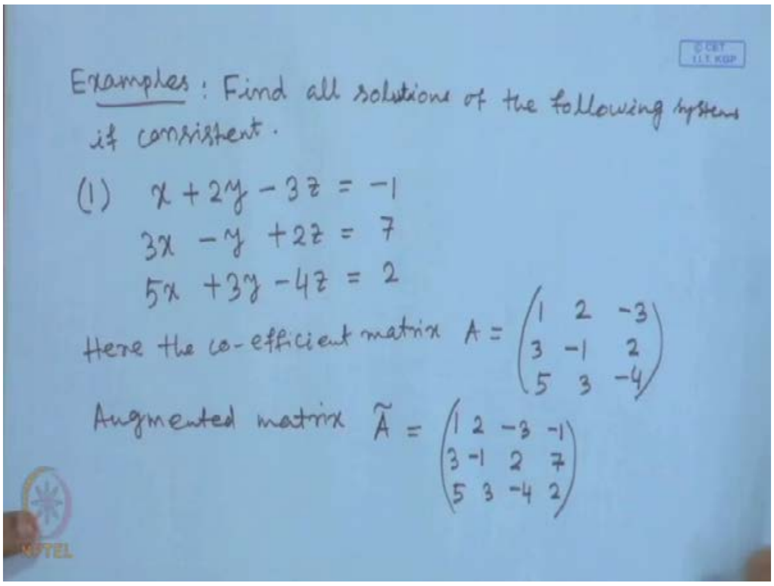

So, we have following examples. So, here we will ask that find all solutions all solutions of the following systems of the following systems if consistent. So, first system is this x plus 2 y minus 3 z is equal to minus 1 , 3 x minus y plus 2 z is equal to 7, 5 x plus 3 y minus 4 z is equal to 2. So, we shall see whether the system is consistent first. Here the co-efficient matrix is co-efficient matrix A is this matrix that is, 1 2 minus 3 3 minus 1 2 5 3 minus 4. Augmented matrix is Augmented matrix A tilta is the matrix: 1 2 minus 3 minus 1 3 minus 1 2 7 5 3 minus 4 2. So, next we shall find out rank of these two matrices, and check whether they are equal or not. So, for this we consider the augmented matrix only.

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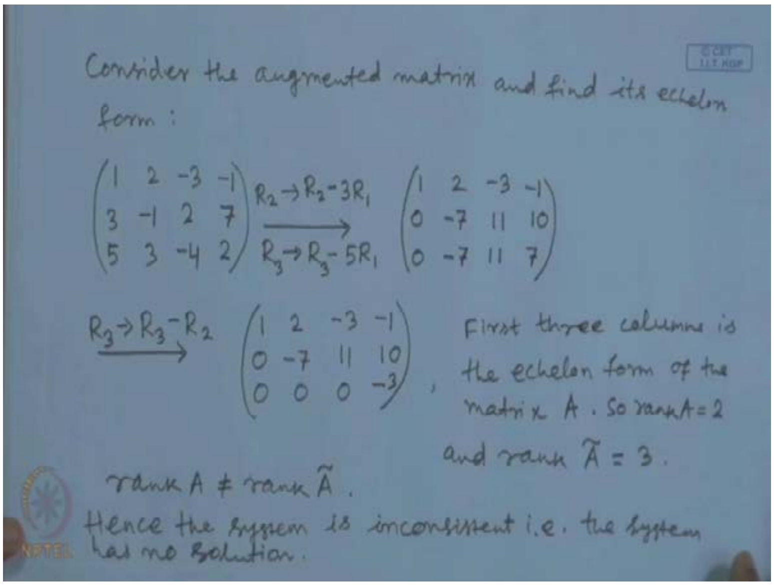

So, now, we consider this consider the augmented matrix, and find its echelon form. So, the matrix is: 1 2 minus 3 minus 1 3 minus 1 2 7 5 3 minus 4 2. So, for finding this echelon form, here we replace this row 2 by row 2 minus 3 row 1 and also this row 3 we replace by row 3 minus 5 times row 1. So, the matrix we get is like this: 1 2 minus 3 minus 1 0 minus 7 11 10 0 minus 7 11 7. So next, again we apply elementary row operations. And here we replace this row 3 by row 3 minus row 2. So, the resultant matrix will be 1 2 minus 3 minus 1 0 minus 7 11 10 0 0 0 minus 3. So, here in this echelon form of the augmented matrix, we can see that first 3 columns of this matrix is the echelon form of matrix A.

So, therefore, the rank of here first three columns is the echelon form of the co-efficient matrix A of the matrix A. So, here in this echelon form of A we are a getting zero row. So, rank of this co-efficient matrix is 2 and rank of this augmented matrix A tilta is equal to 3. So therefore, rank of A is not equal to rank of this augmented A tilta. And hence, the system is inconsistent the system is inconsistent; that means, the system has no solutions.

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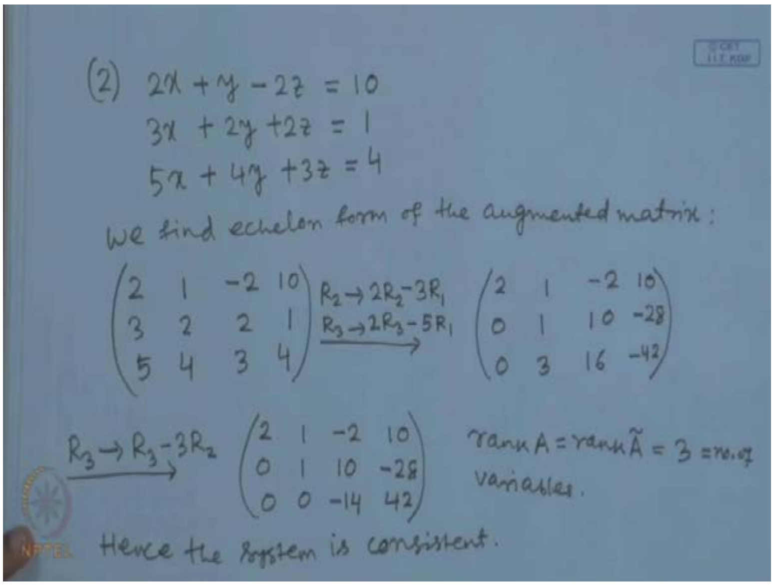

So, next we will see another example; that example two. Here the system is like this; 2 x plus y minus 2 z is equal to 10, 3 x plus 2 y plus 2 z is equal to 1, 5 x plus 4 y plus 3 z is equal to 4. So, here again we will find rank of the co-efficient matrix and rank of the augmented matrix for checking the consistency condition. So therefore, we find echelon form of the augmented matrix. So, here we find echelon form of the augmented matrix. This is given by 2 1 minus 2 10 3 2 2 1 5 4 3 4. So, here we make the following operation: That elementary row operations that row 2 we replace by 2 row 2 minus 3 row 1 3 3 times row 1 and R 3 we replace by 2 R 3 minus 5 times row 1. And we get the matrix be like this that is: 2 1 minus 2 10 0 1 10 minus 28 0 3 16 minus 42.

So next, we apply the following elementary row operation that is row 3 replace we replace by row 3 minus 3 times row 2. And get the matrix be like this that is 2 1 minus 2 10 0 1 10 minus 28 0 0 minus 14 42. So, in the echelon form of the augmented matrix, the first 3 columns is the echelon form of the coefficient matrix. So, notice that rank of this coefficient matrix is equal to 3 as well as the rank of the augmented matrix both are equal to 3, and that is equal to the number of this is equal to number of variables, and hence the system is consistent hence the system is consistent So, here we shall see how to find solutions of the system.

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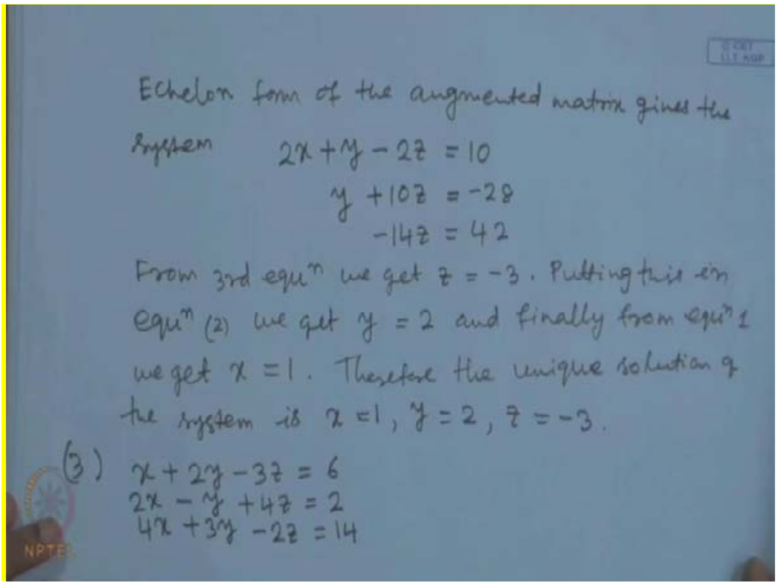

So, from this echelon form of the augmented matrix we get following system that is the echelon form of the given system. So, echelon form echelon form of the augmented matrix gives that gives the system; 2 x plus y minus 2 z is equal to 10, y plus 10 z is equal to minus 28 and minus 14 z is equal to 42. So, from the last equation; from third equation we get z equal to minus 3; putting this in equation 2 We get y is equal to 2 and finally from equation one we get x equal to 1. So therefore, therefore the unique solution of the system of the system is x equal to 1, y equal to 2 and z equal to minus 3. So, next we see another equation. System of equations in this third systems is like this; x plus 2 y minus 3 z equal to 6, 2 x minus y plus 4 z is equal to 2, and 4 x plus 3 y minus 2 z is equal to 14. So, for this system also we find echelon form of the augmented matrix. And here the augmented matrix is like this.

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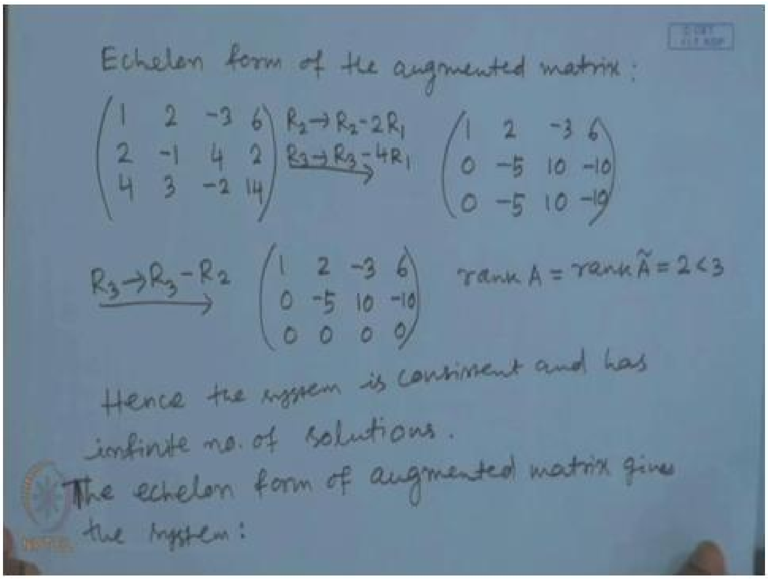

So, here we find echelon form of the augmented matrix. So, the augmented matrix is 1 2 minus 3 6 2 minus 1 4 2 4 3 minus 2 14. So, we apply this elementary row operations, that we replace row 2 by row 2 minus 2 row 1, and row 3 we replace by row 3 minus 4 times row 1, and we get the matrix be like this. 1 2 minus 3 6 0 minus 5 10 minus 10 0 minus 5 10 minus 10, then applying this elementary row operation that, replacing row 3 by row 3 minus row 2, we get the matrix be 1 2 minus 3 6 0 minus 5 10 minus 10 and 0 0 0 0. So, here you see that the rank of the coefficient matrix A is same as rank of the augmented matrix and that is equal to 2, but that is strictly less than 3 the number of variables. Hence the system is consistent, and has infinite number of solutions, infinite number of solutions.

So, next we will find the solutions of this system. So, again from the echelon form of the augmented matrix we get the system be like this. So, here the here the echelon form of echelon form of augmented matrix matrix gives the system

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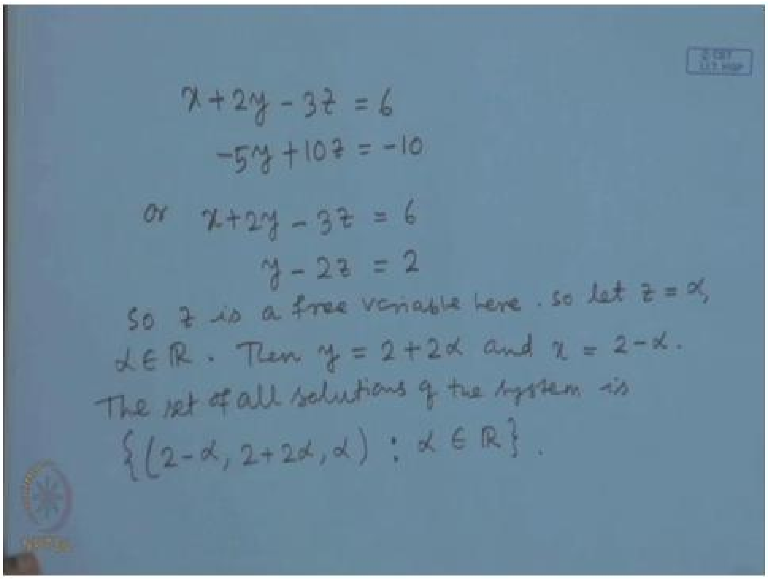

So, this system is like this that is 2 x plus y minus 2 z is equal to 10, y plus 10 z is equal to minus 2. Sorry, that we will get this; sorry, we will not get this one. The system is a like this: x plus 2 y minus 3 z is equal to 6, minus 5 y plus 10 z is equal to minus 10. So this system we get from the echelon form of the augmented matrix, or we can write this system be like this or x plus 2 y minus 3 z equal to 6 and y minus 2 z is equal to that is 2. So, here we have three variables and two equations. So, one variable we take as free variable. We take this variable z as the free variable. In fact, the variables which do not occur at the beginning of any equation in the echelon form is called a free variable. So, z is a free variable here free variable here; that means, it can take any value of the field.

So, let z is equal to alpha and alpha belongs to this real numbers. So, from equation two we get, then this value of y is equal to 2 plus 2 alpha, and value of x will be 2 minus alpha. So, the set of all solutions the set of all solutions the of the system is like this: 2 minus alpha, 2 plus 2 alpha, alpha, such that alpha belongs to the set of all real numbers. Here we are considering the system over this set of real numbers. That is why this alpha will vary over all the elements of this real field. So, next we see another example, and that is a system homogeneous system of linear equations. So, next example is a homogeneous system of linear equations.

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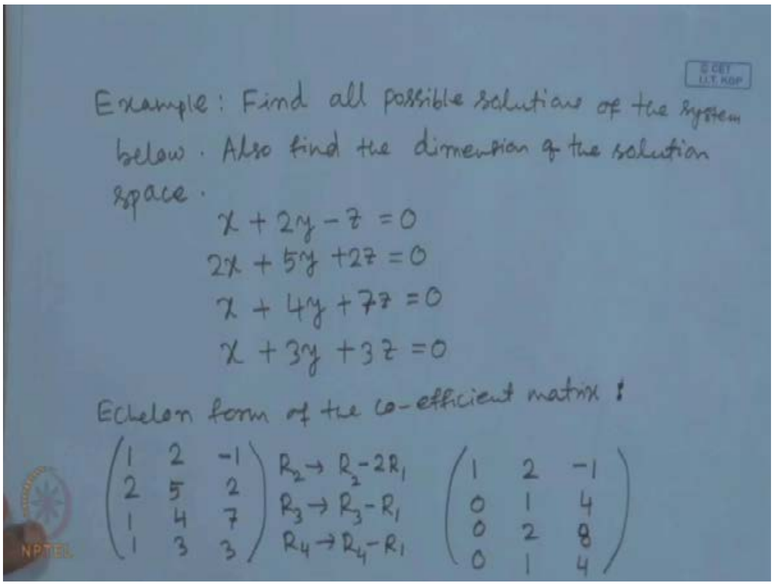

So, here find all possible solutions of the system below. Also find the dimension of the solution space. So, here the system is like this: x plus 2 y minus z equal to 0, 2 x plus 5 y plus 2 z is equal to 0, x plus 4 y plus 7 z is equal to 0, x plus 3 y plus 3 z equal to 0. So, this is the system of linear equations, and this system is a homogeneous system, because all this b i(s) are equal to 0. So, here we shall find out rank of the co-efficient matrix to check whether the system has unique solution or infinite number of solutions. So, echelon form of echelon form of the co-efficient matrix is matrix.

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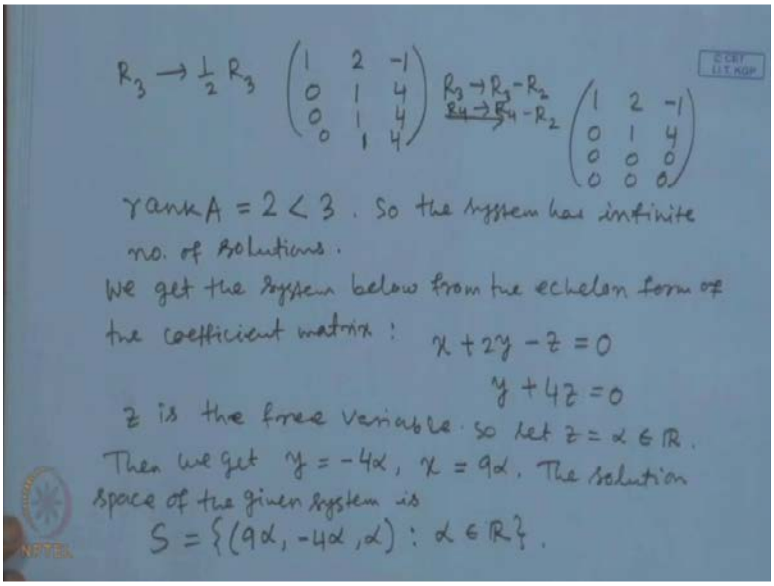

So, the co-efficient matrix is given by: 1 2 minus 1 2 5 2 1 4 7 1 3 3. So, here we apply the following elementary row operations, that row 2 we replace by row 2 minus 2 row 1, row 3 we replace by row 3 minus row 1, row 4 we replace by row 4 minus row 1. And we get this matrix here that: 1 2 minus 1 0 1 4 0 2 4 0 2 this will be 8 and 0 1 4. So here, we next apply the following elementary row operation that we replace row 3 by half of of row 3, and get the matrix be like this: 1 2 minus 1 0 1 4 0 1 4 0 1 4. So here all this 3 rows; row 3 row 2 row 3 and row 4 are identical. So, here we replace this row 3 by row 3 minus row 2, and row 4 also we replace by row 4 minus row 2. And we get the matrix be: 1 2 minus 1 0 1 4 0 0 0 0 0 0. So, from here we get that the rank of coefficient matrix A is equal to 2, and this is strictly the number of variables that is 3.

So, the system has infinite number of solutions infinite number of solutions. And these solutions, we get like this. We get the system below system below from the echelon of the coefficient matrix that that is also called echelon form of the system. And it is x plus 2 y minus z equal to 0, and y plus 4 z is equal to 0. So, again we consider this z as free variable that is variable which do not appear at the beginning of any of the equations in echelon form. So, z is the free variable. So, let z equal to alpha; that belongs to R. Then we get y is equal to minus 4 alpha, and x is equal to 9 alpha. So, this set of all solution are the solution space the solution space of the given system is, this set S that is consists of all vectors like this 9 alpha, minus 4 alpha, alpha, where alpha belongs to this set of all real numbers.

Next, we find out the dimension of the solution space. So, that dimension of the solution space S is also related with rank of the matrix A.

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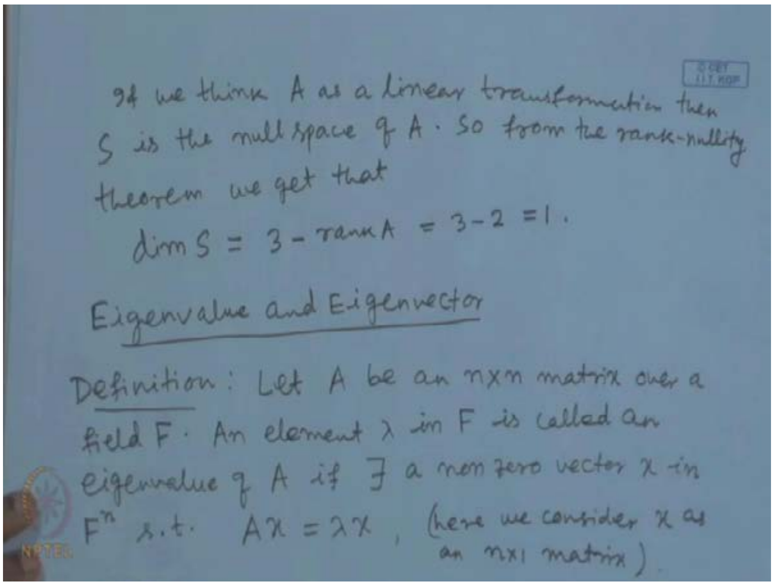

Here, if we consider matrix A is a linear transformation, then this S will be the nullity of this linear transformation A. Therefore, from the rank nullity theorem also we get this result that, if we think A as a linear transformation linear transformation, then S is the null space of A. So, from the rank nullity theorem rank nullity theorem, we get that dimension of the solution space S is equal to total number of variables that is 3 minus rank of A. So, that is 3 minus 2 that is equal to 1. So, this is how we solve system of linear equations; both homogeneous and non homogeneous. And we shall come across the system of equations; latter on while finding eigenvalue and eigenvectors of matrices.

So, next we shall discuss about this another important concept in linear algebra is this eigenvalue and eigenvector of matrices. This is an important concept of linear algebra, and this is very useful in engineering and sciences sciences. Engineers and scientist, they come across this concept of eigenvalue and eigenvectors called open. So, first we shall give definition of an eigenvalue. So, here basically we find eigenvalue and eigenvectors of square matrices. So, let A be an n by n matrix over a field F, then an element of F; an element lambda in F is called an eigenvalue of A if there exist a non-zero vector x in F n, such that this equations holds A x is equal to lambda x. Here, we consider this x as a column vector that is, here we consider x as an n cross 1 matrix are a column vector.

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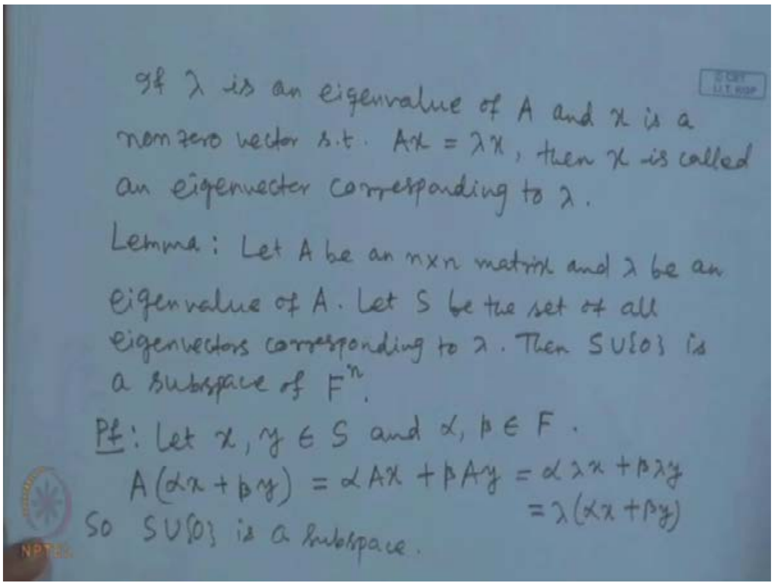

So, if this lambda is an eigenvalue of A, if lambda is an eigenvalue of A and x is a non-zero vector, non-zero vector such that Ax is equal to lambda x , then x is called an eigenvector corresponding an eigenvector corresponding to the eigenvalue lambda. Notice that eigenvectors corresponding to lambda need not be unique. There may be several eigenvectors corresponding to an eigenvalue. So, here in fact, if we we have this following lemma that says that if x 1 and x 2 are eigenvalues to eigenvalue to eigenvectors corresponding to an eigenvalue, then their sum is also an eigenvector corresponding to this same eigenvalue, and also scalar multiple of any eigenvector will be again an eigenvector.

So, in other words, if we include the zero vectors to the set of all eigenvectors corresponding to lambda, then that forms a sub space. So, here we consider that, let A be an n by n matrix and lambda be an eigenvalue of A. let S be the set of all eigenvectors all eigenvectors corresponding to lambda to lambda, then this S union the zero vector is a subspace of this field F n. So, here this proof is not difficult, it is trivial. So let us consider let x and y be two eigenvectors corresponding to lambda and alpha, beta be two scalar from F. So, here this alpha x plus beta y will be an eigenvector corresponding to this eigenvalue lambda or not; that is what we check. S, A of alpha x plus beta y is equal to alpha Ax plus beta Ay, and this is equal to alpha into lambda x plus beta into lambda y or this can be written as lambda times alpha x plus beta y. So, this S union 0 is a subspace. And this subspace is called eigenspace corresponding to the eigenvalue.

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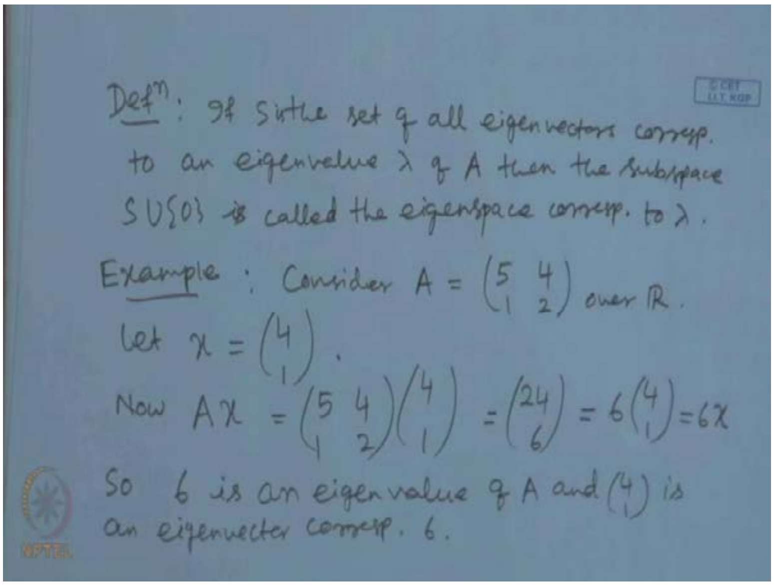

We give this definition of an eigenspace. So, if S is S is the set of all eigenvectors corresponding to corresponding to an eigenvalue lambda of A, then the subspace S union 0 is called the eigenspace eigenspace corresponding to corresponding to this eigenvalue lambda. So, let us see an example quickly. Here we consider this matrix consider this matrix A be like this: 5 4 1 2 over this real numbers of course. And a vector x be like this: 4 1. Now, this Ax that is 5 4 1 2 that is multiplied by 4 1 that gives 24 6 that is 6 times 4 1, so that is equal to 6 times x. So, this implies that 6 is an eigenvalue of A an eigenvalue of A; and 4 1 is an eigenvector corresponding to the eigenvalue 6. So, we stop this lecture here. And in the next lecture, we will discuss about method to find eigenvalues and eigenvectors of matrices. So, that is all for this lecture.

Thank you.