Advanced Engineering Mathematics

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Module No. # 01
Lecture No. # 03
Basic, Dimension, Rank & Matrix Inverse

We shall start, this this is a dimension of a vector space and also we will find rank and matrix inverse. So, basically a basis is very important thing for a vector space. That, A basis characterization vector space or in other words a basis is a sub category representation of a vector space. If basis of a vector space is known, then, we can find the vector space itself by taking all possible finite linear combination of elements in B.

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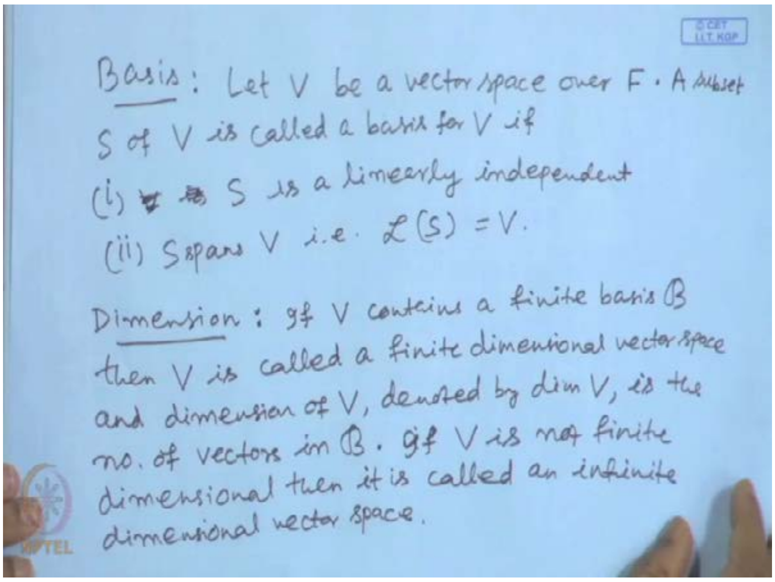

So, let us define a basis first. So, basis of a vector space. So, let V be a vector space over F. Over F A subset a subset S of V is called a basis for V if the following two conditions hold: that is v is a sorry this s is a that is S is a linearly independent set. And second condition is that S spans V that is; L of S is equal to V. That, in other words that every element of V can be expressed as a finite linear combination of elements in S. So, let us see some examples, of course, before that we shall define dimension of a vector space.

That, if V contains a finite basis a finite basis b then V is called a finite dimensional vector space finite dimensional vector space and dimension of V, dimension of V denoted by dimension of V, is the number of vectors in B. If V is not finite dimensional V is not finite dimensional then it is called an infinite dimensional vector space. Infinite dimensional vector space Let us see some examples.

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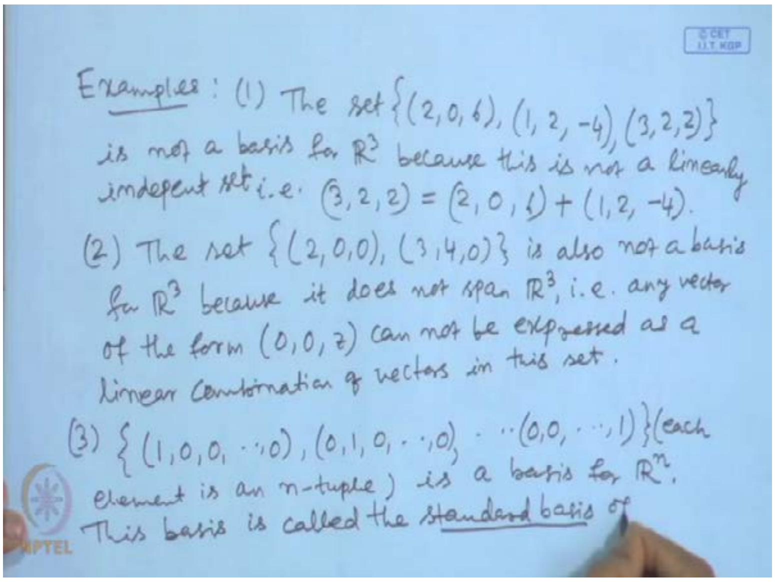

First example is like this, the set consisting of vectors (2, 0, 6) (1, 2, minus 4) (3, 2, 2) is not a basis for R 3, because this is not a linearly independent set because this is not a linearly independent set that one can write this third vector as sum of the other two, that is; (3, 2, 2) this can be written as sum of these two vectors (2, 0, 6) plus (1, 2, minus 4).

Another example is that, the set consisting of vectors (2, 0, 0) (3, 4, 0) is also not a basis for R 3, because it does not span R 3, that is any element in the z axis is, that is any vector of the form that is; (0, 0, z) cannot be expressed as a linear combination of linear combination of vectors in this set. So, another example, third example is like this; however this set, that is consist of vectors like (1, 0, 0, ..,0) (0,1, 0,..,0) up to this (0, 0,..,1) where each vector is an n-tuple, is element is an n duple this set is a basis for R n. One can easily check that this set of vectors forms a linearly independent set. One can also write in echelon form and check this and; obviously, any vector we take in R n that can be written as linear combination of these vectors. So, this basis is called the standard basis of R n this basis is called the standard basis of R n.

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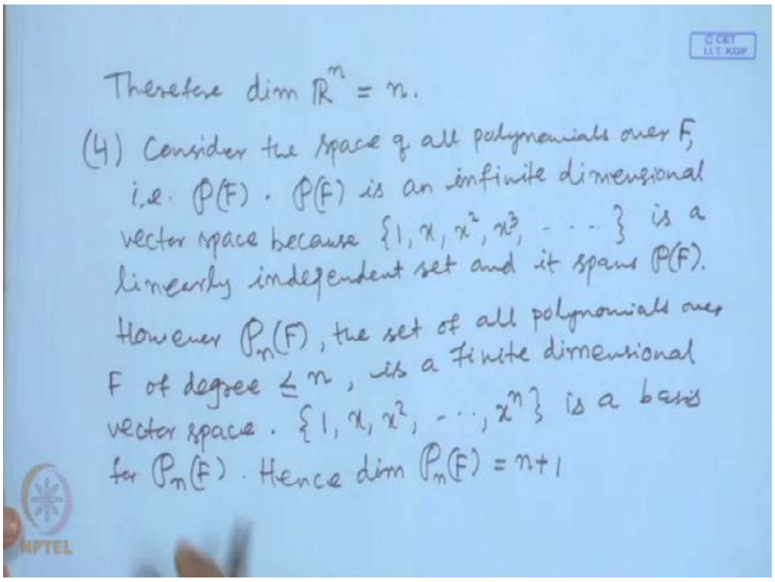

So, therefore, dimension of this vector space R n is equal to therefore, dimension of r n is equal to n. We will see another example, that, here we consider all polynomials over field F consider this consider the space of all polynomials all polynomials over f that is; P of F. So, the P of F is an infinite dimensional vectors space is an infinite dimensional vector space because, because this set consisting of 1, x, x square, x cube is a linearly independent set linearly independent set and it spans this space P of F; however, this however. This P n F that is; the set of all polynomials over F of degree less than or equal to n, is a finite dimensional vector space. Is a finite dimensional vector space. And this set that 1, x, x square, up to x n is a basis for P n F. Hence dimension of P n F is equal to n plus one.

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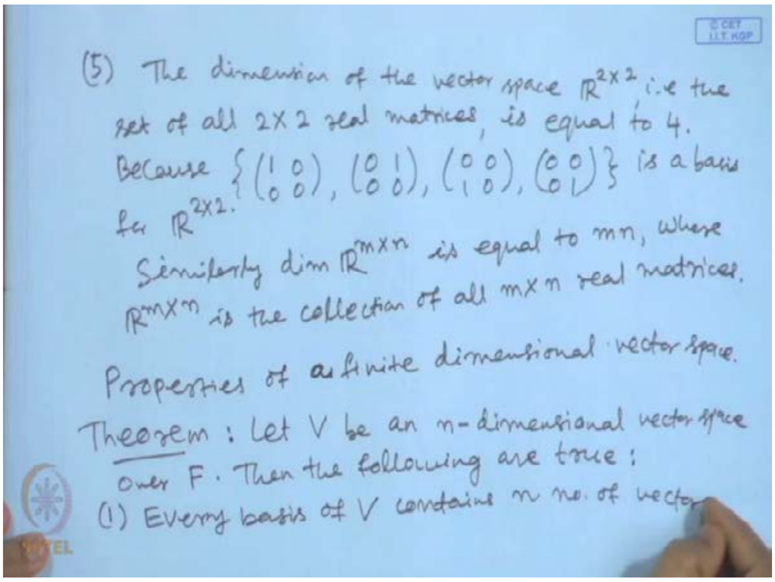

We will see another example of a of a basis, that is example five. Here this, dimension of the dimension of the vector space R 2 by 2 that is; the set of all 2 by 2 real matrices, is equal to 4. Because, this set consisting of matrices (1, 0, 0, 0) (0, 1, 0, 0) (0, 0, 1, 0) (0, 0, 0, 1) is a basis for this R 2 by 2. Similarly, dimension of R m by n is equal to m by n, where R m by n is the collection of all m by n real matrices. So, we will see more examples while preceding this topic later on.

So, next we will see some well known results for finite dimensional vector spaces, we may not able to prove them, but, we may refer those results and they are also very important results. So, all those results we shall write here in the theorem. So that, we can say that properties of a properties of a finite dimensional vector space. Of a finite dimensional vector space. So, here we will write as a theorem. So, let V be an n-dimensional vector space over F. Field F Then the following are true. So, first one is that every basis of V contains n number of vectors. Every basis of V contains n number of vectors.

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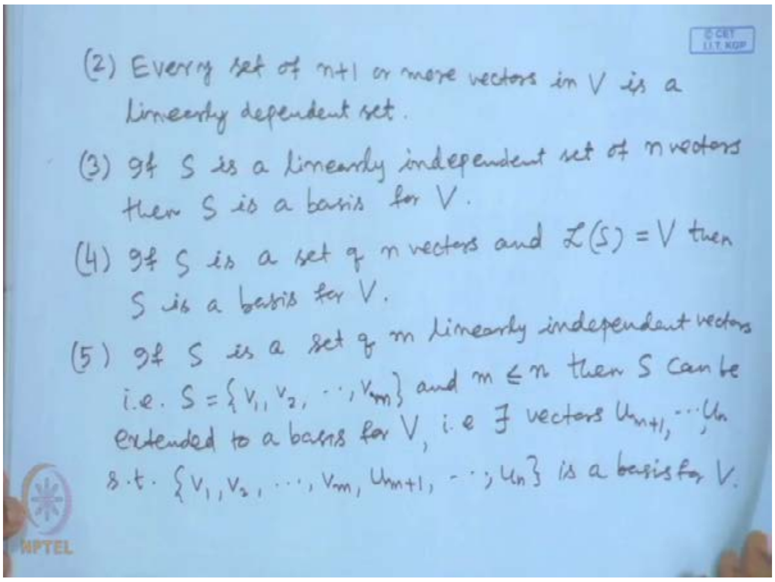

Second property is like this, every set of every set of n plus one or more vectors in V is a linearly dependent set. Linearly dependent set Third result is like this, if S is a linearly independent set linearly independent set independent set and its consist of n vectors of n vectors then S is a basis for V. Fourth property is that, if S is a set of n vectors and L of S is equal to V that is; S spans V then, S is a basis for V. So, here this property third and fourth says that, one can is a set of vectors be a basis for a finite dimensional vector space then, this fifth property is like this, let if S is a set of m vectors s is a set of m linearly independent vectors that is; S is consist of vectors V 1, V 2, to V m and m is less than or equal to n, then S can be extended to a basis for V, That is; there exists vectors say U m plus one, up to U n, such that, this set consisting of vectors V 1, V 2 to V m, U m plus one up to U n is a basis for V.

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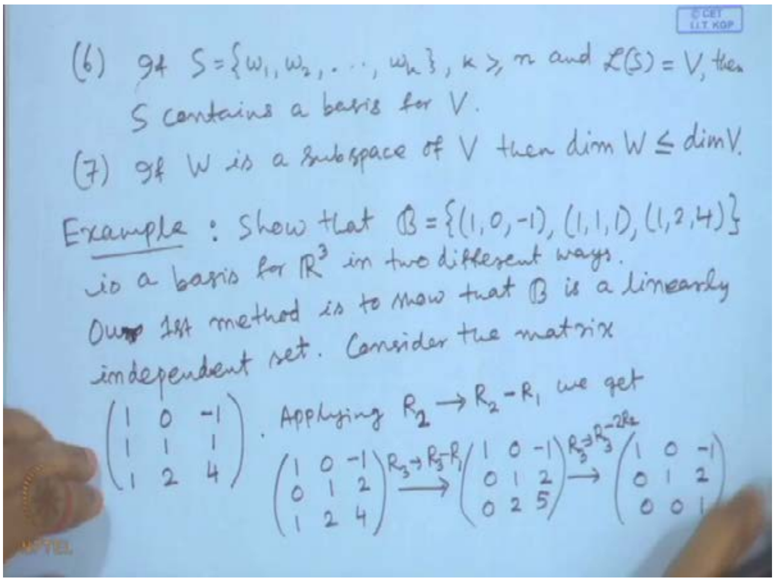

Similarly, we have another property that is sixth one, that if S is consist of k number of vectors say W 1, W 2, up to W k. K is greater than equal to n and L of S is equal to V, that is; S spans V then, S contains a basis for V.

Another property is that, for every sub space that if W is a sub space of V, this vector space V then dimension of W is less than or equal to dimension of V. So, let us see one example to check whether a given set of vectors is a basis or not. So, let us say this example. Here we will show that, this set show that this set B consist of vectors (1, 0, minus 1) (1, 1, 1) (1, 2, 4) is a basis for is a basis for R 3 in two different ways.

So, here actually we shall use that result three and four of the previous theorem or this second. So, this first we shall show that S is a linearly independent set. So, our first method our first method is to show that B is a linearly independent set Linearly independent set. We do this by considering echelon form of these vectors or in other words we form the matrix like this, consider the matrix rows rows of the given vectors (1, 0, minus 1) (1, 1, 1) and (1, 2, 4) converting these to echelon form we get like this. So, from this matrix we apply elementary row operations and get echelon form of it. So, applying this R 1 replaced by this R 2 minus R 1. We get this matrix we get this matrix that is; one zero sorry here we replace this row to (1, 0, minus 1) (0, 1, 2) (1, 2, 4) then we make this elementary row operation R 3 we replace by R 3 minus R 1 and get this matrix that is (1, 0, minus 1) (0, 1, 2) and here (0, 2, 5) and once more we apply this elementary row operations and get this matrix. That is, we have to make at this entry zero to get in echelon form. So, therefore, we replace this R 3 by R 3 minus twice R 2 and get this matrix (1, 0, minus 1) (0, 1, 2) (0, 0, 1).

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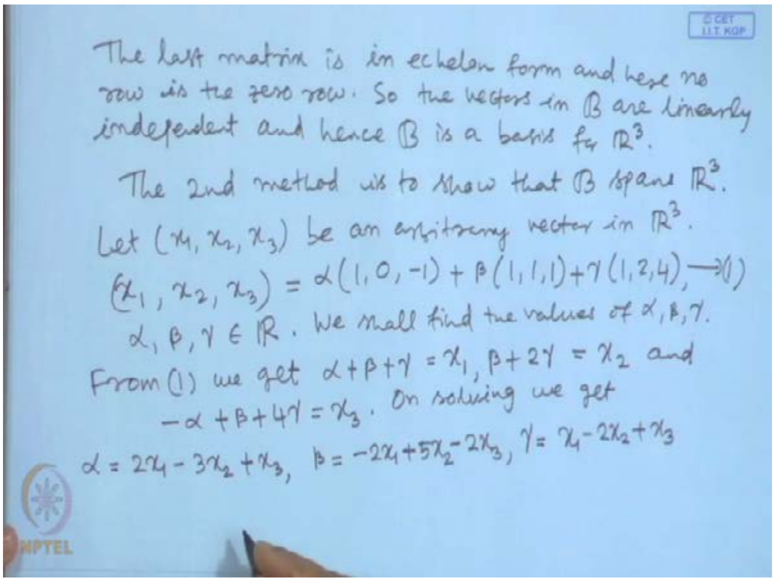

So, notice that this last matrix is in echelon form and here no zero row is there. So, here this last matrix the last matrix is in echelon form echelon form and here no zero row is there and no row is the zero row. So, the vectors in B are linearly independent linearly independent and hence B is a and hence b is a basis for R 3. So, this second method is the second method is to show that B spans R 3. This will work because B is consist of three vectors and dimension of R 3 is also 3.

So, therefore, if we show that, these spans R 3 then B will be a basis for R 3. So, B spans R 3 means; if we consider any arbitrary vector in R 3 that we suitable to write as a linear combination of vectors in B. So, let (x 1, x 2, x 3) be an arbitrary vector in R 3. So, we shall write this arbitrary vector (x 1, x 2, x 3) is a linear combination of vectors in V. So, let us consider these linear combination (x 1, x 2, x 3) and that we write as alpha times (1, 0, minus 1), plus beta times (1, 1, 1), plus gamma times (1, 2, 4). Here alpha, beta, gamma comes from this real fact or they are scalars. So, here we shall find the values of alpha, beta, gamma we shall find the values of alpha beta gamma.

So, from here we get this equation. So, let us say this be one let us say this be one. So, from this equation one, we get alpha plus beta plus gamma is equal to x 1, and this beta plus twice gamma is equal to x 2, and minus alpha plus beta plus four gamma is equal to x 3. On solving for alpha, beta, gamma we on solving we get alpha is equal to twice x 1 minus 3 x 2 plus x 3, beta is equal to minus twice x 1 plus 5 x 2 minus 2 x 3, and gamma is equal to x 1 minus twice x 2 plus x 3. So, that means given any arbitrary vector x 1, x 2, x 3 we can find alpha, beta, gamma in terms of this known values x 1, x 2, x 3.

So, that this x 1, x 2, x 3 can be written is a linear combination of these vectors. So, therefore, this B spans R 3. And hence B is a basis. So, B spans R 3 and is consists of three vectors hence B is a basis for R 3.

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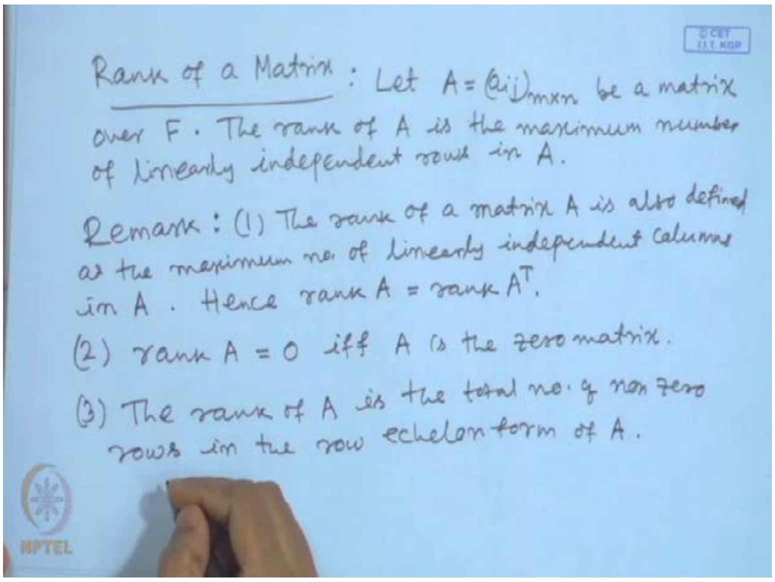

So, next we shell find out another important terminal, that is rank of a matrix and that a rank of matrix is also very useful. So, here we shell find this rank of a matrix and that we find again in terms of this linearly independent matrix.

So, let A i j be a matrix of size m by n matrix over F. The rank of A is the maximum number of is the maximum number of linearly independent linearly independent rows in A of course, one can also take that rank of a matrix is the maximum number of linearly independent columns. So, all those properties we can write as a remark. That row The rank of a matrix the rank of a matrix A is also defined as the maximum maximum number of linearly independent columns in A. Hence rank of A is equal to rank of a transpose.

So, second remark is like this, rank of a matrix A is equal to zero; if and only if A is the zero matrixes. That is all the entries in A are zero. And then, this another way that that is the third one can, this third remark that one can find rank of a matrix from echelon form of this also or one can be find rank of a matrix in this space and also. So, that is the rank of A is the total number of non zero rows in the row echelon form of in the row echelon form of A. In fact, we use this third remark to find rank of of a matrix and. So, this is use full. So, let us see one example that determines rank of a matrix.

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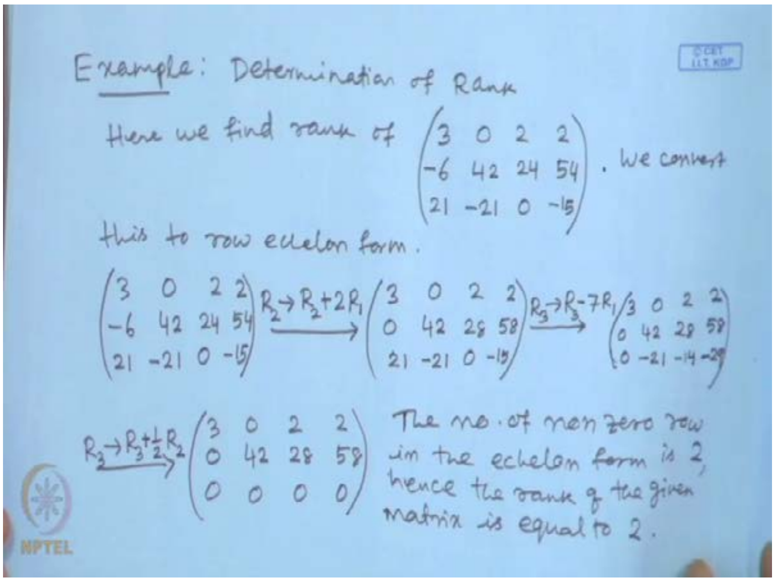

So, the example is like this, here we determine, determination of rank. So, consider this matrix, here we find rank of this matrix that is; matrix are (3, 0, 2, 2) (minus 6, 42, 24, 54) (21, minus 21, 0, minus 15). So, we will convert this matrix to echelon form We convert this row echelon form that is, this given matrix is (3, 0, 2, 2) (minus 6, 42, 24, 54) (21, minus 21, 0, minus 15). So, here we apply this elementary row operation and make this position zero. So, here we shell apply this row operation that R 2 second row we replace by R 2 plus twice R 1 first row. Two times R 1, R 2 plus 2 R 1. So, we get this matrix (3, 0, 2, 2) (0, 42, 28, 58) and the third row is, is it is that is (21, minus 21, 0, minus 15).

So, again we apply this elementary row operation, that here this third row, first entry of the third row we make that zero. So, therefore, we replace row three by this row three minus seven times row one. So, we get this matrix like this. So, first row is (3, 0, 2, 2) (0, 42, 28, 58) and here we get this (0, minus 21, minus 14, and this minus 29).

So, this is not yet in the echelon form that second entry in third row, which we have to make zero. So, here we apply this elementary row operation R 3 we replace by R 3 plus this half of row two. And we get this matrix with (3, 0, 2, 2) (0, 42, 28, 58) and this third row will be completely zero. So, the last matrix is in echelon form, and numbers of non zero rows here is equal to two. So, the rank of this given matrix is therefore, two. So, this the number of non zero row in the echelon form is two hence; the rank of the given matrix is equal to two. So, this is how we will find rank of a matrix by applying this echelon form of it.

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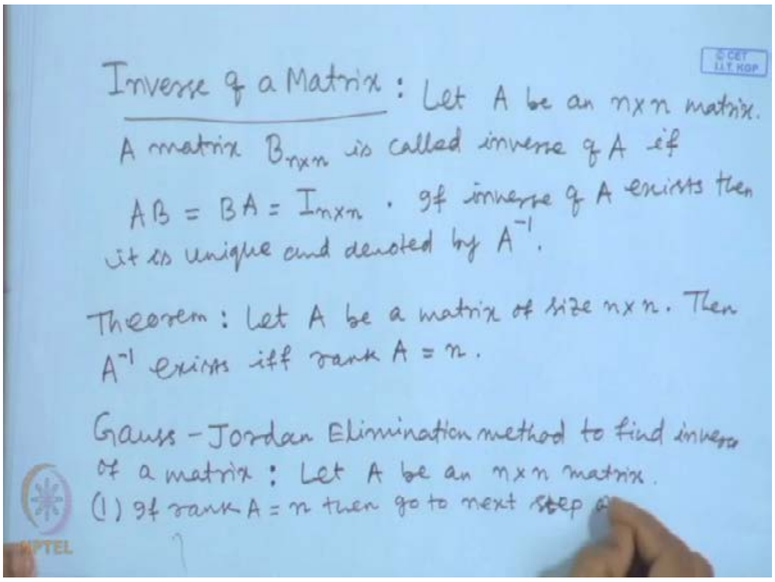

So, next we shell find out inverse of a matrix that is, inverse of a matrix. So, here we considered square matrices only. So, recall that we defined inverse of a matrix like this, let A be an n by n matrix. A matrix B of size n by n is called inverse of A if this AB is equal to BA is equal to this n by n identity matrix. If inverse of a matrix A exists then it is unique if inverse of a exists then it is unique and denoted by and denoted by A inverse. So, inverse of all square matrices may not exist. Here in the following theorem we give a necessary and sufficient condition for existence of inverse of a matrix.

So, let A be a matrix of size n by n, then A inverse exists if only if A is a whole rank or in other words rank of A is equal to n. So, this gives a necessary and sufficient condition for existence of inverse of a matrix. So, here we keep a method to find inverse of A matrix and that method is very famous one. That is called this gauss Jordan elimination method. So, we discuss about this gauss Jordan elimination method elimination method to find inverse of a matrix to find inverse of a matrix. So, here we considered that whether the first the check whether the matrix is full rank or not and we write this method in the following steps. So, let A be an n by n matrix. So, first step is to check whether this A is a full rank or not. If rank of A is equal to n, then go to next step otherwise, the inverse of A otherwise the inverse of a does not exist.

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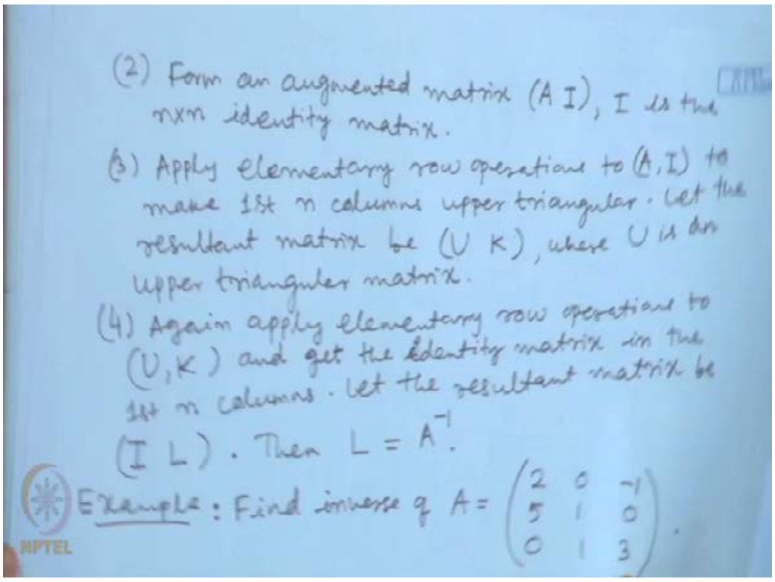

So, this second step is like this, here we form an augmented matrix like this form an augmented matrix (A, I) where A is where A is the given matrix and I is the n by n identity matrix. So, third step is that we apply elementary row operation to this augmented matrix and make this first n columns upper triangular, that apply elementary row operation to (A, I) to this a i to make first n columns upper triangular. And let the resultant matrix be (U, K) where U is an upper triangular matrix. Then fourth step is, again we apply elementary row operations to (U, K) and get the first n columns is identity matrix. So, again apply elementary row operation to this augmented matrix (U, K) and get the identity matrix get the identity matrix in the first n columns. Let the resultant matrix be (I, L), then L is equal to A inverse, this matrix L will be inverse the given matrix. Let us take one example and take this steps in in this process.

So, let us consider one example, where we find inverse of a matrix by apply this gauss Jordan elimination method. That find inverse of this matrix A that is; given by (2, 0, minus 1) (5, 1, 0) (0, 1, 3).

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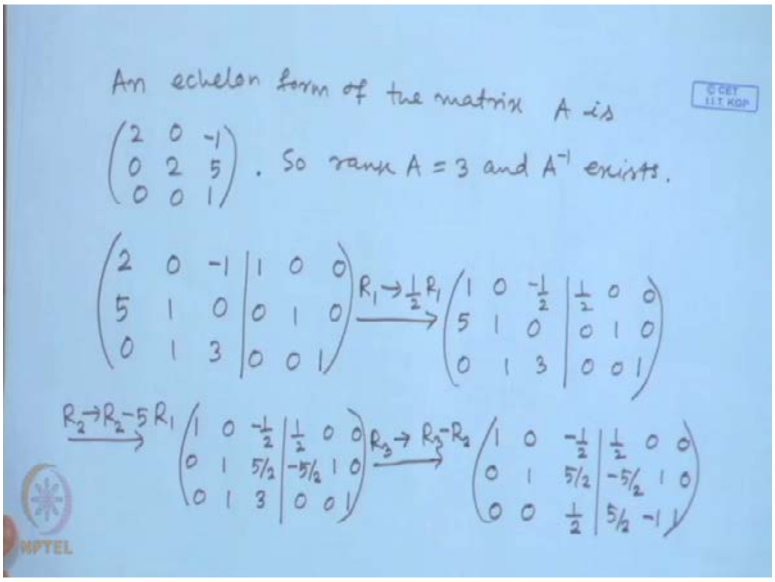

So, first we shall check weather this matrix A is a full rank or not. So, we will get echelon form of this matrix like this. An echelon form of echelon form of the matrix A is given by this (2, 0, minus 1) (0, 2, 5) (0, 0, 1). So, rank of A is equal to 3 and A inverse exists. Then we follow the steps given in the gauss Jordan elimination method and form this augmented matrix form this augmented matrix in the first three columns. We consider the matrix (2, 0, minus 1) the given matrix (5, 1, 0) (0, 1, 3) and here we consider identity matrix, that (1, 0, 0) (0, 1, 0) (0, 0, 1) and we apply elementary row operation to make this first three columns upper triangular. So, we get this matrix, that first we divide first row by 2. So, that we get this first entry be equal to one. So, that is R 1 is replaced by half of R 1 and here we get this matrix (1, 0, minus one-second) (one-second, 0, 0) and this is (5, 1, 0) (0, 1, 0) and where this (0, 1, 3) (0, 0, 1) then again. We apply elementary row operations to make this first entry in the second row is equal zero. So, therefore, we apply this operation that R 2 we replaced by R 2 minus 5 R 1 and get this matrix B like this so, it is (1, 0, minus one-second) (one-second, 0, 0) (0, 1, fifth-second) (minus fifth-second, 1, 0) (0, 1, 3) (0, 0, 1).

So, next again we apply elementary row operations that. We shell replace this R 3 by R 3 minus R 2, that makes the second entry in third row is equal to zero and we get this matrix be like this. So, first row is this (1, 0, minus one-second) (one-second, 0, 0) (0, 1, fifth-second) (minus fifth-second, 1, 0) the second row, and this third row will be (0, 0, one-second) (fifth-second, minus 1, 1). So now, this augmented matrix is every form that, first three rows they form upper triangular matrix. Upper triangular matrix means all entries below the main diagonal they are equal to zero.

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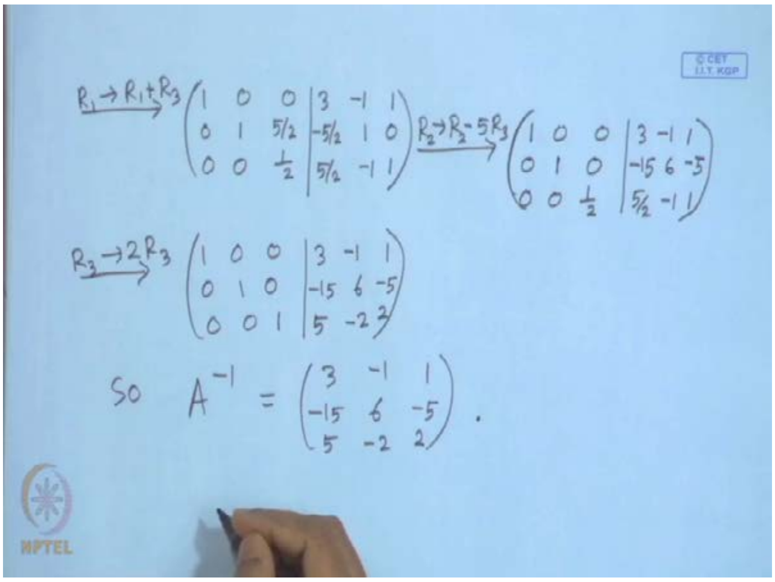

So, next we shall again apply elementary row operations and make this first three columns and identity matrix. And it is like this, from the again we apply elementary row operations R 1 replace by this R 1 plus R 3 to make third entry of first row is zero. So, we get this is (1, 0, 0) (3, minus 1, 1) (0, 1, 5th-second) (minus 5th-second, 1, 0) (0, 0, 1-second) (5th-second, minus 1, 1).

So, next we shell make third entry in the second row that is; equal to 0. So, we apply this elementary row operations that R 2 we replace by R 2 minus 5 R 3 and get the matrix B like this, here we get this (1, 0, 0) (3, minus 1, 1) and (0, 1. So, here we get this (0, 1, 0) (minus 15, 6, minus 5) and this (0, 0, one-second) (fifth-second, minus 1, 1). So, finally, we multiply this row 3 by these 2. So, row 3 we replace by twice row 3 and get this matrix (1, 0, 0) (3, minus 1, 1) (0, 1, 0) (minus 15, 6, minus 5) (0, 0, 1) and (5, minus 2, 2). So, this is required form that the first 3 columns for identity matrix. So, A inverse is this matrix (3, minus 1, 1) (minus 15, 6, minus 5) (5, minus 2, 2). So, this is how we find inverse of a matrix by apply the gauss Jordan elimination method. And this method is very useful and one can we implement in a computer also and find inverse of a matrix. That is all for this lecture will stop here.