Advanced Engineering Mathematics

Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture No # 31
Applications of Fourier Transform to PDEs

Welcome back to the lectures on transform calculus. And in the last lecture, we have studied Fourier transform, and its properties. So, today we will continue this lecture for the application part, and we will discuss its application to partial differential equations. So, the procedure is very similar, what we have done in the case of Laplace transform. So, for a given partial differential equation, we will apply the Fourier transform to both the side of the equation. And then, this partial differential equation will be transformed to a simpler ordinary differential equation that we will solve, and at the end by taking the inverse Fourier transform of the solution of this ordinary differential equation. We will arrive for the solution of the original partial differential equation.

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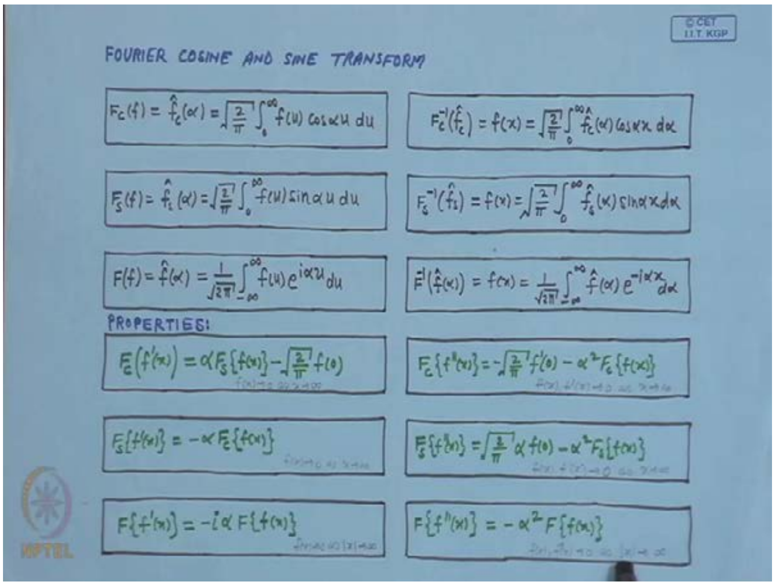

So, before I go for the example let me just summarize some formulas, and the properties of the Fourier transform. So, we have here the Fourier Cosine, and sine transform. So, Fourier Cosine transform was just denoted by F c hat alpha, and the definition was square root 2 over pi and integral 0 to infinity f(u) Cos alpha u du, and its inverse of Fourier Cosine Cosine inverse of this f hat c will be just f(x), and this is given by square root 2 over pi 0 to infinity f c hat f for Cos alpha x d alpha.

Similarly, we have for the sine transform instead of this Cos alpha u, we have sine alpha u; and similarly here this in place of Cos alpha x, we have sine alpha x d alpha, and then we studied this Fourier transform. So, in this case, the Fourier transform of f, we will also denote by this f hat alpha, because is a function of alpha now; 1 over square root 2 pi and the integral over the whole access, and we have f(u) e i alpha u du. And for the inverse, we have again the same constant there, 1 over square root 2 pi and minus infinity to plus infinity, and then we have f hat alpha and this will be e power minus i alpha x on the integral over this d alpha.

So, these are the main properties are mainly the derivative theorem for the Fourier transform, and Fourier Cosine and sine transform. So, first for the Fourier Cosine transform, we will use this for the first derivative it will be alpha, and Fourier sine transform of f and minus square root 2 over pi f(0). And this is the the condition under which we got this result that was that f(x) approaches to 0 as x approaches to infinity; for the second derivatives the Fourier Cosine transform of the second derivative of f, we have minus square root 2 over pi f prime 0 minus alpha square Fourier Cosine transform of f(x). And in this case, we have this result under the conditions that f(x), and f prime x the first derivative, and the function itself approaches to 0 as x approaches to 0.

So, in in these cases… So, basically we will have solve today the partial differential equations mainly, and in that case the function depends on 2 variables, that we will get this Fourier Cosine or sine or Fourier transform with respect to one variable. So, the same formulas will whole the other variable will be treated as constant. One more point we should mention here, before we go for the the examples that we have to see for a particular problem that which one is applicable, whether we should apply the Fourier, Cosine transform or Fourier sine transform or the Fourier transform.

So, one is clear that when the limit of the variable, where we will be taking the Fourier transform is from minus infinity to plus infinity, then we will of course, apply this Fourier transform, but if our range for the variable is given from 0 to infinity. Then we have these two (( )) either Fourier Cosine transform or Fourier sine transform. So, in these cases, if we just look at the properties - the derivative properties here, so for the Fourier Cosine transform; for example, here in the double derivative will be using for the second order partial differential equations.

So, here this f prime 0 appears whereas, in the sine transform this f(0) appears with function value at x is equal to 0, and this is the first derivative of that function at x is equal to 0. So, if this condition is given, and the range of x is 0 to infinity we will apply the sine transform, and if the first derivative - this condition is given a first derivative of 0 is whatever this is given, then we will apply the Fourier Cosine transform. And Fourier transform, when the limit when the range of that variable x from minus infinity to plus infinity.

So, with this information, we continue now for the different partial differential equations, short introduction to partial differential equation, I have already given in Laplace transform case.

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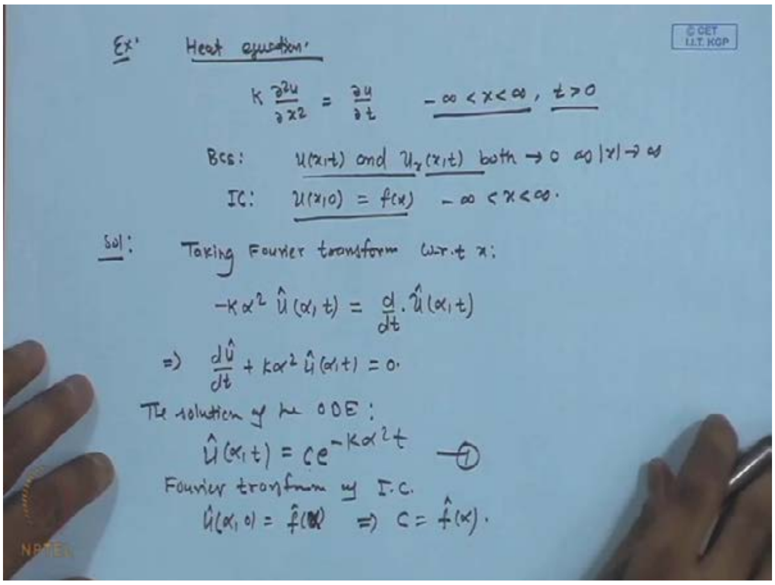

So, we will directly go to the application to be these. So, we will solve first the heat equation heat equation, and that is k del 2 u over del x square del u over del t, and our limits are minus infinity to plus infinity t as given positive. The boundary conditions are given u(x,t) and u x(x,t); both goes to 0 as absolute value of x goes to infinity, and the initial conditions are f(x,0) as f(x). So, we have only the first derivative here is only one initial condition, and for x minus infinity to plus infinity.

So, here the choice of the Fourier transform is clear. So, we will apply here the Fourier transform, because our variable is from minus infinity to plus infinity; just remember for the Laplace transform, we we applied Laplace transform with respect to t, because t is always from 0 to infinity, t can vary from 0 to infinity, but now we will apply for the for the x variable from minus infinity to plus infinity.

So, now taking Fourier transform with respect to x. So, what we will get here, with respect to x is the double derivative, and if we look at the table. So, the double derivatives minus pi square, and the Fourier transform of f(x). So, we have minus k alpha square, and Fourier transform I will denote by this alpha, and t will remain as it is, because we have taken with respect to this x, we have u(x,t). So, with respect to x we have taken, so this x is replaced by alpha. And the right hand side this d over dt will remain as it is, and the Fourier transform of this u will be u hat and alpha t.

So, what we get d u hat over d t plus k alpha square u hat alpha t is 0; note that this boundary conditions for this problem, we have already used here, because the Fourier transform of this double derivative minus alpha square u hat alpha t. We have use these 2 boundary conditions. So, the solution of this ODE will be just the characteristic (( )), here directly we can have this. So, d u hat over we can separate the variable, and we will get this L and u hat. So, clear the the solution is e minus k L pi square, and the integral this side will be t, and we have some constant of integration. So, we now know the Fourier transform of the initial condition given u(x,0) is f(x).

So, Fourier Fourier transform transform of the initial condition, what we will get. So, u hat x will be alpha 0, and the Fourier transform of f(x). So, f hat alpha. Now, we use this condition to get this constant. So, t is equal to 0, we have f hat alpha. So, c is simply. So, this implies that our c is f hat alpha, and then we apply this.

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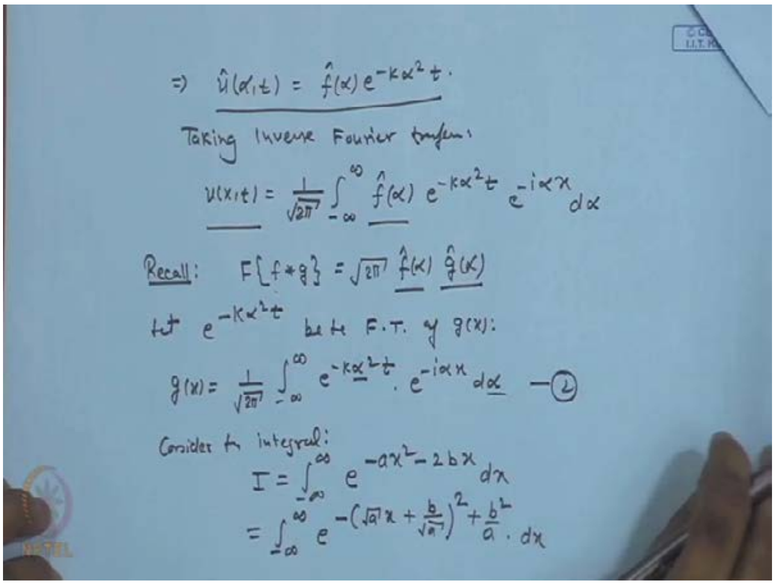

So, we get u hat alpha t is f hat alpha, and e minus k alpha square t. Now, we take the inverse Laplace transform taking the inverse Fourier sorry Fourier transform not the Laplace Fourier transform, what we get u(x,t); if we directly apply the definition of the inverse, what we will get f hat alpha e minus k alpha square t, and e minus I alpha x d alpha. So, now we note that in this solution, if we leave this (( )), it is not a close form at all, because in this integral we are using this Fourier transform of f of the initial condition. So, it is better to have a solution which does not have this Fourier transform, we may have the initial, condition because that is given all already, but we should not expect to have this Fourier transform.

So, in order to avoid this, what we do just recall the convolution theorem. So, we had the Fourier transform of f star g was square root 2 pi, and f hat alpha and g hat alpha. So, if we take the Fourier inverse transform here. So, f inverse of of this the multiplication of 2 Fourier transform like we have here. So, if we know the inverse Fourier transform of this, then we can get u(x,t) just by the convolution of f and g. So, we let now that e minus k alpha square t be the Fourier transform of g(x). So then, by the definition what we have 1 over square root 2 pi, and minus infinity to plus infinity e minus k alpha square t, and e minus i alpha x d alpha.

So, now we need to evaluate this integral. So, for this we consider a simplified form. So, consider the integral I, and then we will come back to this integral again minus infinity to plus infinity e. So, here we have with respect to alpha. So, this is alpha square, so we take here some constant times alpha square, this constant will be k t in our case; and minus just for the simplicity we take 2 b x and dx. So, here we have x here also we have this alpha. So, what we do now, we try to put this in the square form of a so, minus infinity to plus infinity e minus square root a x, this is the whole square, and p b over square root a well, k was the multiplication of these 2 times; 2 x b with minus so, we have this extra. What extra we have here b square over a with minus, so we have 2 add here b square over a and then d x. So, this e power b square by a, we can take out of this integral.

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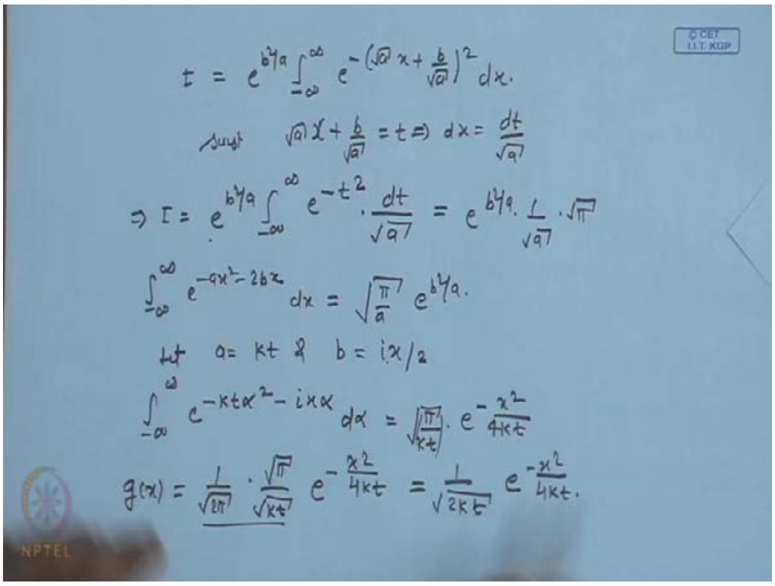

So, we have this integral e b square over a minus infinity to plus infinity minus square root a x plus b over square root a square and dx. Now, we substitute this new variable a x plus b over square root a to t. So, that we have dx is equal to d t over square root a, and this implies that I is e b square over a, and we have minus infinity to plus infinity e minus this is t. So, we have t square, and dx is dt over square root a. So, we have then this square root a, we can take out of this integral. So, e b square over a, and 1 over square root a, and this integral minus infinity to plus infinity minus t square dt that is the Gaussian standard integral, we have a square root pi the value. So, we got this integral we had minus infinity to plus infinity e minus a x square minus 2 b x dx is equal to pi over a with square root, and e b square over a. So, if we let now, because you want to go back to this integral.

So, we will choose our a is k t, and this 2 b is i x. So, a as k t, and this 2 b are b i x by 2. So, we will get the our required integral, and we change this integral variable to alpha. So, e minus k t alpha square for this x square, and minus 2 b; so, b is 2 b is i x. So, minus i x, and this x we replace to alpha. So, d alpha and we have pi over a is k t, and e b square i square x square by four. So, here we have then b square. So, that is minus x square over pi by a is k t. So, a b square over a is k t. So, we have k t, and this 4 comes from here. So, we have minus x square over 4 k t is square root pi over k t. So, now we go back to this integral g(x) to equation 2. So, this we have evaluated, and 1 over square root 2 pi will come.

So, our g(x) is 1 over square root 2 pi, and we have a square root pi and square root this k t, and e minus x square over 4 k t. So, we simplify this is square root pi will cancel out and we have a square root 2 k t from here, and we have e minus x square 4 k t.

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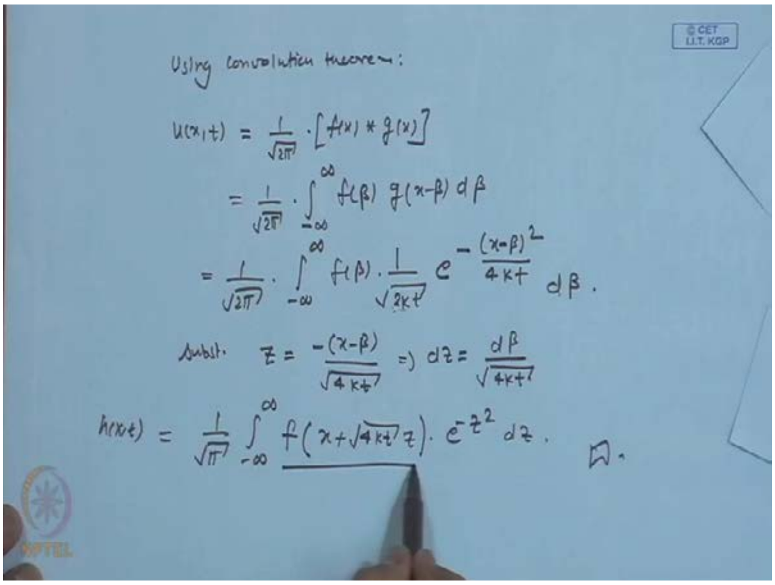

And now we can use the convolution theorem, because we had this u alpha t is f hat alpha, this the inverse transform we know, and now for this also we know that g(x) is the inverse transform of this. So, use the convolution theorem. So, using convolution theorem, we get u(x,t) is 1 over a square root 2 pi and this f(x) convolution with g(x). So, this is 1 over square root 2, pi and we have this convolution minus infinity to plus infinity f beta, and g(x) minus beta d beta. So, 1 over square root 2 pi, and minus infinity to plus infinity, we have f beta, and g beta is given 1 over 2 k t e minus x square.

So, x is now, x minus beta over 4 k t, and we have d beta. Again let us simplification we can made, if we take z is equal to minus x minus beta over 4 k t. So, this 1 here. So, that we have this square again. So, this t z will be d beta over 4 k t. So, square root 2 square root 2, we have this square root 4 k t, and this d beta will be. So, we have 1 over square root pi u(x,t), because this square root 2, we can have with this is square root 2. So, that we have exactly this term, and minus infinity to plus infinity f for the beta we have to get from here; that will be x plus square root 4 k t and the z, and we have e minus z square d z. So, this is a solution of the of the problem, and we have in terms of the given function f.

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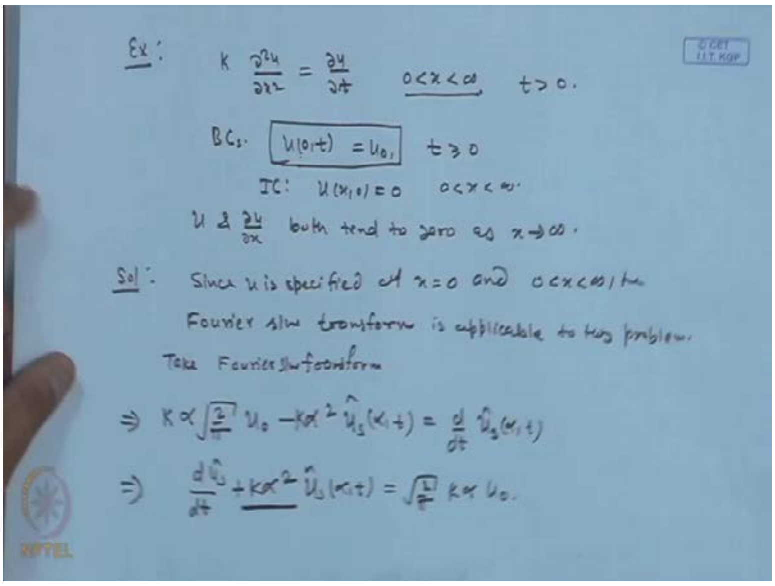

So now, we take another problem, where we will apply the Fourier sine (( )) sine transform. So, we have the problem k del 2 u over del x square is equal to del u over del t, and x is between 0 and infinity, t is positive. So, the boundary conditions are given u(0,t) is u 0 for t positive greater than equal to 0, and the initial conditions are given u(x,0) is 0. And also that information del u and del u over del x and u both tend to 0, as x approaches to infinity. So, now the solution, and now we note that this u is specified at x is equal to 0. So, we have this boundary condition u is given at 0, and our range for the x is 0 to infinity. So, we have twice for Cosine or sine transform, but this u is given. So, let us have a look again for these properties. So, if we have the Cosine transform, then we need here f prime 0, but this information is not given in the problem for the for the sine transform we need f(0).

So, this is given, so we will apply the sine transform the twice is very clear. So, let me just also write, since u is specified at x is equal to 0 and x is between 0 to infinity, the Fourier sine transform is applicable to this problem. So, we take the Fourier transform, taking Fourier transform - Fourier sine transform sorry Fourier sine transform, what we get. So, this k is there and for the del 2 u over del x square for the second derivative sine transform square root 2 pi alpha f(0), and minus alpha square Fourier sine transform. So, we have alpha, and a square root 2 over pi the function value at 0, that is this is u(0), it is given minus we have alpha square, and k is there already; so, k alpha square, and the Fourier sine transform of u.

So, Fourier sin transform of u, and the right hand side we have with respect to t. So, this will remain as it is this differentiation, and we have the sin transform of u. So, now what we have you a set over d t plus k alpha square u s head alpha t, and this is square root 2 over pi, and we have k alpha and u naught. Now the, now we need to solve that equations, so far that what we get we get the integrating factor.

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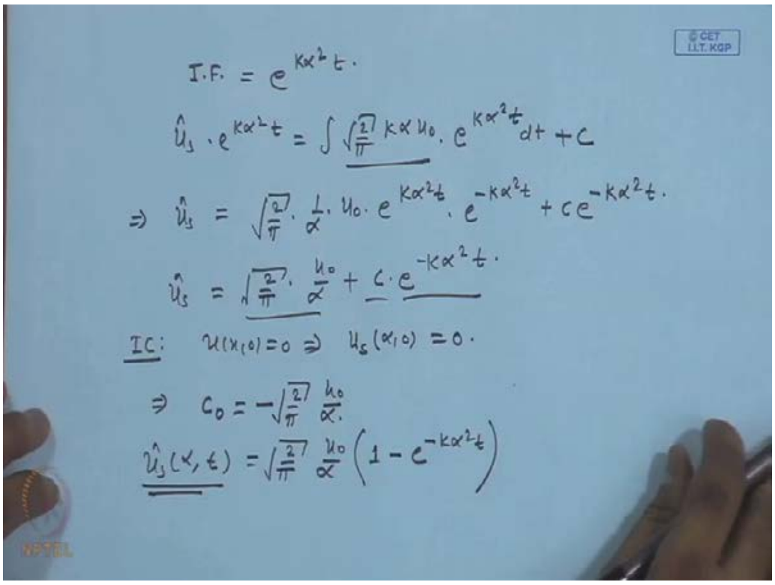

So, this is the linear equation – linear ordinary differential equation. So, integrating factor will be simply a k alpha square t, and then the solution we can get now e k alpha square t, and the right hand side we have 2 over pi k alpha u naught, and e k alpha square t dt, and plus a constant. Over we have u s hat, this we can take to the right hand side. So, k alpha square t, if we integrate this with respect to t this is any way a constant. So, we will get e k alpha square t over k alpha square. So, this k alpha will be cancelled, and we will get 1 over alpha. So, would we have 2 over pi, we will get 1 over alpha also u naught, and this e k alpha square t; and from this side, we have with minus k alpha square t, and c e minus k minus k alpha square t.

So, this u s is 2 over pi, and we have u naught over alpha plus c e minus k alpha square t. Now, the initial condition to get these this constant, we have u(x,0) is 0 that is given, if we take the Fourier transform with respect to x, Fourier sine transform. So, we will get alpha 0 is 0. So, with this condition if we set here this t 2 is 0; so, this will be 1, and we have C naught. So, this implies that C naught is minus 2 over pi, and u naught over alpha. So, we have this u s alpha t square root 2 over pi u naught over alpha, we take common, because c has also having this factors, we have 1 minus e minus k alpha square t.

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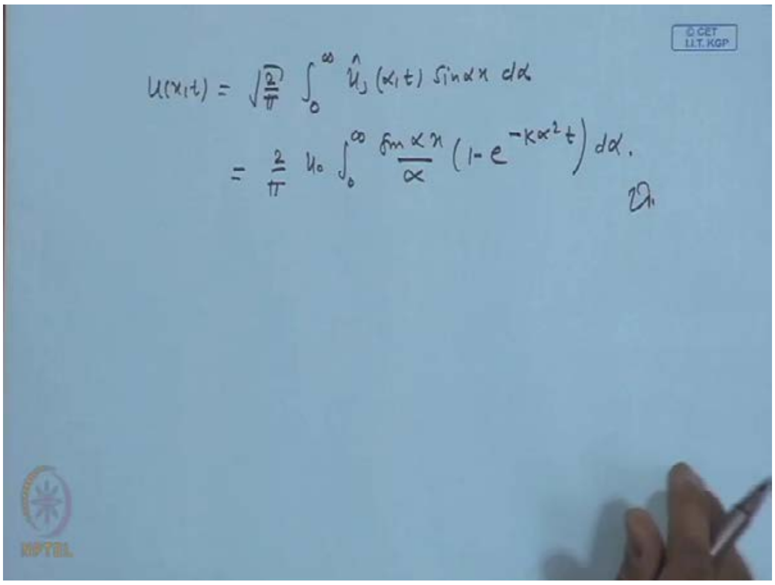

So, we have this and now we take the inverse - sine inverse transform. So, we will get straightaway this u(x,t), and that is 2 over pi, and we have 0 to infinity u as hat alpha t sine alpha x d alpha. So, we can substitute that the 2 over pi, and also u naught will come 0 to infinity we have sine alpha x over alpha and, we have 1 minus e minus k alpha square t and d alpha, so this is the solution.

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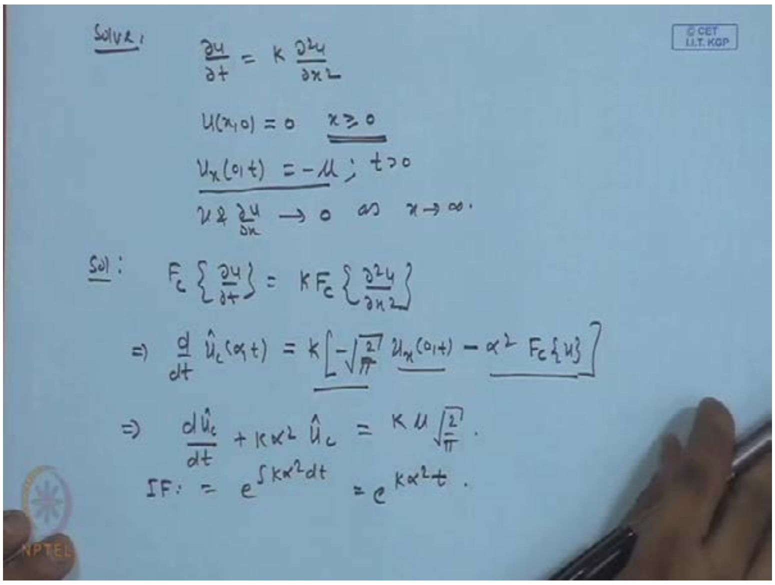

The next problem, where we will see that we need to apply the Cosine transform. So, next problem solve del u over del t is k del 2 u over del x square, and subject to the conditions we have u(x,0) as 0, and for x greater than or equal to 0. And we have u x the first derivative 0 t as minus mu; that is given for t positive, and u and again this del u over del x, this both goes to 0 as, x goes to infinity. So, we have again in the half range. So, we can we have possibility to apply Fourier Cosine or Fourier sine transform, but this u x is given. So, we can apply only Fourier Cosine transform, because we have this first derivative information there.

So, taking this Fourier Cosine transform del u over del t, and k Fourier Cosine transform of del 2 u over del x square, what we get here, d over d t and this is u Cosine we denote it by this alpha t, and we have this k. And now, this Fourier Cosine of this. So, by that formula we have 2 over pi and u x at this 0, t and minus alpha square and Fourier Cosine transform of u. So, we have d u hat c over d t, and this we will take or this one, because it is a u hat only. So, k alpha square, and u Cosine transform, and here we have k this is given minus mu; so, minus minus will be plus.

So, k mu and this 2 over pi, again we solve this linear equation. So, the integrating factor is e k alpha d t. So, we have e k alpha square t, and then over get the solutions; I am writing the solution directly.

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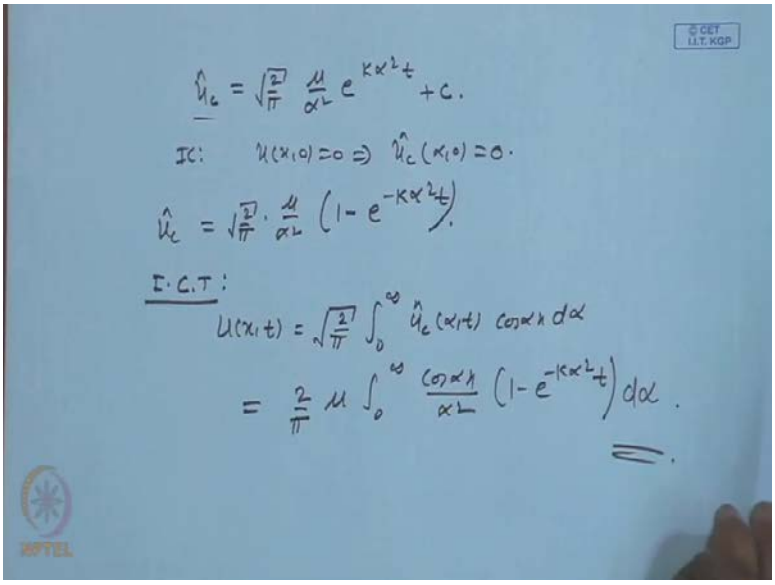

So, that solution will be for that in terms of the u Cosine transform 2 over pi mu over alpha square e k alpha square t plus constant, and again this initial condition is given that u(x,0) is 0. So, we will get u c hat alpha 0 is 0, and if we put it here, we will get this c. So, this is 0, and this will be 1. So, c will be minus of this. So, then we get this u c hat is square root 2 over pi mu over alpha square 1 minus e minus k alpha square t, and now taking the inverse of sine inverse Cosine transform. So, inverse Cosine transform now to get the u(x,t) that is 2 over pi, and 0 to infinity u Cosine alpha t and Cos alpha x d alpha. So, we substitute here. So, we get 2 over pi, we get this mu, and 0 to infinity Cos alpha x over alpha square, and we have this 1 minus e minus k alpha square t and d alpha.

So, these were the three applications, where we have use the heat equation in the first was by applying the Fourier transform directly to the problem, in the second we applied the Fourier sine transform depending on the conditions. And in the last example, which we have just an we applied the Cosine transform again depending on the condition.

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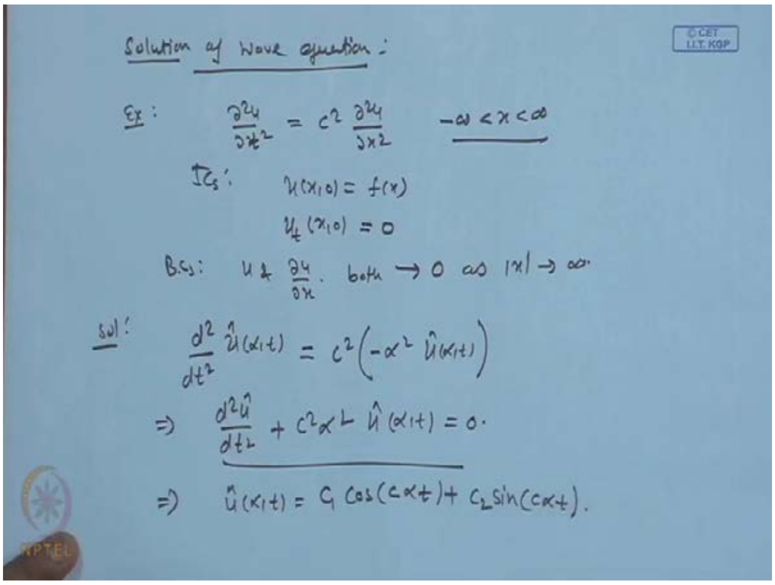

So, we go now for the solution of wave equation solution of wave equation. So, we take this example that solve the wave wave equation del 2 u over del t square c square del 2 u over del x square, and x is given minus infinity to plus infinity. So, initial conditions that u(x,0) is f(x) for this x, and u t(x,0) is 0, and we have boundary conditions that u and del u over del x both goes to 0 as absolute value of x goes to infinity. So, we take the Fourier transform, because this ranges came from minus infinity to plus infinity. So, we have d t d 2 over d t square and u hat alpha t c square, and for this we have simply minus alpha square, and u hat alpha t.

So, equation is d 2 u hat over d t square plus c square alpha square u hat alpha t is equal to 0, and it is general solution one can write u hat alpha t (( )) square plus c square alpha square is equal to 0. So, the roots will be plus plus minus this c alpha I. So, for that we have the solution C 1 constant, and Cos c alpha t plus C 2 sine c alpha t.

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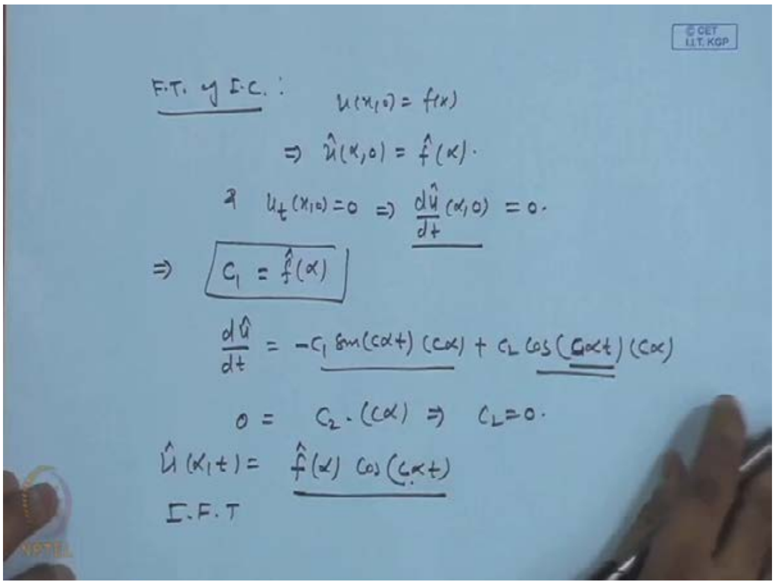

So, now, the initial conditions the Fourier transform of initial condition, we have the u(x,0) is f(x) first condition. So, here we get u hat alpha 0 is f hat alpha, and the second condition we have u t(x,0) is equal to 0.

So, from here we will get d u hat over dt alpha, and approaches to where this t approaches to 0, and this is again 0. So, form the first condition, so our solution has this u hat alpha t c one Cos, and c 2 sine. So, when we put t is equal to 0; this is 0, and we have here one. So, we get c 1 straightaway from the first condition, we get c 1 is f hat alpha; and from the second condition, so we need to get the derivative first of this. So, d u hat over d t is minus c 1 and sine C alpha t and, we have c alpha plus c 2 and for sine we have Cosine of C alpha t and again the derivative of this.

So, c and c c alpha. Now, again if you put this t to 0 in this case this will disappear, this is one. So, this is 0. So, we have c 2 and c alpha this is 1; so, in any case this c 2 is 0. So, we have u hat alpha t is f hat alpha, and Cos c alpha t. Now, we take the inverse Fourier transform- inverse Fourier transform.

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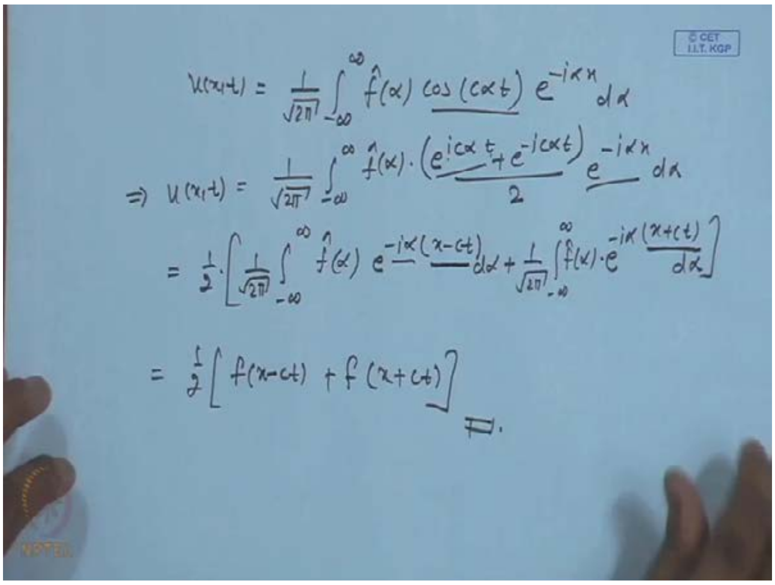

So, what we get u(x,t) 1 over square root 2 pi minus infinity to plus infinity f hat alpha, and Cos c alpha t, and then we have minus i alpha x and d alpha . So, u(x,t) is 1 over square root 2 pi, and this we take minus infinity to plus infinity f hat alpha as it is, this we write in terms of the again exponential function. So, i c alpha t plus e minus i c alpha t divided by 2, and e minus i alpha x d alpha. So, we have 1 over 2, and this 1 over square root 2 pi, then minus infinity to plus infinity f hat alpha, and this and this we combine in to 1. So, we have e minus i alpha i alpha. So, we have x minus c t d alpha plus again this (( )) factor 2 pi, 1 over square 2 pi minus infinity to plus infinity; we have f hat alpha, and e minus i alpha with this. So, we have x plus c t and d alpha.

So, if we see now that half, and this is the definition of the Fourier inverse. So, we have minus i alpha instead of x, we have here x minus c t here we have x plus c t. So, we will get f(x) minus c t plus f(x) plus c t, and this is also known as (( )) solution of the wave equation.

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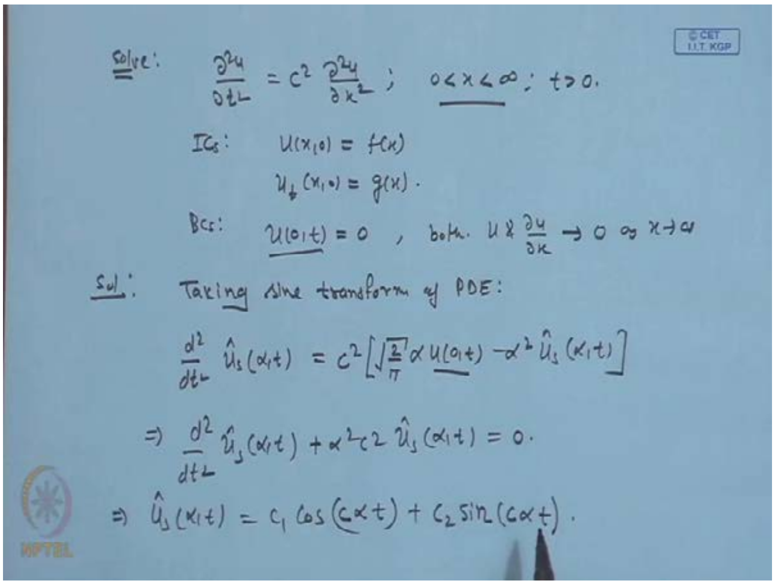

So, just one more example for the wave equation, then we will solve the Laplace equation. So, we have del 2 u over del t square c square del 2 u over del x square, and this is given for t positive. So, in the half range we have this problem. So, initial conditions are u(x,0), f(x), and u t(x,0) is g(x), and boundary conditions are u(0,t). So, u is given as 0, and again the both u and del u over del x, they goes to 0 as x goes to infinity. So, now since the value of u is given as for the formula we will apply the sine transform, because this function value is given at 0.

So, we take the taking sine transform of the PDE. So, we get d 2 over dt square, and sine transform of this u will denote by u s hat alpha t, we have c square; and then by the derivative theorem we have square root 2 over pi alpha u(0,t); and minus alpha square u s hat alpha t. So, this is d 2 over dt square u hat s alpha t, and u(0,t) is given 0. So, this term will disappear, and this will come to the left hand side; this is alpha square c square, and we have u s hat alpha t is equal to 0. So, now again we can find general solution, and that will be given by c 1 Cos c alpha t plus c 2, and sine c alpha t. So, we use this initial conditions now to get these constants.

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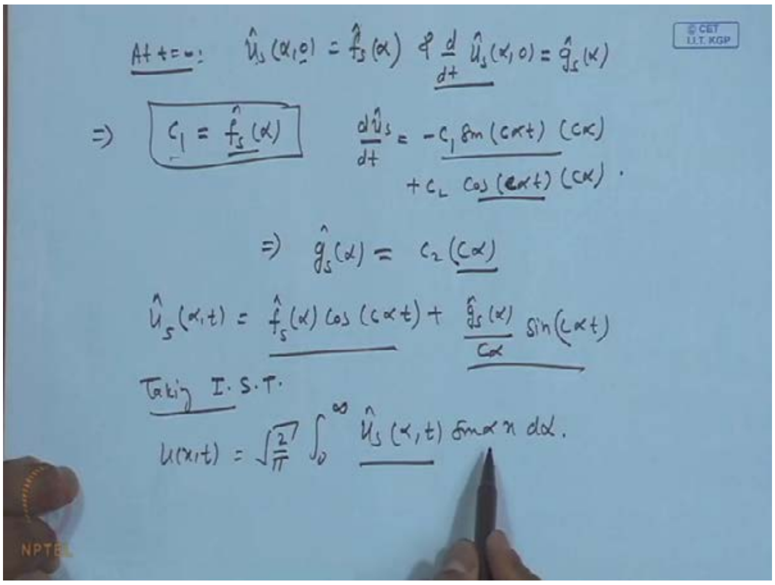

So, what we have the first. So, at t is equal to 0, the initial condition we have first u s alpha 0 is f hat alpha, because this was the initial condition. Now, u(x,0) is equal to f(x). So, we take the Fourier sine transform. So, we get u hat s alpha 0, and the Fourier transform of this f. And from the second one we have the derivatives, so d over d t and u s hat alpha 0 is g s hat alpha. So, from the first initial condition, when we put t is equal to 0. In this, so this is 0. So, we have c 1; this implies c 1 is this given f s hat alpha. So, this is one; for the second one we need to get the derivative of this first, and then we will go to the t is equal to 0.

So, we have d u hat as over d t is minus c 1, and Cos will be sine c alpha t, and then we have c alpha plus c 2 and Cos Cos c alpha t, and we have C alpha. So, we put t is equal to 0; this (( )) disappear, this is 1 and this is g hat s alpha is equal to c 2, and we have c alpha. So, c 2 is g s hat alpha over c alpha, and c 1 is this. So, we can get now the solution u s alpha t that is f hat s alpha Cos c alpha t, and plus we have g s hat alpha over the c alpha, and then sine c alpha t.

Now, we take the inverse taking inverse sine transform we get u(x,t) will be square root 2 over pi, we have 0 to infinity and this function here. So, we have u s hat alpha t and sine alpha x d alpha. So, this is our u s alpha t, and then multiplied by sine alpha x. So, we write this now.

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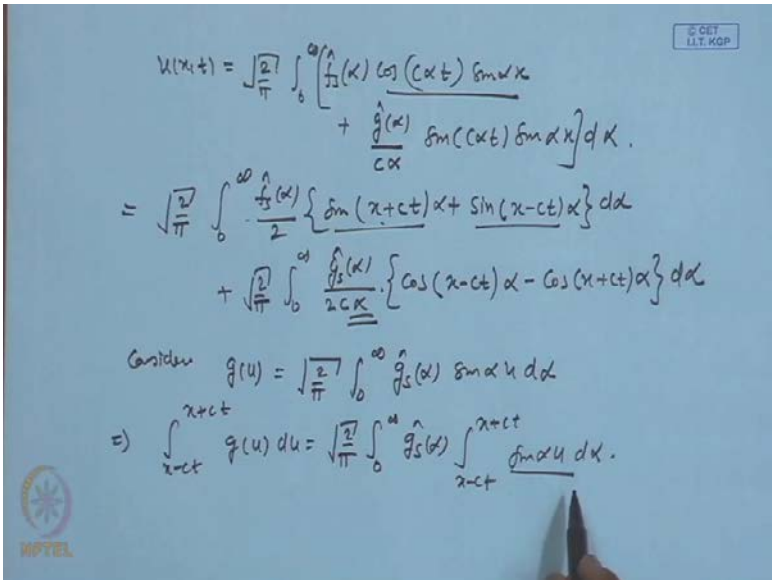

So, we have u(x,t) as the square root 2 over pi, we have 0 to infinity, we have f s alpha and Cos c alpha t sine alpha x, in one term plus we have g hat alpha over this c alpha, and sine c alpha t and sine alpha x, then d alpha . So, now we expand this, because the two sine alpha, and Cos beta form. So, we take 2 over pi and we have 0 to infinity, and this f(x) hat over 2, and this will be now 2 times sine a Cos b will be sine a plus x plus c t alpha plus sine x minus c t alpha d alpha plus; again here 2 over pi with that integral, we have 0 to infinity g hat alpha 2 c alpha. So, 2 sine a sine b we have the Cos terms then Cos a minus b our b minus a does not matter. So, x minus c t we write, and then minus Cos a plus b; so, x plus c t alpha d alpha.

Now for the first round, we can easily a write in terms of f taking inverse sine transform, because it is given exactly in that form of a hat alpha, and the sine and again this other term with this sine we will get is straight away f alpha; and here also this f sorry f(x) plus c t and this will x minus c t divide by 2, but for this term this alpha is appearing here. So, in this case what we take consider a function g(u). So, g(u) is the Fourier transform of inverse Fourier transform of this g hat alpha. So, we have 0 to infinity Fourier sine transform. So, g s hat alpha, this is wholes Fourier sine transform, and then we have sine alpha u d alpha, and if we integrate this form x minus c t to x plus c t g(u) du 2 over pi, and we also integrate change the order of integration.

So, first 0 to infinity will come g s hat alpha, and then this x minus c t to x plus c t will come and sine alpha u d alpha.

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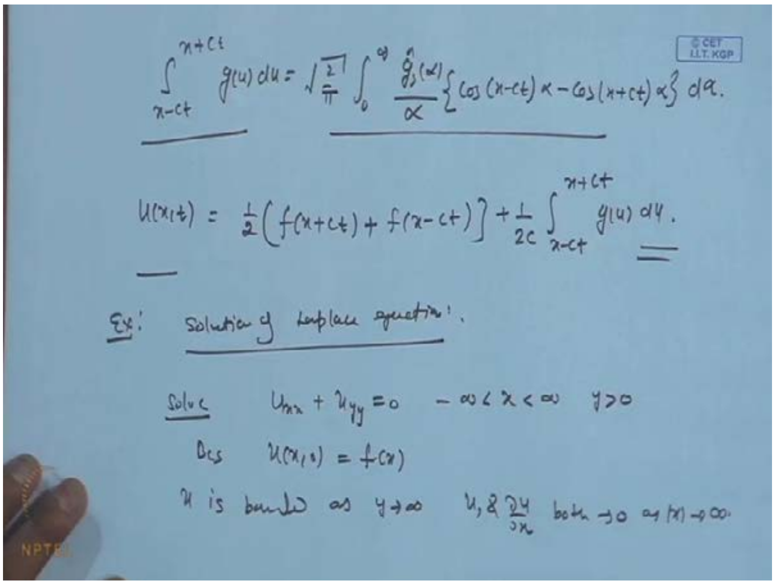

So, if we integrate this we will get Cos alpha u over alpha, and then put the upper limit, and the lower limits, so what we get in this case that this integral x minus c t to x plus c t g(u) du is 2 over pi, we have 0 to infinity g hat s alpha over this alpha, we are getting exactly the term we need there. So, Cos x minus c t alpha minus Cos x plus c t alpha, and then we have d alpha; thus the solution is given by. So, if we just… Now, go back to this, this we know what will be the the inverse what will be this, and here also we know now from this integral that is equal to that. So, we take with get the solution u(x,t) is half half here, and square root 2 over pi 0 to infinity f s hat alpha sine x plus c t, we will get the f(x) plus c t.

And then for the second one we have x minus c t f(x) minus c t, and for the second term what we just write it will be just x minus c t over x plus c t, and g(u) du. So, this is the solution. Now, we go for the last example of this lecture, and as a solution of Laplace equation, Laplace equation. So, in this case, we solved u xx plus u yy is equal to 0 is given from minus infinity to plus infinity by positive, and the boundary conditions are u(x,0) is f(x). Again in the same range, and this is given that u is bounded is bounded as y approaches to infinity, and again u and del u over del x both goes to 0 as mod x goes to infinity.

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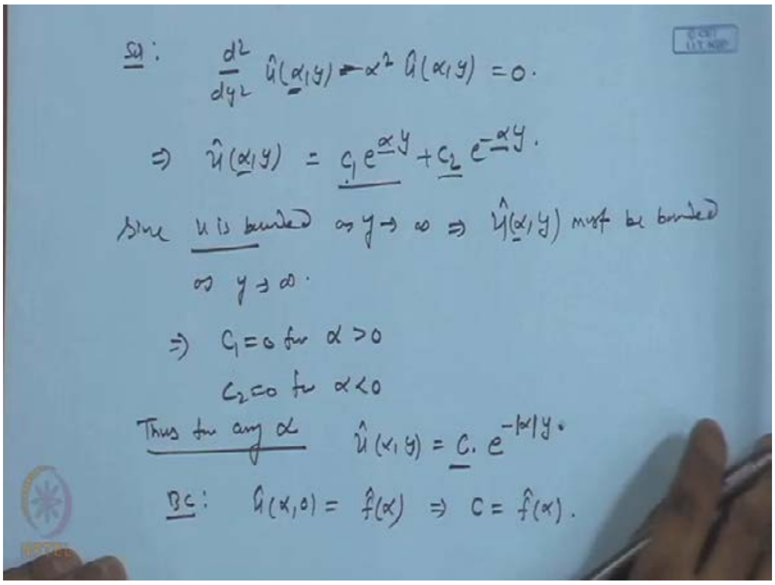

So, we take the Fourier transform both the side we will get this transformed ordinarily differential equation, and then I will directly write the the after taking the transform Fourier transform, we will d 2 over d y 2; you get alpha y, and we get minus alpha square you get alpha y is equal to 0, and its solution we have alpha y c 1. So, with respect to x we have taking again this Fourier transform: e alpha y plus c 2 e minus alpha y, and the condition is given that u is bounded u is bounded as y approaches to infinity, and this will imply straightaway that this it is Fourier transform.

And that we take with respect to x naught with respect to y must be also bounded as y approaches to infinity, and this will tell us because here we have this alpha y minus alpha y that c 1 is 0 for alpha positive, because otherwise this will flow up. So, to have this boundedness, and this c 2 will be 0 for alpha negative in that case this will beyond bounded. So, for this boundedness. So, in any case thus for any alpha, what we can write we can eliminate one constant there, because we are anyway getting this alpha y is some constant times alpha y.

The the absolute value of alpha y, we cannot get this alpha, because this c 1 will be 0; if alpha is positive, and if alpha is negative this c 2 will be 0. So, what we have some constant, and e power minus alpha y. Now, with the boundary condition we have u hat alpha 0 is f hat alpha, this is given the boundary condition we have taken the Fourier transforms as y(0), we will get this c. So, c will be f hat alpha.

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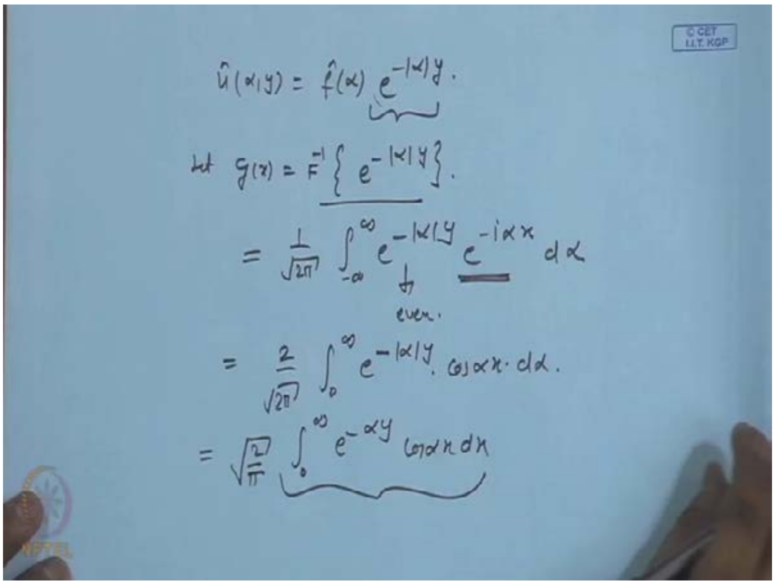

And now our solution of the transform over the e has f hat alpha, and e minus alpha y. And now, we again apply the same trick to get this inverse this convolution theorem, but we need to get the function before here inverse of of this. So, let that g(x) is the Fourier inverse of e minus alpha y, and then we take as per the definition 1 over square root 2 pi minus infinity to plus infinity e minus alpha y e minus i alpha x d alpha. And then, this we write Cos alpha x and i sin alpha x. So, since this is the even function, and here Cos is also even function, and with these sine odd function; so, this over the symmetric integral will be 0. So, we have 2 times 2 pi, and 0 to infinity e minus alpha y and Cos alpha x will remain.

And now this, we can have 2 over square root pi, because this will be square 2 here, and we have 0 to infinity e minus alpha y Cos alpha x. And this is a very simple integration, one can just get integrating by parts.

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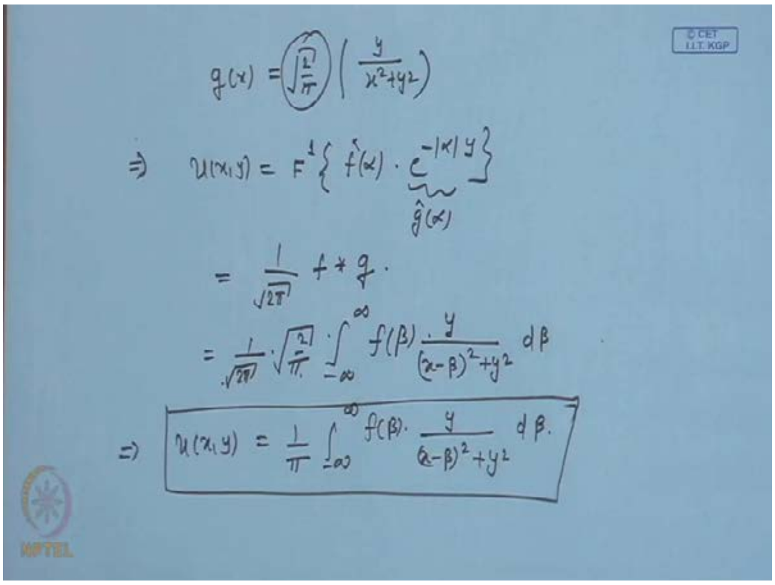

So, we will get in this case g(x) as 2 over pi, and this will give us y over x square plus y square. So, this integral will give y over x square plus y square, and now we go back to the solution now, and that is the Fourier inverse of this f at alpha and e minus alpha y.

So, this was the g hat alpha, this is f hat alpha. So, by the convolution theorem, we have 1 over square root 2 pi, and we have f on the convolution of this g. So, 1 over a square root 2 pi, and this convolution we can write down. So, we have 2 over pi, and because this g alpha we have this vectors. So, we have taken this, and minus infinity to plus infinity f beta introduce this integrating variable, and then we have y over x, we replace this x by x minus beta for the convolution x minus beta x square plus y square, and we have this d beta. So, what we get u(x,y) is 1 over pi square root, two will cancel this square root pi square root pi you have 1 over pi minus infinity to plus infinity.

We have f beta y over x minus beta whole square plus phi square d beta, and this solution is also known as the Poisson integral formula. So, that is the here we we complete the discussion on Fourier transform. As well as on this rather introductory lectures on transform calculus, and we have mainly discuss the Fourier transform, and the Laplace transform. And each topic was implemented by by various well chosen exercises. So, I hope that this lecture which was other introductory of course, will have to understand advance topics in related areas. Thank you.