Advanced Engineering Mathematics

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Lecture No. # 29

Fourier Integral Representation of a Function

Welcome back to lectures on Fourier transform. And now today, we will continue the idea of Fourier series for a function, which are not a periodic function. So basically, this Fourier series representation as given for a function, which is a periodic, and now we will go for the extension of this period. So, if we let this period goes to infinity, then we will get a non periodic function, and we will see what will be the representation in that case.

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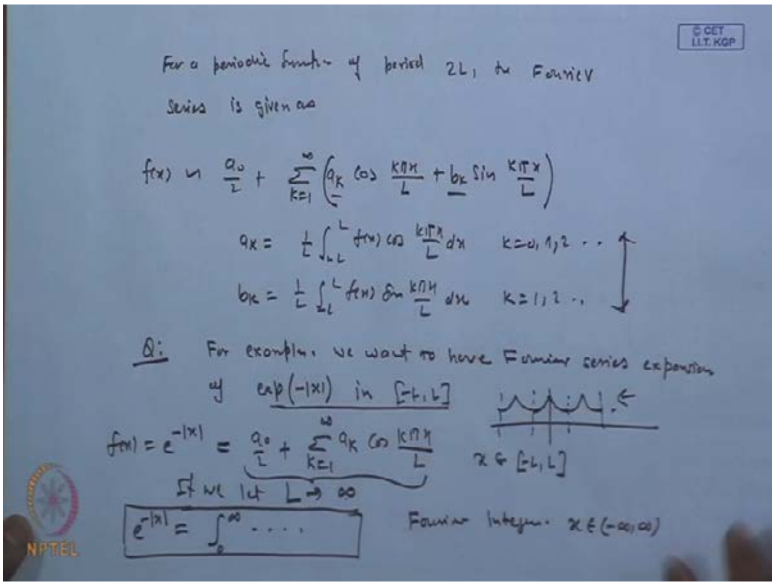

So, let me just continue with this Fourier series are representation of function. In that case, we had for a periodic function of period 2 L, the Fourier series is given as f (x) a naught by 2 and k from 1 to infinity a k cos k pi x over L plus b k sin k pi x over L, where this a k and b k is this Fourier coefficients are given by the integrals. So, 1 over L minus L to L f (x) cos k pi x over L dx k 0, 1, 2 and so on; b k was again1 over L minus L to plus L f (x) sin k pi x over L and so on.

Now, the question is that, if want to have a Fourier series expansion, let say for example, if we want to have Fourier series expansion of… Let say exponential minus modulus x, which is valid in minus L to L. So, we look at the graph of this function, so we have period, we have a symmetric function, because e power minus modulus x in this side also we have this. And took at the Fourier series of this function, we will go for its continuation as a periodic function and so on, to this side as well and then we find the Fourier series as given here. So in that case, we will get this representation a naught by 2 and this a k and cos k pi x over L, and this will be evaluate for the function minus L to L. This can represent this function in this range or in this interval. This a k and b k, one can calculate with the help of this given integrals. But now the point is that, this is the function was given from minus L to L so as this extension we have.

So the function this becomes a periodic function of period 2 L. But now what we will happen, if we let; so if we let this L tends to infinity; for any how this we can represent this function by this Fourier series, but this is the limiting case that what we will happen as L tending to infinity. Just note that, in that case, we cannot have this periodic continuation, in that case, we have a non periodic function. So, we will see to today that we can still represent this function, as L tending to infinity and then instead of sum and series, we will get a integral form something, and this is called the Fourier integral; and this will be evaluate for all x from minus infinity to plus infinity. This is the main objective of this lecture that have to represent a non periodic function with the idea, what we had already in the last lecture Fourier series.

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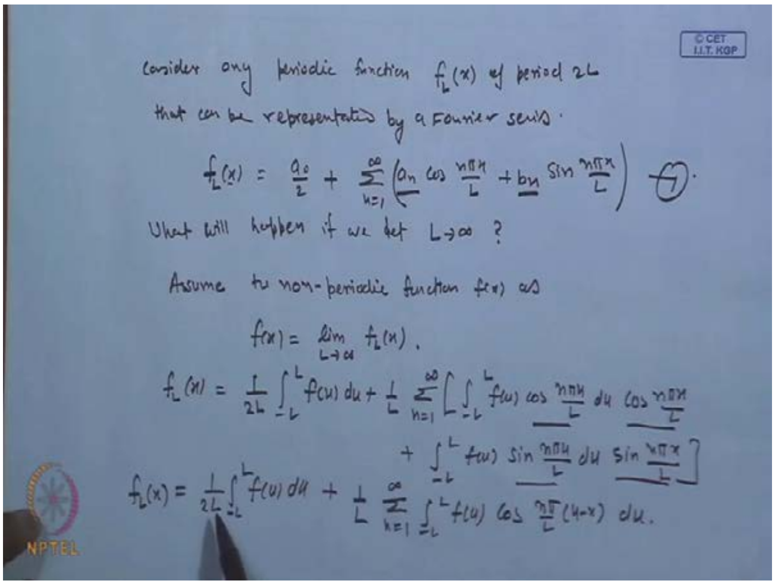

Let me just continue now with this for; so again we consider any periodic function, we denote now, f L(x) this L denotes the period, because of a period 2 L, and we have the following extension. So, f L is a function of period 2 L that can be represented that is what is assume this can be represented by a Fourier series. That f L(x) is a naught by 2 plus n 1 to infinity and we have a n cos n pi x over L and plus b n and we have sin n pi x over L. So this is the Fourier series, corresponding to this function of period 2 L, we note this function f L(x). In more general case that, this f L(x) we can are replace by the mean value of this function at this x. So, if the function is not defined or discontinuous at x, so we can have here f L(x) plus an plus f L(x) minus divide by 2, so we can have this separate value here. Now the question is that, what we happen, if we let L to infinity? So first we assume the non periodic function, because as L approaches to infinity you will get a non periodic function obviously, so periodic function f (x).

As this f (x), we denote the limit as L approaches to infinity and the function f L(x). So, if we assume this and now we pass, first let substitute this a n and b n into the function; so, what we have, that this f L(x) is a naught 1 over L; so, we have 1 over 2 L then minus L to f (u) du; here we have for a n 1 over L, I take out of this summation and we have then minus L to L f(u) cos n pi u over L du and then we have this cos n pi x over L and plus the other term here, minus L to L; for that, we have f (u) and sin n pi x over L du and then this sin n pi x over L.

Now what we do? So, the first term which as it is and f (u) du plus, for the second term here we have this cos and cos; so cos a, cos b and this plus sin a, sin b; so this we can combine into 1 cos function. So, we have here n 1 to infinity and we have this integral minus L to L also, f (u) is come on to both the integrals; and here we write this cos a minus b, so we have n pi over L and u minus x and du. So, this is still the Fourier series representation, just we have rewritten in this form. And now, we will have to limit L, this period L to infinity.

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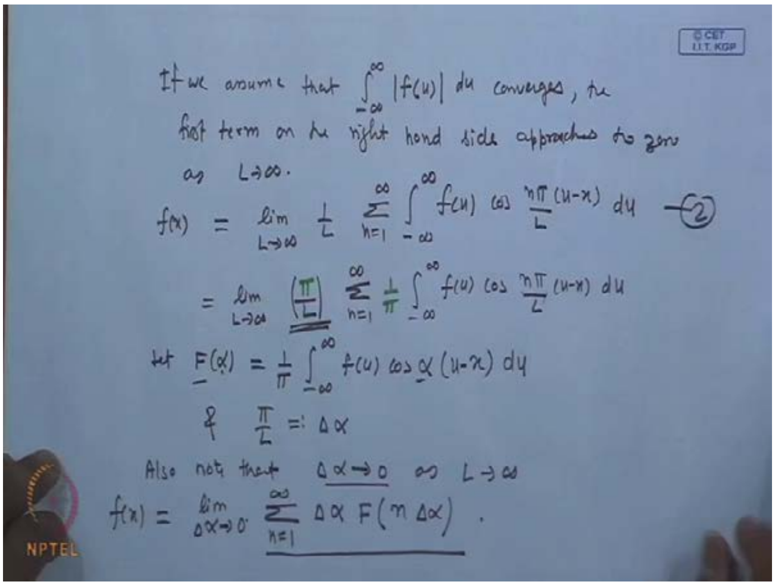

So for that, we also assume let me just write down that if we assume that, this integral minus infinity to plus infinity and the absolute value of f (u) and du converges, the first term on the right hand side approaches to 0 as L approaches to infinity. let us take we look again, so with this term as L we let to infinity, since we have here 1 over L and this we assume a finite quantity, so this will go to 0. So we have then, this f L (x) and the left hand side as we take the limit L tend to infinity and we assume that this will be a non periodic function f (x). So we have now, first term is 0 and then we left with L to infinity 1 over L and n 1 to infinity; we have minus infinity to plus infinity f (u) cos n pi u minus x over L du put it to equation number 2 that was the equation number 1. So now there is a trick here, let me explain very clearly, L to infinity we multiplied by pi here; so pi over L and then we have this summation n 1 to infinity, so this 1 over pi we have to accommodate here; so 1 over pi and then this minus infinity to plus infinity f (u) and we have cos n pi over L and u minus x du.

So now the point is that just for that simplicity, we assume that, this F (alpha) is 1 over pi integral minus infinity to plus infinity f (u) and cos alpha u minus x alpha u minus x d u; and also we assume that, this pi over L term, which is sitting here and also we have here pi over L. so, this pi over L we take as delta alpha, if we denote by this. So what we get after this new notation and also we also note that, as this delta alpha approach to 0, as L approaches to infinity. So we have a L approaches to infinity, because we have the relation between this L and delta alpha; so as this L approaches to infinity, delta alpha approaches to 0. What yet f (x) be get now, so f (x) this limit L to infinity, we have now limit delta alpha to infinity, delta alpha to 0 for this L infinity and we have pi over L and that is delta alpha. So, let me put again inside this summation, so n 1 to infinity and we have delta alpha and this 1 over pi and this quantity in terms of this f, so it is just simply where alpha is here we have n pi over L; pi over L is delta alpha, so this is n delta alpha, so we have F (n delta alpha). Now interesting to see that, what an exactly the summation is? So this f is some function of this alpha for a given x actually.

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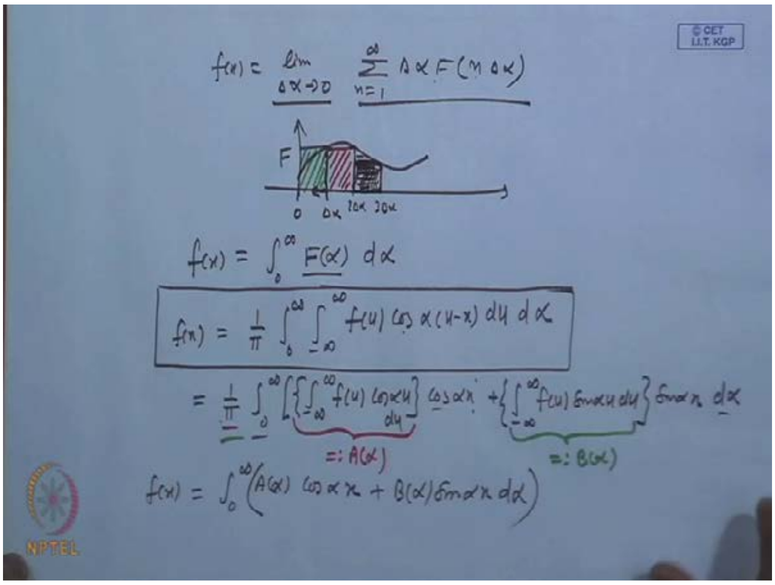

So, what we see now, let me just write again, so we have limit delta alpha to 0 and n 1 to infinity and delta alpha F (n delta alpha). So we just look at this as summation, so we have n is equal to 1, so the value of this F at d alpha. So we, let me start from this, so 0 and we have here this delta alpha, so value of this function as delta alpha; lets so we can have this form of this f, so let us assume that, this is F, this is the graph of this F.

So what we have doing here that; F when n is 1, so f delta value and then multiply delta alpha; so basically getting this area here this and then again n is 2, we summing up with F 2 alpha, so again alpha and we add, now we take this integral, so this area of this rectangular and so on; for the 3 alpha, here we have 2 delta alpha, we have 3 delta alpha and this area. So now the question is that, if this delta alpha we let this delta alpha to 0 and this is the exactly the definition of (( )) go L that is this delta alpha is tends to 0, this area which represent have this is be here, so 3 delta alpha and then multiply by delta alpha, so area of this rectangular. So in that case, now if we let this delta alpha to 0, this summation will give as the area under this curve. And that we can write exactly in the integral form, so this f (x) is nothing then, its 0 to infinity this integral 0 to infinity of this function f alpha d alpha.

Now, we go back to the original notations. So F(alpha) was defined 1 over pi and 0 to infinity and here we have f alpha minus infinity to infinity, we have f (u), we have cos alpha u minus x du and d alpha. And then, we can so this is in fact we got representation and that is already here, that we have 1 over pi and 0 to infinity minus infinity to plus infinity f (u) cos alpha u minus x du d alpha. But if we expand this cos, we get another form; so we have 0 to infinity and then minus infinity to infinity f (u) cos alpha u cos alpha x plus; so this I can cos here and then we have again, cos alpha u cos alpha x we have here and then this sin alpha u sin alpha as will go to the second integral minus infinity to plus infinity and f (u) sin alpha u d u; so, the du is missing and then du and sin alpha x and then we have alpha for this integral.

Now, what we do that, if we assume that this get 1 over pi and this one is A alpha and again with 1 over pi and with this, if we assume this B alpha. So, we can write in a complex form that this f (x) is 0 to alpha and we have A alpha cos alpha x cos alpha x plus B alpha sin alpha x and d alpha. And where alpha is 1 over pi minus infinity to plus infinity f (u) cos alpha u du and we have B alpha is 1 over pi minus infinity to plus infinity f (u) and sin alpha u du, so this is the Fourier integral representation of a function. And now we will go again discuss the convergence issue. So, It is more over less, we started with the Fourier series idea, so the convergence most of the assumption will remain same, but we have made one extra assumption and that was the integral mass converse that absolutely.

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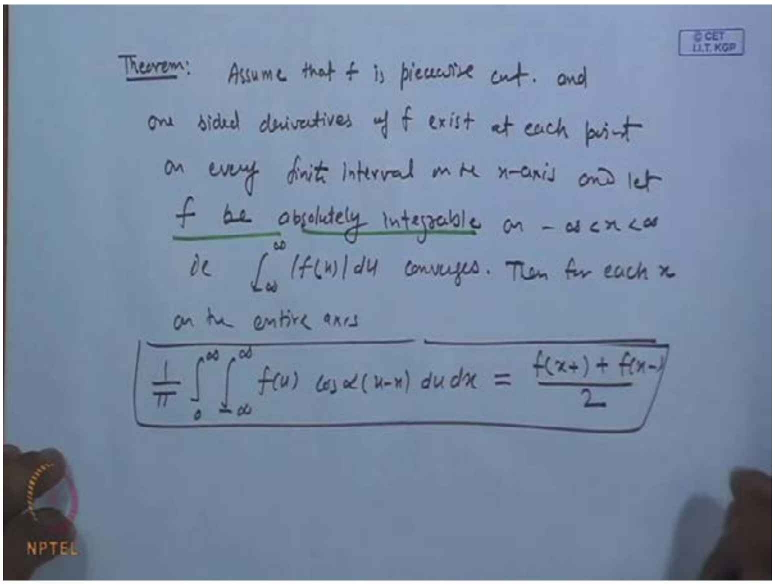

So, let summarize this theorem, we assume that f is piecewise continuous and one sided derivatives of f exist at each point on every finite interval on the x axis and let f be absolutely integral on minus infinity to plus infinity, that is this f (u) du converges. Then for each x, on the entire x axis, but we have that this 1 over pi 0 to infinity minus infinity to plus infinity, I am writing again, this complex form cos alpha u minus x du d alpha this Fourier integral will converts to the average value of the function, as the case of case of Fourier series, so f (x) minus and divided by 2. We have all other assumptions, what we had Fourier converges of Fourier series other than this f should be now absolutely integral, because we have use in the proof that, if f is absolutely integral then only, we have this representation for the function.

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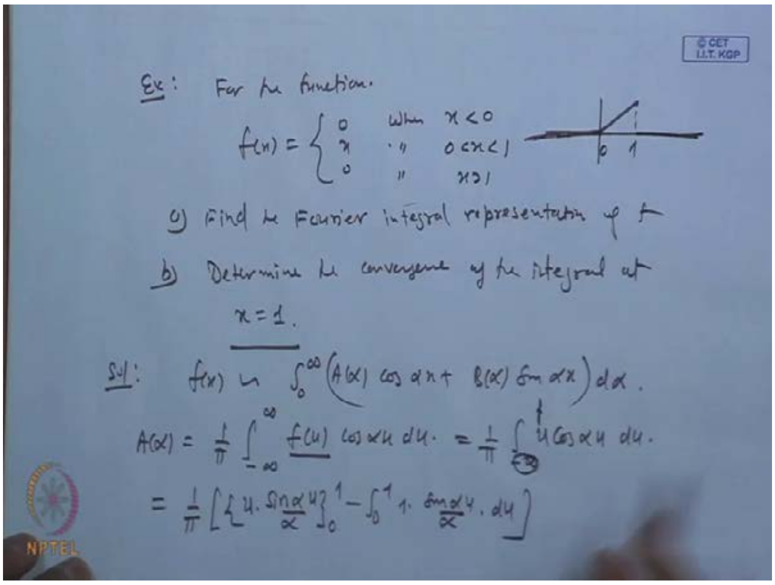

Now we go for the example, for the function we have f (x) 0, x, and 0; when x is negative and when x is between 0 and 1 and when x is greater than 1. So, find the Fourier integral representation of f integral representation of f and then determine the convergence of the integral at x is equal to 1. So here we look at this function here, which is defined from minus infinity to plus infinity, its 0 and the negative x is a greater than x is greater than once again 0; so we have this is x between 0 and 1 and then we have 0 and also here 0. So, in this case, this Fourier series representation in fact is not possible, because this function is not periodic and we do not have possibility to make a periodic, because its function is given on the whole x's.

So we have this Fourier integral representation and to get that, so we have this f (x) we can represent by Fourier integral representation A alpha cos alpha x plus B alpha and sin alpha x as d alpha. And we calculate this A alpha that is 1 over pi minus infinity to plus infinity and f (u) cos alpha u d u; this is 1 over pi minus alpha to we can have the function is 0 and then negative x axis x greater than 1. So what we have basically, a 0 to 1 and cos alpha u du of this function x, we have u cos alpha u du and this we can integrates 1 over pi and we have this u as it as and the integral of this sin cos alpha u alpha and 0 to 1 minus 0 to 1 the differentiation of this is 1 and now, sin alpha u over alpha and du.

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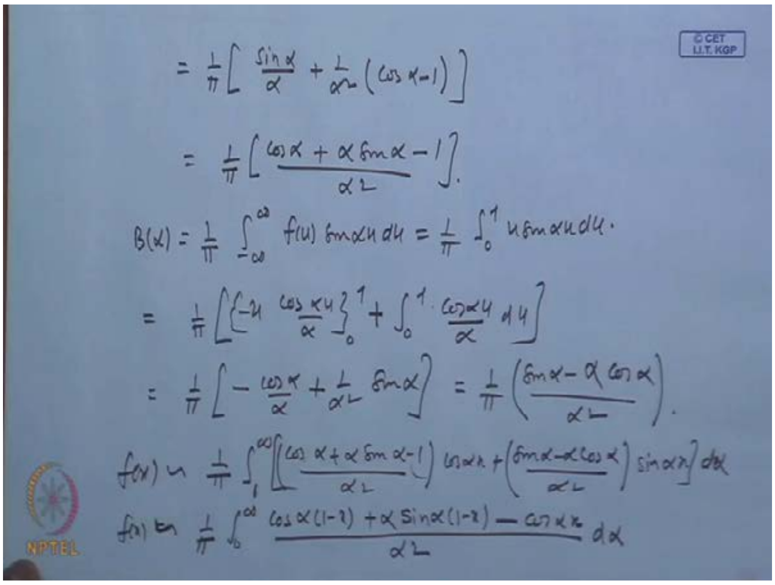

So we go for the, so we have 1 over pi. So, 1 this u is 1 we have sin alpha u over alpha and then 0 will be 0; so we have sin alpha over alpha plus 1 over alpha square. And so once we integrate again, here we will get 1 over alpha square and cos alpha u this will be cos in that case minus will come after this integration. And then we substitute the limit, the upper limit will give cos alpha and then minus 1. So 1 over alpha square and cos alpha minus 1, so we have 1 over pi cos alpha plus alpha sin alpha and minus 1 over alpha square. Similarly, we calculate the B alpha that is 1 over pi minus infinity to plus infinity and f (u) sin alpha u d u; 1 over pi 0 to 1 u sin alpha u du.

So again, we integrate by we have u minus cos alpha u over alpha and this limits 0 to 1 minus 0 to 1 and this minus minus will be plus, so we have here u will be 1, so we have cos alpha u over alpha and d u; so we get 1 over pi and minus cos alpha over alpha minus minus plus u to 0. This is round here, so we have 1 over alpha square and sin alpha u, so that is 1, u is 1 we will get sin alpha and 0 we get 0. So this is nothing, but 1 over pi and sin alpha minus alpha cos alpha over alpha square. So, this Fourier integral representation of that given function is 0 to infinity, we have cos alpha plus alpha sin alpha minus 1 divided by alpha square, so this was A alpha and then we have cos alpha x plus this B alpha sin alpha A alpha cos alpha over alpha square and we have sin alpha x and d alpha.

So we can combine this again, 1 over pi 0 infinity and this cos alpha cos alpha x plus sin alpha sin alpha x will give us, cos alpha 1 minus x and plus, this alpha we take common, then we sin alpha cos alpha x and cos alpha sin alpha x will give us, sin alpha 1 minus x and then this minus cos alpha x and divide by alpha square and d alpha. Well, so this is the Fourier integral representation of the given function and now the question was that, determines the convergence of the integral at x is equal to 1.

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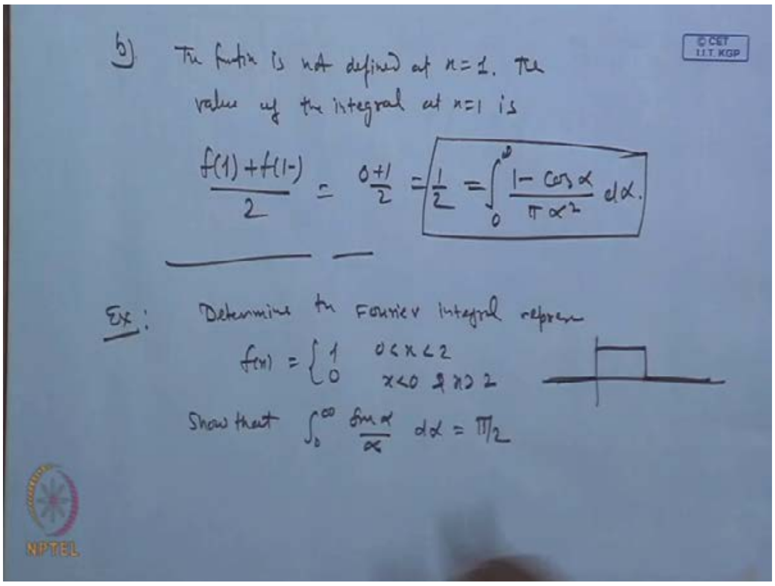

So now, the b part and if we look at the function this function is not defined at x equal to 1, but we this integral will convergence in that case, to this average value, so that is 0 here and 1, so 1 plus 0 by 2 that integral will convergence. So the function is not defined at x is equal to 1, so the value of the integral at x is equal to 1 is, f (1) plus f (1 minus) divided by 2, so 0 plus 1 by 2. So, what is the integral at x is equal to 1, let us just see, we will put here x 1; we will get cos alpha, if we put cos alpha into cos 0 that is 1 and this will be 0, because sin 0 and 1 minus cos alpha.

So this is just, 1 minus integral we have 0 to infinity 1 over pi and 1 minus cos alpha over by pi and alpha square d alpha, so integral of this as half. So we continued in the next example, determine the Fourier integral representing this function f (x) 1 and 0, then x is between 0 and 2 and then x is less than 0 and x is greater than 2, it is 0. So between 0 and 2 the function is given by this 1 and then 0 and this side also we have 0, so further we show that this integral sin alpha over alpha d alpha is pi by 2.

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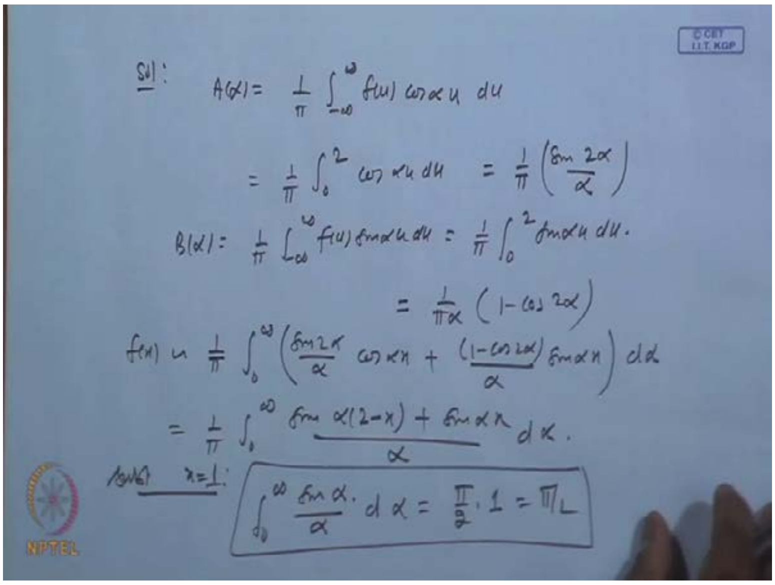

So it is a very similar example, what we have seen this, so we get this A alpha 1 over pi minus infinity to plus infinity and f (u) cos alpha u du. So, we have 1 over pi and function was defined 0 to 2 only other than this it was 0, so we have cos alpha u du and that we can simplify to give sin 2 alpha over alpha. Similarly, the B alpha we get 1 over pi minus infinity to plus infinity f (u) sin alpha u du is 1 over pi 0 to 2 and sin alpha u du and then we have cos alpha u over alpha, so we get this over pi alpha 1 minus cos 2 alpha. This f (x), we can represented by this Fourier integral 0 to infinity and sin 2 alpha over alpha cos alpha x plus 1 minus cos 2 alpha over this alpha pi is there and then we have sin alpha x d alpha. So, we have 1 over pi 0 to infinity, we can again combine these 2 cos 2 alpha cos alpha and minus cos 2 alpha sin alpha x.

So sin, we get alpha and 2 minus x plus this sin alpha x let and divided by alpha and d alpha. So, we substitute here it x is equal to 1, we have f 1 and the value of this function at 1, so here 0, it was 2, so at 1, this value is 1, because here we have point of continuity. So we will get just the functional value there, so we have this integral 0 to infinity sin alpha and here also, sin alpha so, we get 2 sin alpha to will take to the right side. So, 2 over pi will be pi by 2 and the function value at 1 and this is 1, so this integral value is pi by 2. This was sin, this was the Fourier integral representation of a function.

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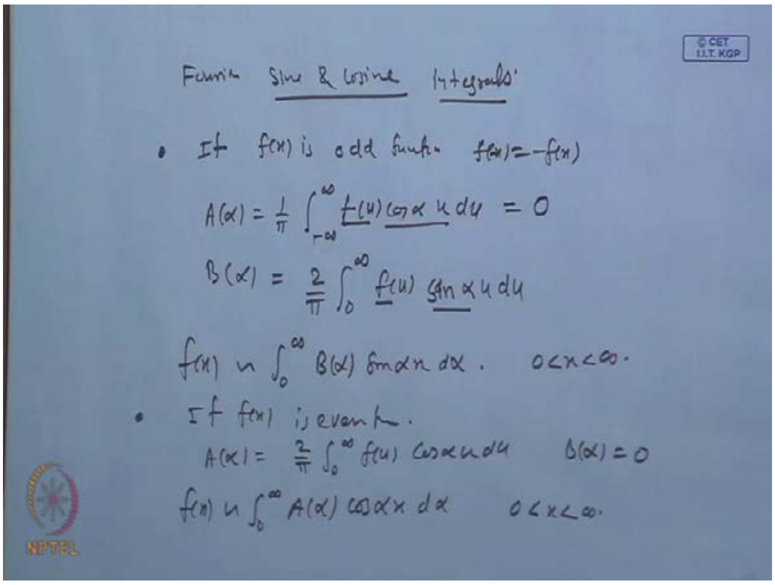

And now, we will go for the particular case is that, Fourier sine and cosine integrals. And it's basically, the area what we have Fourier sine and cosine series. If this f (x) is odd function that means, this f (x) is minus f (x), f minus x is minus f(x) odd function. In that case, this A alpha, we have minus infinity to plus infinity and f (u) cos alpha u du. So, f is odd then this is even so we have odd integral minus infinity to plus infinity is 0. And this B alpha will be 2 over pi and 0 to infinity instead of minus infinity to plus infinity, because now the integral is even function, sin alpha u du, so this is odd and this is odd so we have even functions, so the integral 2 over pi and f (u) sin alpha u du. In that case, this Fourier sine representation of that function will be given by 0 to infinity B alpha and sin alpha x d alpha. And similarly, if the function f (x) is even function then d alpha will be 0 and this A alpha will be given by 2 over pi 0 to infinity and we have f (u) cos alpha u du, B alpha will be 0 and this f (x), we can represent by this 0 to infinity A alpha and cos alpha x d alpha and this x is between to 0 to infinity.

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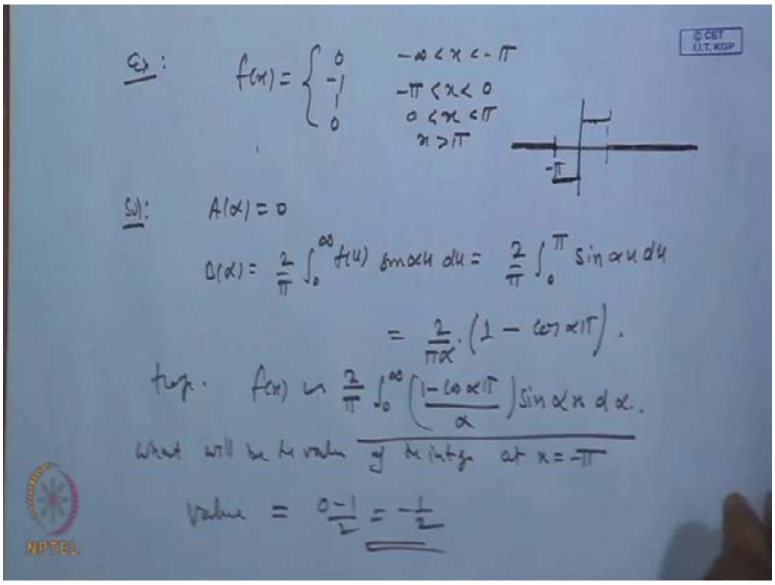

So, let us let us quickly go for one example, we consider this function f (x) 0, minus 1, 1, and 0; minus infinity to minus pi and minus pi to 0 and 0 to pi and this is 0 x is greater than pi. For this function, if we just look at , we have 0 less than minus pi then the function value is minus 1 then we have plus 1 and then again 0. So, this function is an odd function, because this value minus of this value this function is odd function and then A alpha will be 0.

So solution you want to have this Fourier integral representation and A alpha will be 0 and then B alpha if we get such 2 over pi 0 infinity f (u) sin alpha u du and this is 2 over pi, we have 0 to pi simply that is value 1 for f sin alpha u d u; and this is 2 over pi alpha, we have cos alpha u, so with minus sin 0 will put first we get minus 1 minus cos alpha pi. Therefore, the Fourier integral representation of Fourier sin integral representation will be given by 2 over pi 0 infinity 1 minus cos alpha pi over alpha and we have sin alpha x d alpha. So there was a question that what will be the value of the integral at x is equal to minus pi. So value of this integral at x is equal to minus pi, we will take the average of these 2 values, such as 0 and minus pi by 2, so this value will be 0 minus 1 by 2 so minus half.

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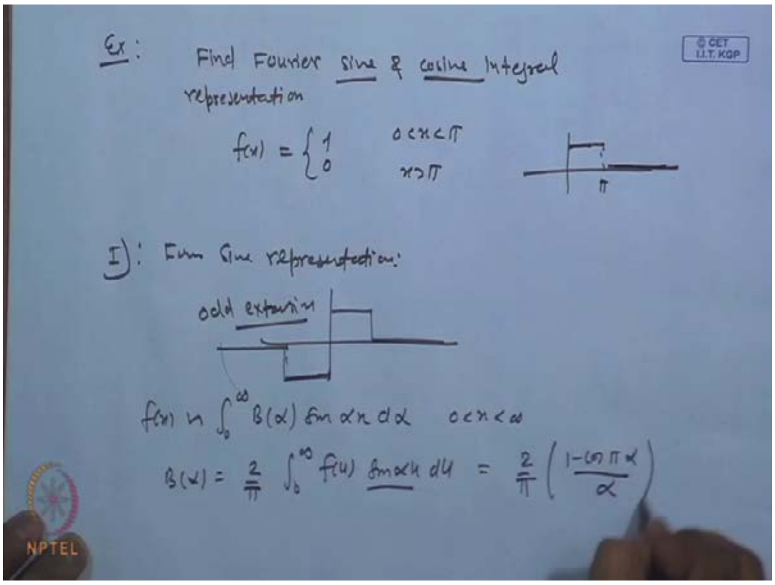

So one more example, we go; find Fourier sine and cosine integral representation for f (x) 1 0 for 0 to pi and when x is greater than pi, it is 1. So this function is given, it is define from 0 to infinity only we have 1 value and then when greater than pi 0, this is the function given and we are interest to find this sine and cosine integral. So for the sine integral, we will have the even the odd extension and for the.. So, let us just go for the Fourier sine representation. In this case, so we can extend that function which is given here as an odd function this and then 0. So now for this function with new function, this is the continuation to that function the given function in the left x axis. We have this odd extension and in the case, we will get sin integral representation, because for this function A alpha will be 0 and we will get only the f (x) 0 to infinity B alpha and sin alpha x d alpha. And for this we will calculate this B alpha that is 2 over pi 0 to infinity f (u) and sin alpha u du. So this integral, so the f (u) we can 0 to pi and we have sin alpha u du, so this will be 2 over pi 1 minus cos pi alpha over alpha.

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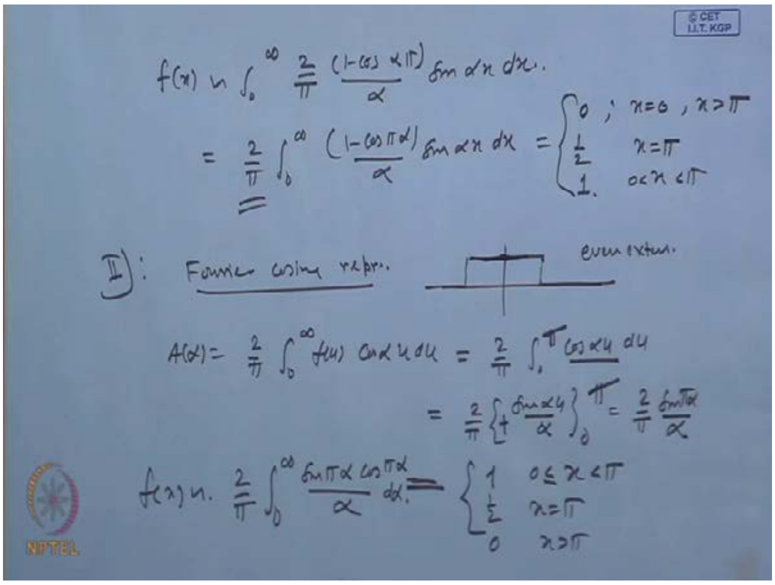

So this Fourier series representation, Fourier integral representation 0 to infinity and we have this B alpha 1 minus cos alpha pi pi alpha divide by alpha sin alpha x dx. Or it is 2 over pi 0 to infinity 1 minus cos pi alpha over alpha and sin alpha x dx. If we want to see value of this integral, what will be the value of this integral? So at x is equal to 0 or x is greater than pi the value would be, so here the value would be 0 and at this point if we take the average is going to again 0; so, the value of this function will be 0 and at x is equal to pi, we will take the average, so we have 1 and we have 0.

So, 1 plus 0 by 2 that means 1 by 2 and what left now between 0 and 1. So, if x is between 0 and pi then the value is 1. So for different values, we can get particular integral and can find the value of the function. So now the second Fourier cosine representation; so in this case, we will extend this function as given functions even extension. And in this case, now B alpha will be 0 and this A alpha, we can calculate by 2 over pi and 0 to infinity f (u) cos alpha u du, so we have 2 over pi and 0 to 1 cos alpha u du or 2 over pi, we have sin alpha u over alpha with negative sin and 0 to 1. So, we have 2 over pi with plus only, because the integral of this cos will sin alpha u over alpha 0 to, so we have 2 over pi and we have sin alpha over alpha. And then this f (x) we can represent by this 2 over pi 0 to infinity and this function was defined between, it was 0 to pi, so we have to have here 0 to pi. So in that case, it will be just pi alpha.

So this Fourier representation, Fourier integral representation will be sin this pi alpha and cos pi alpha and divided by alpha, d alpha. So if we want to have the value of this integral, so it will be 1, as usual when this x is between 0 and pi. So in this case, at this 0 also we have the value 1, because its function is not continuous. At that point also we have the half value at x is equal to pi and we have 0, when x is greater than pi.

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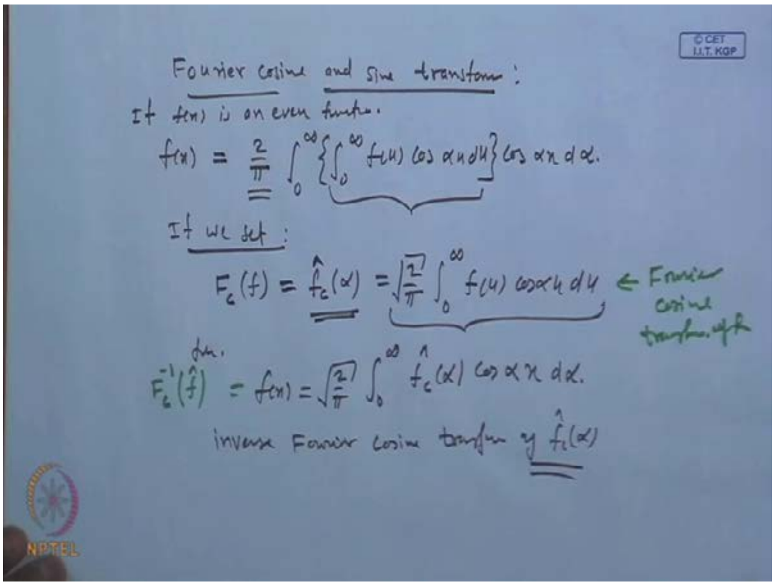

So now, we go to very important topic of this which is just extension of this is study, what we are doing now; so Fourier cosine and sine transform. So what we have this Fourier integral of a function f (x) we assume this equality there, so 2 over pi. In general, we have this average value 0 to infinity and let us write in this form. So this is the first case for the cosine 1, so if function f (x) is an even function, then this Fourier cosine representation will be 0 to infinity and 0 to infinity f (u) cos alpha u du and then we have this cos alpha x d alpha; this is A alpha and this is 2 over pi this and this. Now, if we set here, if we set let F c, of course we denote this F c (f) or f c hat function of alpha is square root 2 over pi and this integral 0 to infinity f (u) cos alpha u du. So, basic value this part and we have taken here is root 2 over pi du. So, this part of we name with F c alpha with this hat or later on we will call it the Fourier cosine transform of f. And then what we have the f (x) is equal to again the same factor leftover there 0 to infinity, f hat c alpha and cos alpha x d alpha.

So this is called the Fourier cosine transform of f and this is here is called, we can write this f cosine inverse of this f hat c alpha; so, this is called the Fourier or inverse Fourier cosine transform of f hat c alpha.

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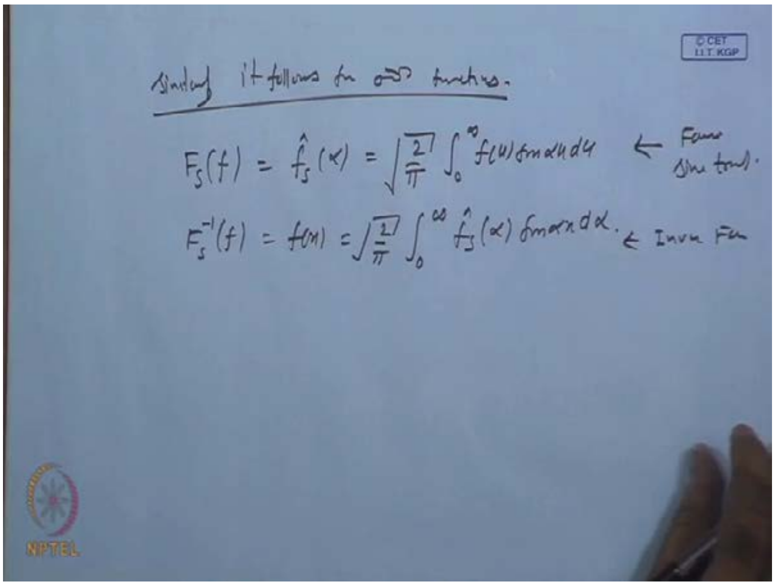

So now same for the sin representation; so similarly, it follows for odd functions; for odd functions what we have that, Fs (f) f hat s alpha denote it and we are 2 over pi and 0 to infinity f (u) sin alpha du. And we have similarly, F s inverse f this is f (x) and we have 2 over pi, we have 0 to infinity f hat s alpha sin alpha x d alpha; so this is the Fourier sin transform Fourier sin transform Fourier sin transform and this is inverse Fourier sin transform. They are different persons available in the literatures, so we can have here completely 2 over pi and then can remove here this factor or vice versa; so, but we will follow this that same factor sitting for both.

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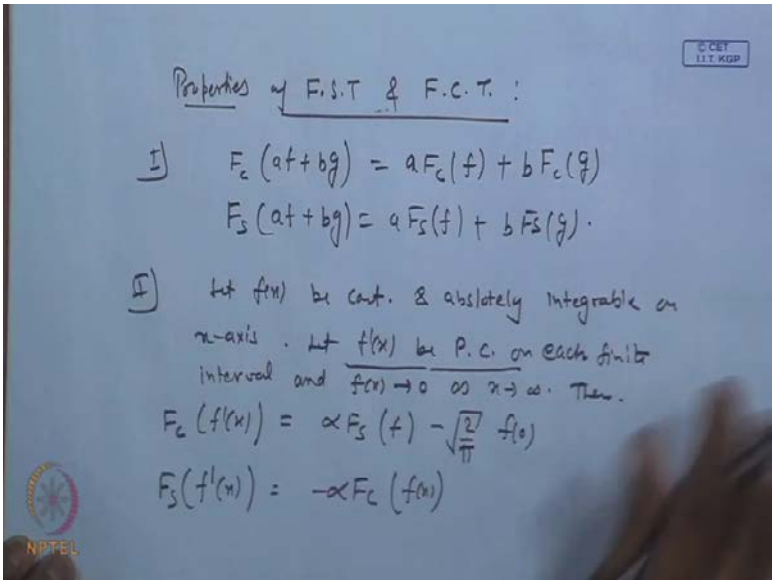

So we now quickly go for some important properties of these transforms, properties of Fourier sine transform and Fourier cosine transform. So the one linearity which holds for all these integral transforms, so we have a f plus b g and this will be a the constant time Fourier cosine transform of f plus b, the Fourier cosine transform of g. Or similarly for this sine, if we have a f plus b g a F s plus b F s g. The second which is important have property and we will not discuss more properties for this sine and cosine transform, but this will be useful. Let f (x) be continuous and absolutely integrable on x axis, also we let this f (x) be piecewise continuous on each finite interval and this f (x) tends to 0, as x tends to infinity. Then, so this is similar condition what we have this left and right derivative exists, so this is slightly stronger condition than that. So, what we in this case that the Fourier cosine of the derivative will be f Fourier sin of the function f minus 2 over pi and f (0) and we have f f sin f prime x as minus alpha Fourier cosine transform of f.

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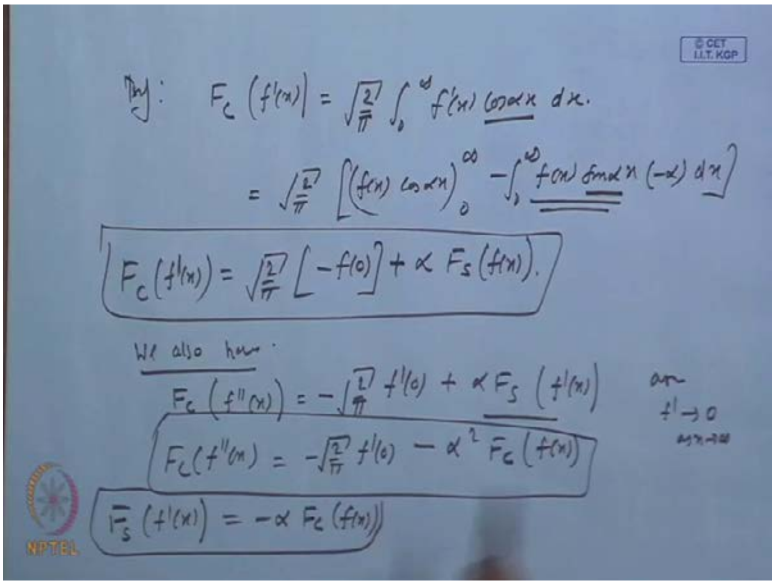

Let us go quickly to the proof, so we have Fourier cosine transform of a prime x is square root 2 over pi 0 to infinity f prime x and cos alpha x dx. So we have 2 over pi and then we take the integrate by parts here cos alpha x limit 0 to infinity minus 0 to infinity, we have f (x) sin alpha x into minus alpha, because this is the differentiation of this with minus sin alpha will appear. So what we have 2 over pi, as x approaches to infinity this f will go to 0, so we have minus f 0 and minus minus plus alpha and this is the Fourier sin transform, Fourier sin transform with this one over. So we take this 2 over pi with this term and this is Fourier sin transform of f (x); so this was Fourier cosine transform of f prime x. So, we also have, if we go for the second order derivative that means f prime x, so we have minus square root 2 over pi, first derivative by this formula is first derivative and plus alpha f sin and f prime x; and now we assume here, in addition to f f prime goes to 0, as x goes to infinity then only will get this. And then we apply 2 over pi f prime 0 minus, this is will see that of sin of a prime or a function alpha square and f cosine we will get f (x), so this is this. And similarly, we can go for the sin function, so for cosine, we have f sin and the f prime x with a similar trick, we can have will get minus alpha f cosine f (x) and for the second derivative.

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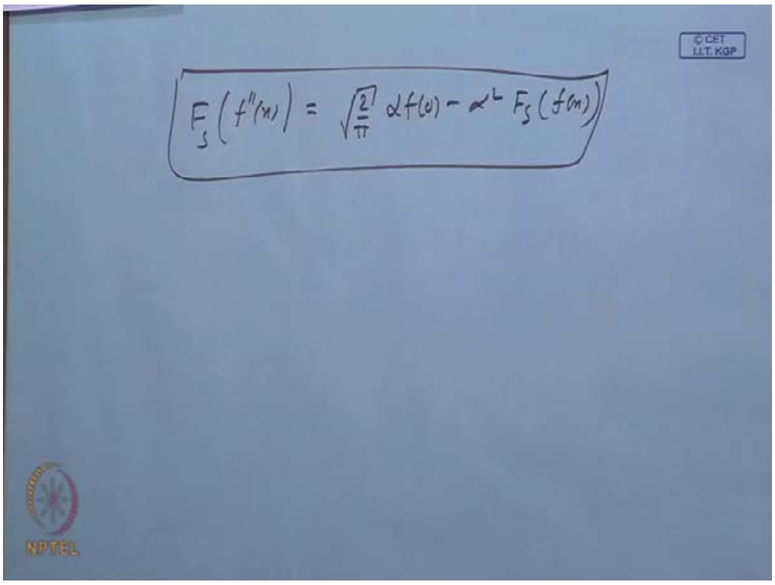

We have the F sin for the second derivative of x is square root 2 by 0 alpha f(0) minus alpha square F sin f (x). So here we have introduce this Fourier cosine and sine transform we started with the Fourier integral our representation of a function, and then we introduce this Fourier cosine and sine transform. So in the next lecture, we will continue with this Fourier sine and cosine transform with the help of this example, and then go for the Fourier transform. So more on this in the next lecture, thank you. Good bye.