Advanced Engineering Mathematics

Prof. Pratima Panigrahi
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture No. # 04

Linear Transformation, Isomorphism, & Matrix Representation

So, in this lecture we shall study linear transformations, isomorphism and matrix representation of linear transformations. Actually, when we compare mathematical structures of same type that I am we study some order preserving mappings. And such order preserving mapping in case of a linear algebra are called linear transformations.

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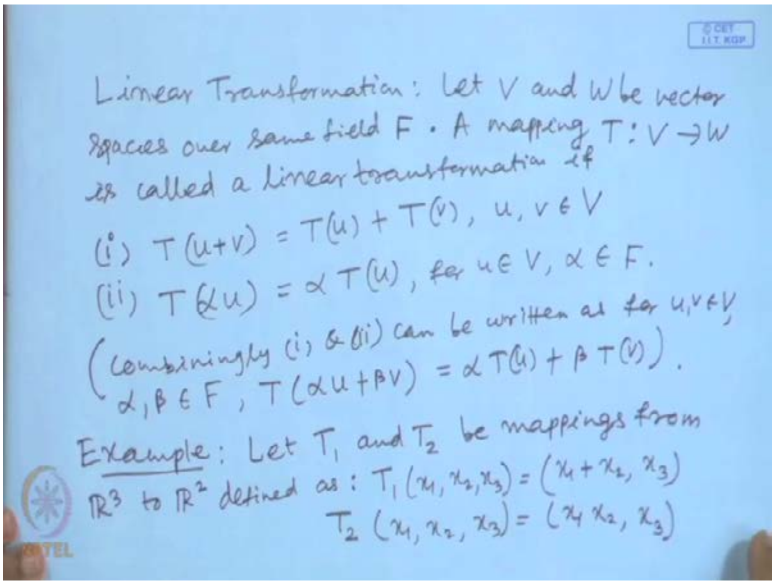

So, let us define this linear transformation. So, let V and W be vector spaces over same field F, A mapping T from V to W is called a linear transformation. If the following that first condition is that T of u plus v is equal to T u plus T v for any vectors u and v in the vector space v here of course. This plus operation in the left hand side represents plus operation in the vector space v and this plus operation in the right hand side represents plus operation in the vector space w. Or in other words this first condition means that this mapping T preserves addition operation of the vector space. Here T of alpha u is equal to alpha times T u for u belongs to the vector space v and alpha comes from this field F.

The second condition is also means that the this mapping T preserves scalar multiplication combining this first and second can also be written like this. So, combining first and second can be written as for u v belongs to vector space v and alpha beta belongs to F, T of alpha u plus beta v is equal to alpha times T u plus beta times T v. One can check this condition also for mapping T is linear transformation or not. So, let us see one example that let us check so, let T 1 and T 2 be mappings from R 3 to R 2 defined as defined as T 1 of x 1 x 2 x 3 is equal to x 1 plus x 2 and x 3, T 2 of x 1 x 2 x 3 is equal to x 1 x 2 here x 3. We shall check whether T 1 and T 2 are linear transformations or not.

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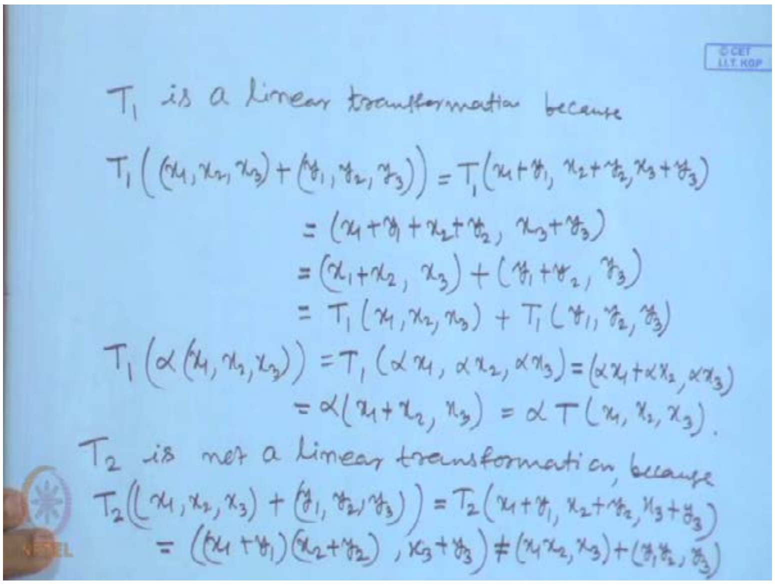

So, this T 1 is a linear transformation because it preserves addition and scalar multiplication like T 1 x 1 x 2 x 3 plus y 1 y 2 y 3 is equal to t of x 1 plus y 1 x 2 plus y 2 x 3 plus y 3 and according to this is T 1. So, according to definition of T 1 it is like this x 1 plus y 1 plus x 2 plus y 2 x 3 plus y 3 and this can be written as x 1 plus x 2 x 3 plus y 1 plus y 2 y 3 and that is equal to T 1 of x 1 x 2 x 3 plus T 1 of y 1 y 2 y 3. So, T 1 preserves addition operation and similarly, T 1 also preserves scalar multiplication because T 1 of alpha times x 1 x 2 x 3 that is equal to T 1 of alpha x 1 alpha x 2 alpha x 3 and according to definition of T 1 this is equal to alpha x 1 plus alpha x 2 alpha x 3 and this can be written as alpha times x 1 plus x 2 x 3 and that is equal to alpha times T of x 1 x 2 x 3.

So, therefore T 1 is a linear transformation, but one can said that T 2 is not a linear transformation because it does not preserve addition scalar multiplication also. So, because T 2 of x 1 x 2 x 3 plus y 1 y 2 y 3 this is equal to T 2 of x 1 plus y 1 x 2 plus y 2 x 3 plus y 3 and according to the definition this is equal to x 1 plus y 1 into x 2 plus y 2 x 3 plus y 3 and of course, this is not equal to x 1 x 2 x 3 plus y 1 y 2 y 3. So, therefore this T 2 is not a linear transformation.

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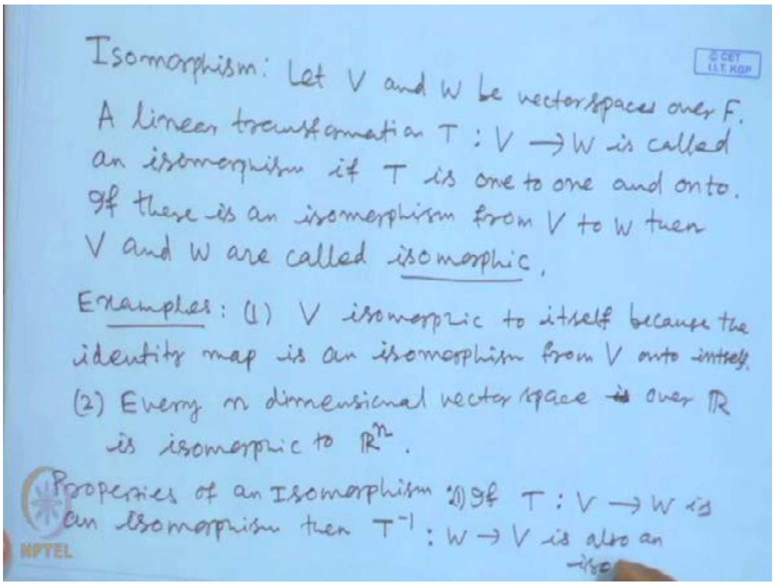

So, linear transformations are very useful and here we have one specific type of linear transformation that is called isomorphism. So, that here we consider again vector spaces V and W be vector spaces over the same field F a linear transformation T from V to W is called an isomorphism. If T is one to one and onto, if there is an isomorphism from V to W, then V and W are called isomorphic. So, isomorphic means that V and W have same structure algebraically they are the same structure. So, for every algebraic structures we consider this concept of isomorphism this verifies whether given vector spaces same structure or not.

So, we can have some examples of isomorphism so, this the first is condition is that V is isomorphic to itself because the identity map is an isomorphism from V onto itself. So, every another example is that every finite dimensional n dimensional we can say every n dimensional vector space is vector space over R is isomorphic to R n of course, we will see some more example of isomorphisms later on. So, here we will see some properties of an isomorphism, that properties of an isomorphism. So, if properties are like this if T from V to W is an isomorphism then T inverse from W to V is also an isomorphism.

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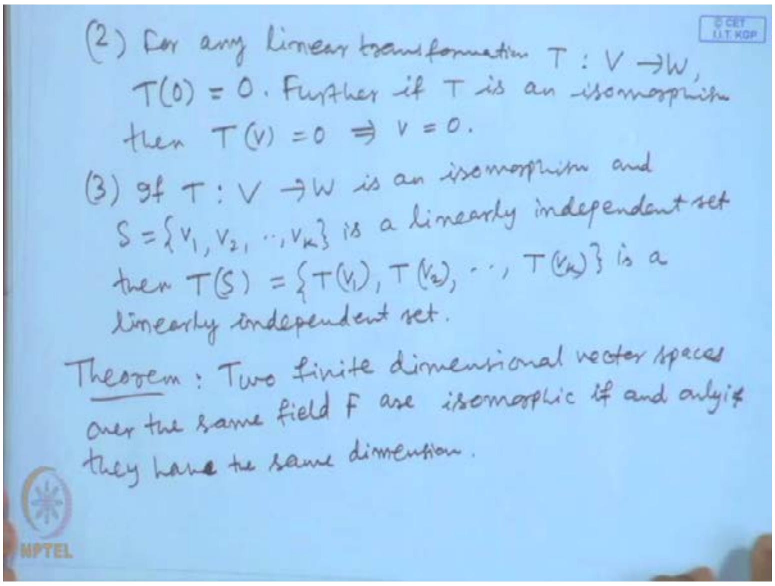

So, for any linear transformation we have for any linear transformation T from V to W that is T 0 with 0 vector again and further if T is an isomorphism then T of V equal to 0 implies that V is the 0 vector. And third property is like this and that an isomorphism third property says that an isomorphism maps are linearly independent set to a linearly independent set. That is if T from V to W is an isomorphism and S is consist of a set of linearly independent vectors S is consist of vectors V 1 V 2 to V k is a linearly independent set. Then this inverse of S T S that is T of V 1 T of V 2 T of V k is a linearly independent set.

So, of course this property we have already given in the example so, this theorem gives a result which checks whether I mean one two finite dimensional vector spaces are isomorphic. So, the result is like this two finite dimensional vector spaces over the same field F are isomorphic if and only if they have the same dimension. So, this gives a criteria for checking whether finite dimensional vector spaces over the same field are isomorphic or not. So, next we shall see another important point that we defined some spaces associated with a linear transformation they are called Rank's spaces and Null's spaces. So, they play an important role.

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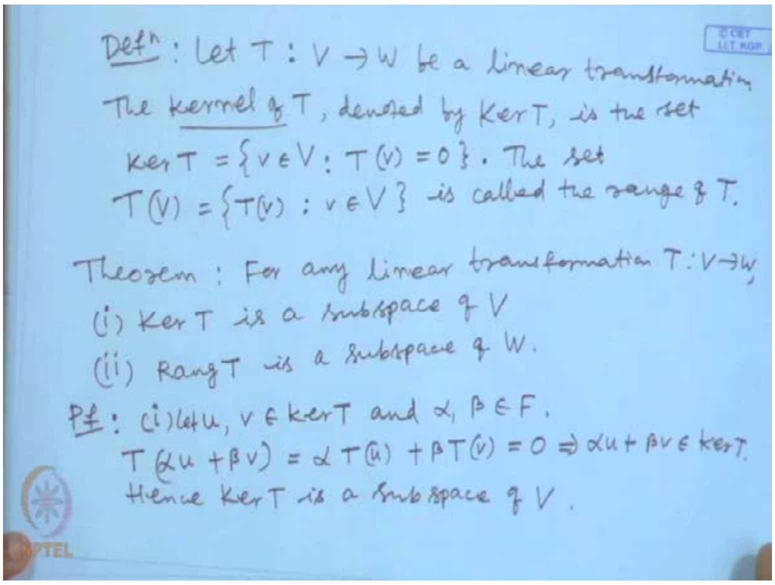

So, let us have this definition that let T from V to W be a linear transformation. Then the Kernel we define the Kernel of T denoted by Ker T is the set Kernel of T is consist of all vectors in the vector space V such that T of V is equal to 0. And the set that is T of V that is consist of T of V such that v belongs to V is called the range of T. And if this T is so, if for any linear transformation T we have this result one can check easily that for any linear transformation T from V to W. Kernel of T is a subspace of V and second result is that range of T is also a subspace of is a subspace of W that you can that one can check easily by using the definition of a linear transformation and it is not hard to check.

In fact, we can also give briefly group of this the first one is say we have to consider that vectors u and V in kernel. So, let u and v belongs to Kernel of T and alpha beta they comes from the field F. So, this T of alpha u plus beta v that is equal to alpha times T u plus beta times T v this is equal to 0. So, this implies that alpha u plus beta v belongs to this Kernel of T and hence Kernel of T is a subspace of V. Similarly, one can also prove this range of T is a subspace of the V.

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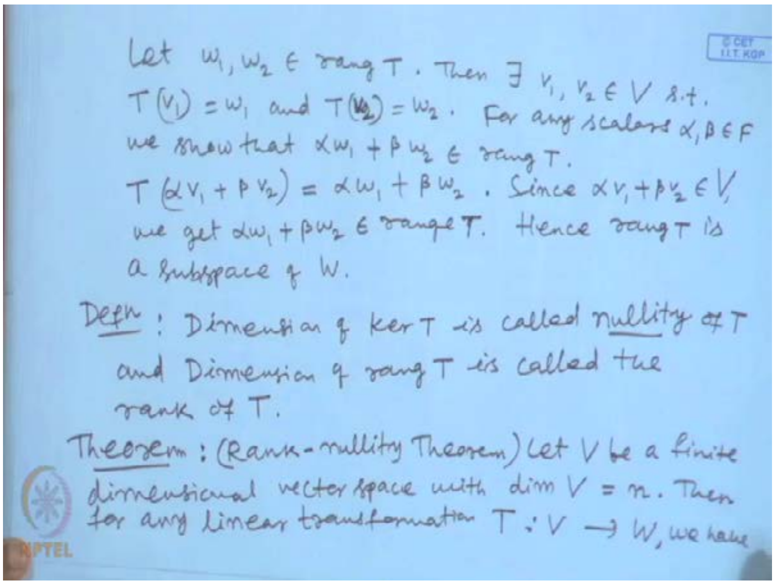

So, let w 1 w 2 belongs to this range of T then there exist vectors v 1 v 2 in the vector space v such that T of v 1 is equal to w 1 and T of this v 2 T of v 2 is equal to w 2. So, we will we shall show that this for any scalars for any scalars alpha beta in F we show that alpha w 1 plus beta w 2 belongs to range of T. So, now the inverse of that alpha v 1 plus beta v 2 is equal to alpha w 1 plus beta w 2. So, since this alpha v 1 plus beta v 2 is an element of v we get that alpha w 1 plus beta w 2 that belongs to this range of T hence this range T is a subspace of w. So, here we give this another definition that the dimension of this dimension of Kernel of T is called nullity of T and dimension of range of T is called the rank of T.

So, here we are having an important result that relates this nullity and rank and dimension of the vector space of course, the dimension of the vector space has to be finite. So, that result is known as this rank nullity theorem so, this is there is (( )) theorem called rank nullity theorem. So, let V be a finite dimensional vector space with dimension of V be equal to n. Then for any linear transformation T from V to W we have the following relation, we have this following relation.

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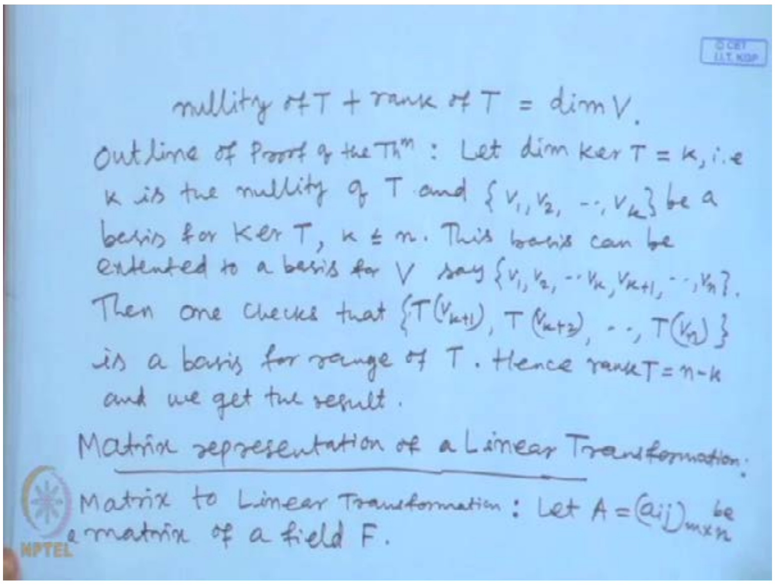

That nullity of T plus rank of T is equal to dimension of V. So, of course one can prove this theorem easily it is a not difficult you will keep your outline of proof of the theorem. So, we have one considers that since dimension of V is finite, that dimension of Kernel of T is also finite let dimension of Kernel of T that is also nullity of T be equal to k. This is that is k is the nullity of T and v 1 v 2 to v k be a basis for Kernel of T of course, this k is less than or equal to n. So, now this basis of kernel of T can be extended to a basis the this basis can be extended to a basis for v say v 1 v 2 v k v k plus 1 up to v n. Then one checks that this T of v k plus 1 of these vectors v k plus 1 v k plus 2 up to T of v n is a basis for range of T.

Hence rank of T is equal to n minus k and we get the result so, this rank nullity theorem is also very useful. So, next we shall see matrix representation of a linear transformation that is matrix representation of a linear transformation. So, here we see that every linear transformation can be expressed as a matrix and every matrix gives a linear transformation there is some set of correspondence among matrices are linear transformation. So, some time people refer therefore, people refer of matrices linear transformations also so, first we shall see from matrix to linear transformation. So, this matrix to linear transformation so, if we have we are having a matrix m by n matrix.

So, say A the entries a i j be an m by n matrix over let A be a matrix over a field F then find a linear transformation associated with this matrix. So, here we cal that an m dimensional of vector space over F.

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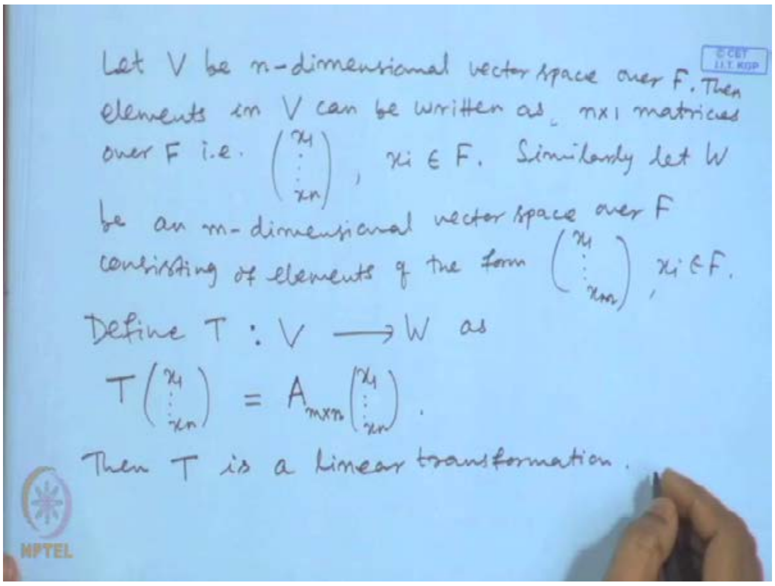

So, we consider that so, let V be n dimensional vector space over F then elements in V can be written as n (( )) of elements in V or that is n minus 1 matrices over F that is x 1 x 2 of 2 x n here x i s they come from F and similarly, let W be an m dimensional vector space over F consisting of elements of the form x 1 up to x n x m with x i comes from F. Then we define a say T from V to W as T of any element x 1 x 2 to x n is this matrix A this is m of size m by n and that is multiplied by this matrix x 1 x 2 to x n. So, this is a matrix multiplication and it is resultant will be an ten by one matrix or that is an element of W then easily one can verify that T is a linear transformation. Then T is a linear transformation this can be checked by (( )) a properties of this matrix A also and this is how we linear transformation T from given m matrix n notice. That if A is a matrix of size m by n then we get a linear transformation from a m dimensional vector space V to an n dimensional vector space W.

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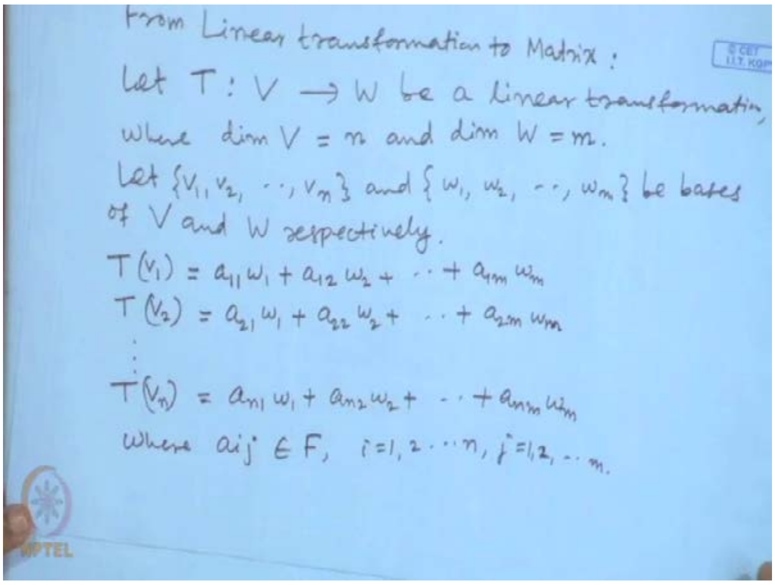

So, next we shall consider the converse of this that is from linear transformation to a matrix that converse of this from linear transformation to matrix or in another words we see matrix representation of a linear transformation. So, here we consider this linear transformation let T from V to W be a linear transformation where dimension of V is equal to n and dimension of W is equal to m. And we consider bases in the vector space V and W. So, let this v 1 v 2 v n and w 1 we 2 w n be basis for V and W respectively. Actually we find matrix representation of linear transformation T with respect to this bases; that means, we fix a bases in the vector space V and a bases in the vector space W.

And with respect to these two bases we find matrix representation of this linear transformation. So, here we find this matrix corresponding to this linear transformation like this so, T of v 1 we shall see images of the vectors v 1 v 2 to v m with respect to this linear transformation. So, T of v 1 is a is an element in the vector space W so, therefore T of v 1 can be written as a linear combination of the basis vectors w 1 w 2 to w n. So, let us consider that linear combination be like this T of v 1 is equal to a 1 1 w 1 a 1 2 w 2 plus a 1 m w m and T of v 2 is a 2 1 w 1 a 2 2 w 2 plus a 2 m w m like this T of v n of v n is equal to a n 1 w 1 plus a n 2 w 2 a n m w w m so, this is the w m. So, here the all these a i j where all these a i j they are in the field F i from 1 to 2 n and j from 1 2 to m they all belongs to this field F.

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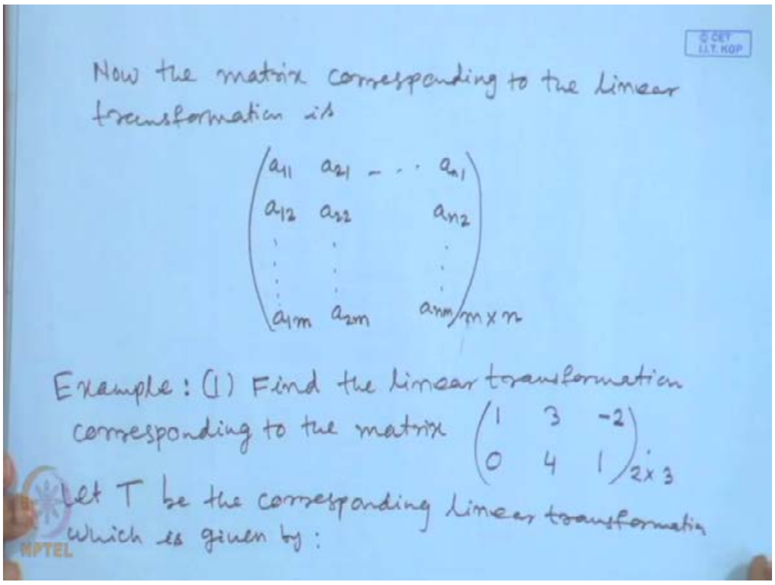

So, now from this expression this we find the matrix corresponding to the linear transformation T and it is like this. That now the matrix corresponding to the linear transformation is, this is matrix that here we consider the coefficients of first linear combination as first column that a 1 1 a 1 2 and this a 1 n a 2 1 a 2 2 a 2 n and like this a n 1 a n 2 a n m. So, this is this matrix here we are having this basically and one this is 1 a 1 n so, this 1 m 2 n a n m so, this is basically an m by n matrix. So, size of this matrix is m by n so, here we will consider an example how to find this matrix representation of a linear transformation. We notice that if we consider different bases for the vector spaces V and W then we may get different matrix corresponding to this linear transformation.

Of course those matrices are not totally different they are also related and they are in fact, similar and that we are not going to prove. So, let us see one example so, we consider this example first example is that we shall consider a matrix and from there we get a linear transformation. So, that is find the linear transformation associated with or corresponding to the matrix say that is 1 3 minus 2 0 4 1. So, here this matrix is of size two by three so, we get a linear transformation from R cube to R square. So, the corresponding linear transformation so, let T be the corresponding linear transformation this is given by.

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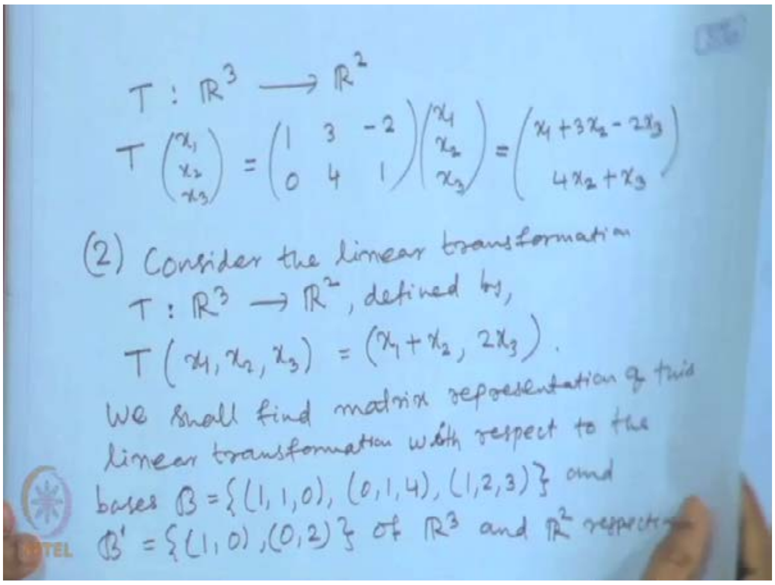

So, T is basically a linear transformation from R cube to R square and T of any x 1 x 2 x 3 that is given by 1 3 minus 2 0 4 1 that we multiply with this x 1 x 2 x 3 and we get this as x 1 plus 3 x 2 minus 2 x 3 and here we get this 4 x 2 plus x 3. So, this is the linear transformation associated with the given matrix then example two is like this here we consider a linear transformation T from R cube to R square defined by T of x 1 x 2 x 3 is x 1 plus x 2 twice x 3. We will we shall find matrix representation of this linear transformation. So, here we this linear transformation with respect to the bases say that is B consistently of 1 1 0 0 1 4 1 2 3 and B prime is consist of 1 0 0 2 of R cube and R square respectably.

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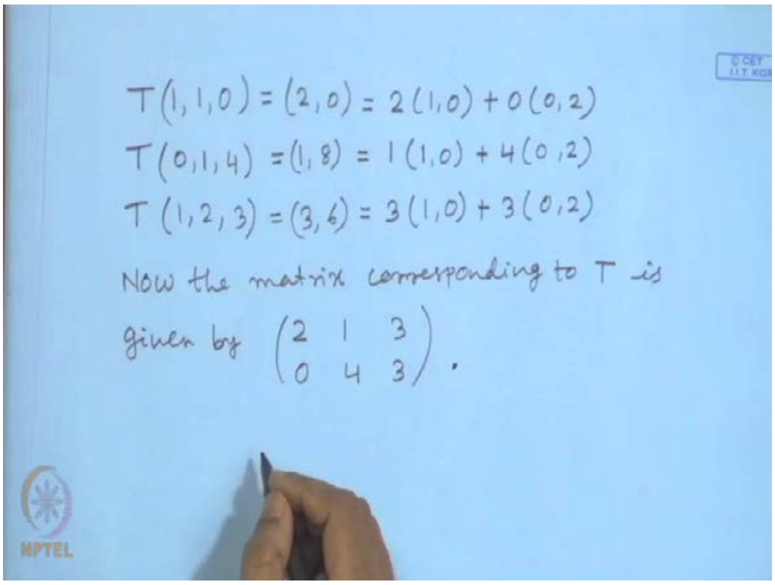

So, now we shall find image of this of bases vectors that T of 1 1 0 that is according to a definition of this linear transformation these values 2 0 and that can be written as linear combination of bases vectors in the space is like this 2 into 1 0 plus 0 times this 0 2 image of the next bases vector that 0 1 4 is 1 8. And this can be written as linear combination of bases vectors in R square or in the range of T S 1 times 1 0 plus 4 times 0 2. Then image of 1 2 3 is 3 6 from the definition of this T and this can be written as 3 times 1 0 plus 3 times 0 2.

So, now this coefficients from these coefficients we get the matrix of T now the matrix corresponding to T is given by that 2 0 1 4 3 3 or in another words that also we can think is transpose of this coefficient matrix. So, this is how we find matrix representation of a linear transformation. So, of course, here we are getting this matrix representation of a linear transformation with respect to some bases. We change this bases in the range space of this T, then we get different matrix that is all for this lecture here we stop thank you.