Advanced Engineering Mathematics
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Lecture No. # 27
Fourier Series

Welcome back to the series of lecture seven; transform calculus and so, for we have covered Laplace transform including its application to solving partial differential equations ordinal differential equations and in the rest of this lecture we shall discuss about the Fourier transform. And before we continue with this Fourier transform it is very important to to introduce Fourier series because it gives a pathway to understanding Fourier transform and Fourier series has it is own wide range of applications for example, in analysis of the current flow and sound waves and in many more. They are also used to solve differential equations and in a general sense, we can say that we use Fourier series to represent a, to approximate a periodic function and in need not be a periodic function, but, a function which is defined on a finite intervals.

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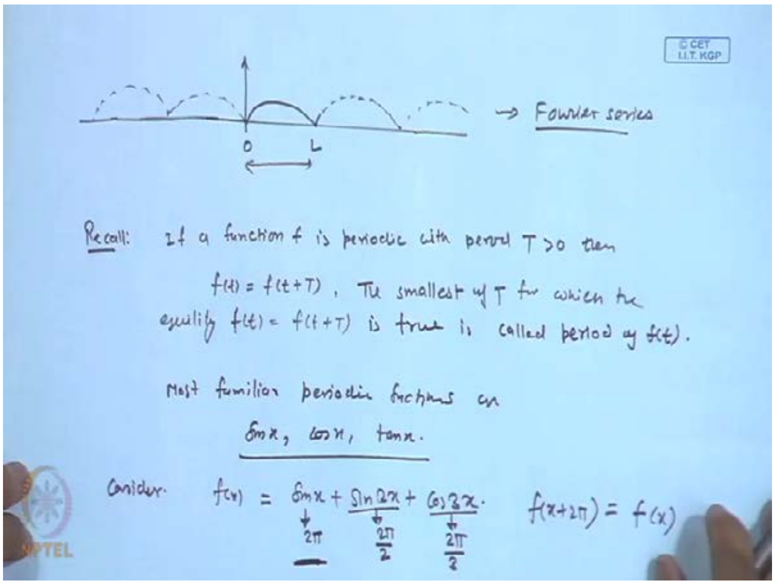

So, let us let us see that if a function is defined for example, in a interval 0 to l. So, what we do in that case since is Fourier series represents a periodic function that we extend this as a periodic function on the whole real axis and then we approximate this function which is periodic now by the Fourier series by the Fourier series and then this Fourier series will approximate this function which was define 0 to l. So, in that way we basically can have this Fourier series for for any function which is defined on a finite interval 0 to l. So, since we are or we will be talking about the periodic functions. So, let us again we just recall this.

So, if a function f is periodic with period T then, we have f t is equal to f t plus this period. So, the function value is again the same after this period capital t and the smallest of t for which the equality holds or which for which the equality f t is f t plus t is true is usually called period of ft. So, most familiar periodic functions are most familiar periodic functions are like we know already sin x, cos x, tan x and so on. So, the one point here are the property of this periodic functions that the sum difference of product or the coefficient of 2 periodic function is again a periodic function. So, consider for example, a function f x as sin x plus sin 2 x plus cos 3 x. So, what is the period of the sin x, is 2 pi the period of the sin 2 x is 2 pi divided by 2 and the period of this cos 3 x is 2 pi and divided by this frequency three. So, we have 2 pi by 3 so we have 2 pi by 3 that is the period of this function. So, what we see that the common period of all these three functions is this 2 pi. It has 2 circle in this 2 pi range it has 3 circles or cycles in this 2 pi range. So, the common period is basically 2 pi of this function and basically what we have that f x plus 2 pi will be. So, because f x plus 2 pi is again sin x and sin 2 x plus 2 pi will be again sin 2 x because this 2 pi is also period for this sin 2 x function and same for this cos 3 x, we have again this the value will repeat after this 2 pi. So, we have this common period for all these three functions is 2 pi and that is the period of this f x function. So, f x is periodic and it has the period 2 pi.

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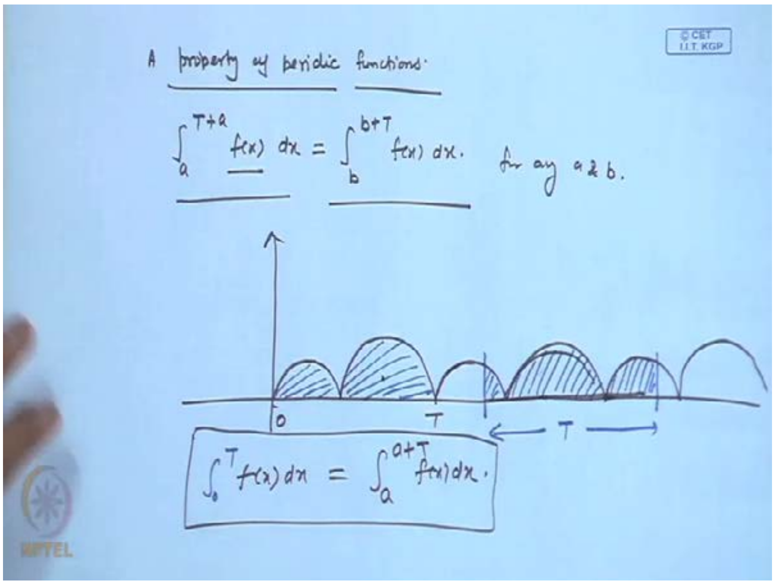

So, let me just introduce an interesting property of periodic function which we shall be using in this lecture. Periodic functions a property of periodic function. So, if we have a periodic function of period capital t then and the function is integerable. So, a to t plus a. So, we are integrating over this length T, this function f t which is a periodic function of period this capital T. So, this will be equal to we take any other b here b to b plus T f x T x for any a and b. So, if a function is integerable on any interval of length t then this is integral on any other interval of the same length and the value of the integral is the same and this we can without going to formal proof what we can see? What we can see here that for example, you consider that this is the periodic function we are talking about and so, on. So, this is from 0 to capital T we have this period and so, on. So, if we integrate for example, from here in interval of this t. So, this will end up here. This is again t. Now, we see that this area what we get after this integral, it is basically equal to to this area if we integrate over 0 to t. So, basically what we have that is 0 to T f x d x 0 to T f x d x this is equal to any a we can take up to a plus t as we have taken here and this will be equal because the part left here of this small circle that is included here. So, we are getting actually this area. So, this result is is true.

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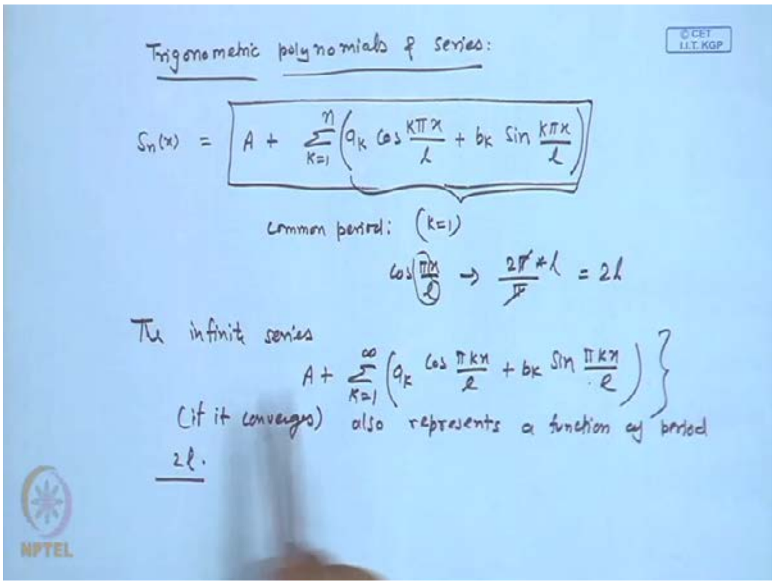

Now we talk about the trigonometric polynomial and trigonometric series. Trigonometric polynomials and series. So, trigonometric polynomial of order n is defined or call it as an x some constant plus k from 1 to n another constant a k and cos k pi x over L plus b k and sin k pi x over l. So, what is the common period of of these functions here? So, the common period or the largest period what we will get while putting k is equal to 1. So, this is we will get here cos pi x over L and this sin pi x over L and here the period is 2 pi over this number pi and l. So, we have 2 l. So, what is represents? This sum, this represents a function of period 2 l. However, if we keep on increasing this n; this will it still represent a function of period 2 l. In fact, there is true infinite series as well if it converges of course, so, the infinite series if we talk about a plus is k from one to infinity a k cos pi k x over L plus b k and sin pi k x over L if it converges. So, if it converges also represents a function of period 2 L. Now, the question is can any function, can any given function of period are 2 L be represented as a some of trigonometric series and the answer to this question is yes because it is possible for a wide variety of or wide class of functions. So, will be talking about today that how to get such an infinite series for a given function.

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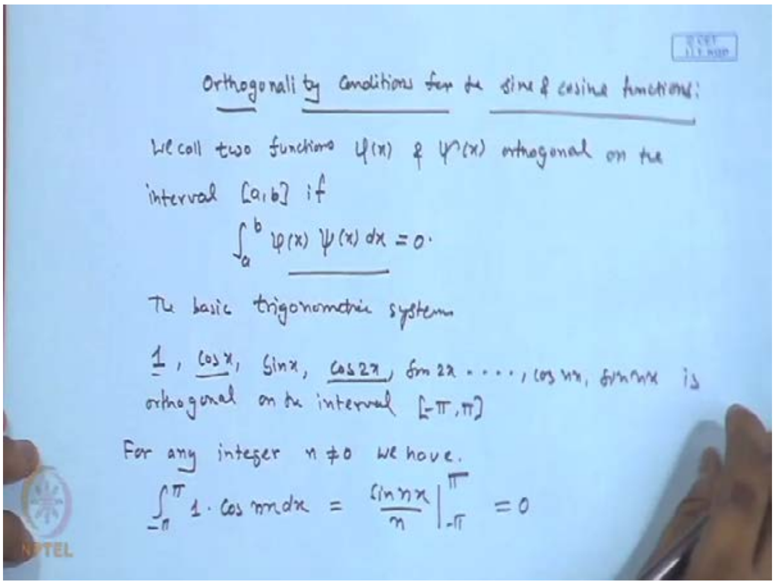

So, before we go in to that detail, we just consider the Orthogonality property of conditions for the sine and cosine function. So, we call two functions phi x and psi x orthogonal on the interval a to b if a to b phi x psi x if this integral is 0 and with this definition what we can say that the basic trigonometric system triogonometric system of period 2 pi. So, one cos x sin x cos 2 x sin 2 x and so on cos n x sin n x is orthogonal on the interval minus pi to pi. So, we shall prove that any two distinct functions if we take then they are orthogonal. So, let us go to the proof quickly. So, for any integer if we take which is not is equal to 0 what we have that minus pi to pi if with this function one we take any other function cos n x g x what will be the integral? This is sin n x and over n and we have minus pi 2 pi n sin n pi 0 sin minus n pi is 0. So, we get here equal to 0.

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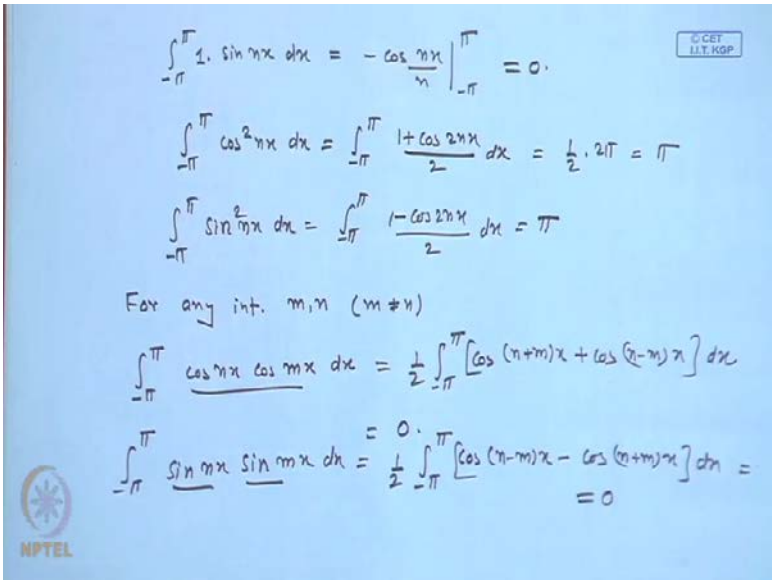

Similarly, if we take here the multiplication with one and sin n x for any n not equal to 0; because for 0 is going to be 0. So, what we have here minus cos n x over minus over n and minus pi to pi and this cos n pi will be minus one power n and again this cos minus pi will be are the same minus one power n. So, here this will also give us 0. Let us see what will happen if we take the same functions from that system that must be minus pi to pi and if we take cos square n x. So, cos n x and cos n x the product; so, minus pi to pi and we have this one plus cos 2 n x over 2 d x. And in this case this will give 0 because sin 2 n x and pi and minus pi both will be 0, but, we have here half and the integral that is 2 pi. So, we get pi and similarly, we if we take the sin n x and sin n x. So, sin square n x d x we will also get this pi because we have one minus cos 2 n x over 2 d x and this will also give then pi.

Now we take for any integer m and n where m is not is equal to n what we have if we integrate minus pi to pi and cos n x and cos m x. So, any two function, we have taken here. So, what we can write this as 2 cos n x cos m x. So, minus pi to pi that will be cos n x plus m x. So, n plus m x and plus cos n x minus n m x. So, n minus m x and then d x. So, again we have sin after this integration here also we have sin. So, in that case this pi and minus pi both will give as 0 again. And similarly, if we take here sin n x and sin m x the product of these two functions and we can again see that this will be also 0. So, two times this will be cos n x minus m x. So, n minus m x and minus cos this sum of this two, n plus m x d x and for the same reason we have also this is equal to 0.

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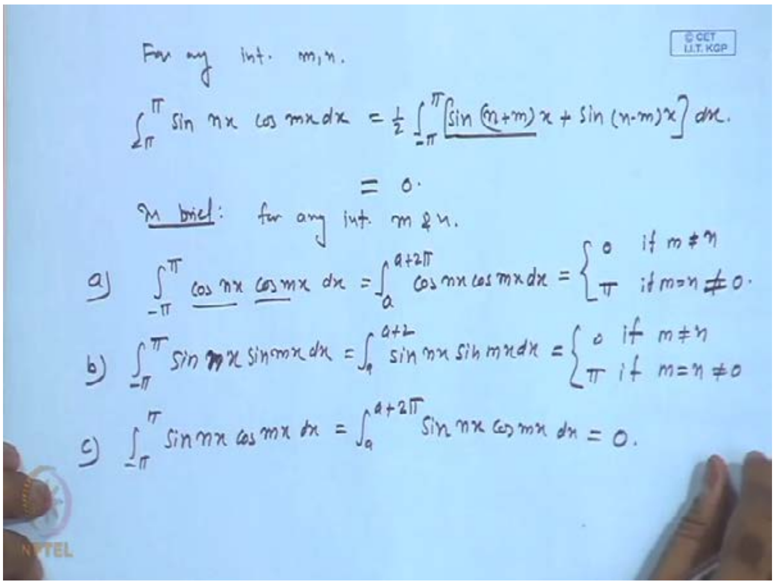

Now we take the left combination and that is again for any integer we can take m and n and if we take the product of sin and cosines. So, n x and the cos m x t x so, in this case we have minus pi to pi sin n plus m x and 2 sin cos. So, we have plus this sin a minus b. So, n minus m x and d x we will get here cos and cos pi over minus pi it is the same value. So, we will get again 0. So, in brief what we got? For any integer m and n we have the following result that, from minus pi to pi if we integrate this cos n x cos m x d x or which is the property of the periodic function that instead of minus pi to pi we can go for any a to a plus 2 pi; it will be the same. We have cos n x and cos m x d x and this will be 0 if m is not is equal to n and will be pi if m is equal to n because in that case we have this square here and of course, this these should not be equal to to 0. Otherwise we will have to pi value here. So, this is one result we have the second one minus pi to pi if it take sin m x over n x m x area to a plus 2 pi same value we will get n x and sin m x d x. In this case also we have this 0 if m is not is equal to n and we have pi if we have m is equal to n and is not is equal to 0. So, the last case we have minus pi to pi where we have combinations of sin n x n cos m x t x and this will be a to a plus 2 pi and we have sin n x cos m x d x and this is always 0. Whatever is m is equal to n or not equal to n; we will get 0. So, this was for the family or for the system where the common period was 2 pi and we are integrating from minus pi to pi.

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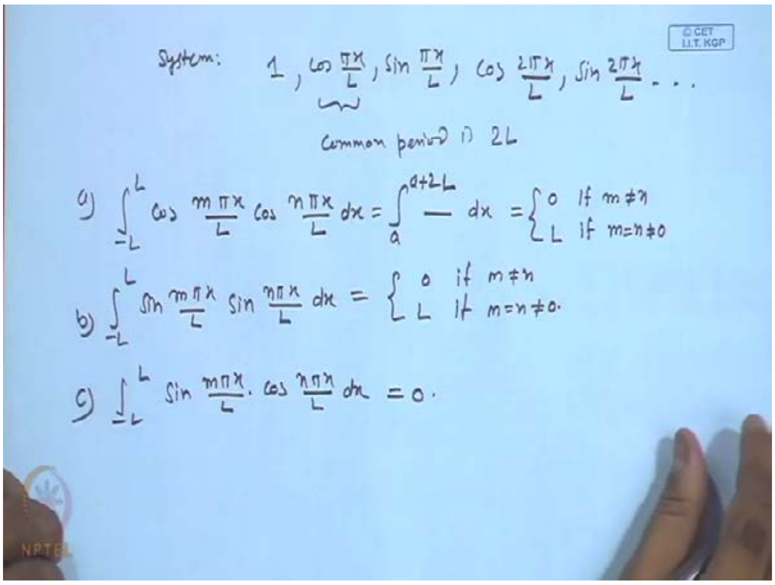

A similar result we have basically more general for the system if we take one cos pi x over L sin pi x over L cos 2 pi x over L and sin 2 pi x over L and so on. So, here the common period common period is is 2 L and now we have again similar results that if we integrate from minus L to L or from 0 a ton 2 L over 0 to 2 l. So, we have cos m pi x over L and cos n pi x over L d x or we take this from a to a plus 2 L 2 L in the same integrant and this will be equal to 0 if m is not is equal to n and will give us L if m is equal to n and not is equal to 0. The analogous we have this second result minus L to L and we have sin m pi x over L and sin n pi x over L t x over a to a plus 2 L. We can integrate. So, if m is not is equal to n otherwise its L if this n is equal to n n not equal to 0. And the last one if we have the product of the two; that sin m pi x over L and cos n pi x over L d x and this will be zero.

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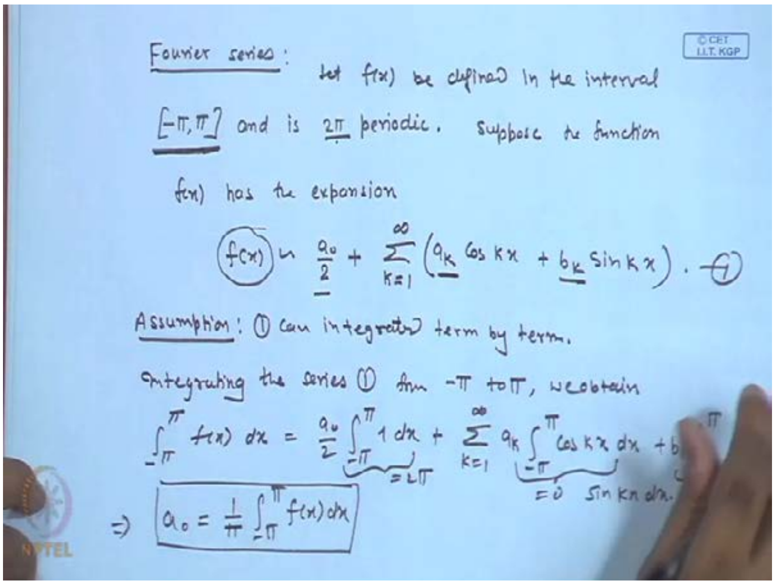

So, now, we have done this preparation to work for the Fourier series. So, Fourier series now we will introduce can here. So, we will start here. So, let f x be defined in the interval minus pi to pi and as 2 pi periodic two pi periodic. So, let me just mention this that for simplicity we are taking the symmetric interval, but, definitely this is not necessary at all or that all the functions to be defined on a symmetric interval. So, this is just for the simplicity and later we will see while discussing the examples that we will go for basically for any interval but, the function should be periodic. Or later on for the Fourier series also we will generalize this when the period is not 2 pi, but, for any given general period what will be the Fourier series.

So, now, suppose the function f x has the expansion f x a naught by 2 k one to infinity a k cos k x b x sin k x. So, we have taken this instead of this constant a naught by 2 to have this consistency with this a k later on we will see why this vector 2 we have introduced here. So, now, question is or the problem is that how to determine these coefficients for a given function f x yeah. So, for this we have taken one assumption; that the series here can be integrated can be integrated term by term. That is we have assumed that for the series, the integral of the sum is equal to the sum of the integral. So, now, we integrate the series here from minus pi to pi for a general case when we will be taking minus L to L for the function of period 2 L we will integrate again from minus L to l. But, in this case we integrate this minus pi to pi and let us assume that the integral of this is equal to the integral of this f x. So, integrating the series one from minus pi to pi what we obtain?

Minus pi to pi f x d x. So, this is a by 2 and the integral minus pi to pi and we have one d x one plus k one to infinity. We have here a k and then minus pi to pi cos k x d x plus this b k and we have this integral minus pi to pi sin k x d x. So, both these integral of this one is 0 and the other one both are 0 we have just seen. So, what we get from here; that a naught because this will be 2 pi, this is 2 pi. So, we have a naught pi. So, a naught is one over pi and minus pi 2 pi f x d x. So, we have obtained this a naught, the one coefficient of that series and now we multiply to get the other constant.

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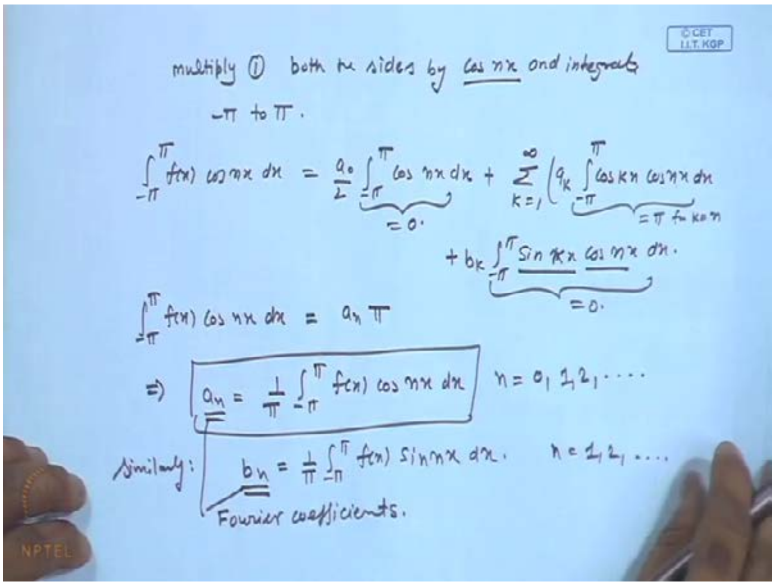

So, we multiply one that series one both the sides. So 2 f x and that the series. So, both the side by cos n x and integrate. So, from minus pi to pi of course, so, what will happen? In this case we have minus pi to pi and f x cos n x d x is a naught by 2, we have minus pi to pi and cos n x d x plus k 1 to infinity a k minus pi to pi cos k x cos n x d x and plus b k minus pi to pi sin k x. So, sin k x and cos n x d x. So, this is the sin cos. So, this will be always 0. That is what we have seen and here we have cos k x cos n x. So, when this k will be n, this value would be pi for k is equal to n and the rest all the values are 0. And this is again this is zero. So, what we got; that minus pi to pi and f x cos n x d x is equal to we have this a n and pi. So, we get basically a n is equal to one over pi minus pi to pi and f x cos n x d x. So, this is the formula for the a n and here n is a 0 we already got, but, if you see that a 0 now we have this, we can also get from here because a 0 is going to be one over pi minus pi to pi f x d x. So, this is basically true for 0 one 2 and so, on and similarly, we can get this b n. So, we multiply instead of this cos n x by sin n x and in that case this term will get n here we will get pi because sin k x sin n x. So, in this case b n we will get one over pi minus pi to pi f x and sin n x d x. So, where the n 1 2 and so on, so, these coefficients a n and b n they are called. So, this a n and b n they are caller Fourier coefficients Fourier coefficients.

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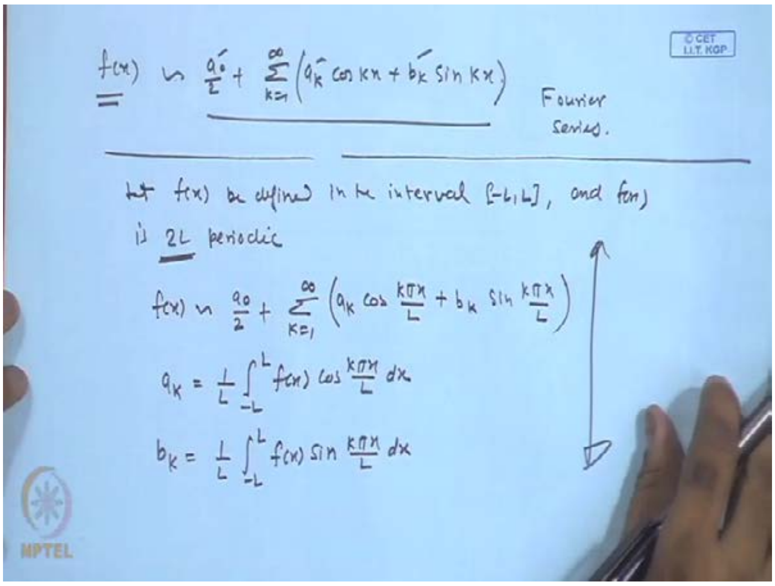

And the trigonometric series; this a naught by 2 and this k 1 to infinity a k cos k x plus b k sin k x is called the Fourier series and we normally denote by this (( )) is. So, f x this is the Fourier series for for f x and we we are not writing here the equality because first of all this is clear from the construction. We have evaluated these coefficients a k and b k by putting the interval equals and the does not mean that the effects is equal to the sum of the series. So, that part we will be discussing more in detail that when we can have the equality; that means, the series converges to the function value f x. So, just remark here. So, if we just take a look at the coefficients is Fourier coefficients in b n this is minus pi to pi f x cos n x t x and if we change the value of the function at finite number of points for this f x at finite number of points then, this integrals defining these coefficients are unchanged.

So, the functions which differ at finitely many points have exactly the same Fourier series. So, this was the Fourier series for the function of period 2 pi and we have define or we have assumed for simplicity that have a function which define from minus pi to pi. And now we are go for a general case we will skip definitely the proof because all the steps are very similar what we have done. But, now we will consider when a function is of general period and we let say 2 L again for simplicity and defining the interval minus L to l. So, let f x be defined in the interval minus L to L and f x is 2 L periodic is 2 L periodic then the Fourier series of this function will be given by a naught by 2 k one to infinity and we have to take now the family of the trigonometric function which has the common period 2 l. So, that will be cos k pi x over L plus b k and sin k pi x over L and where this Fourier coefficients will be given by one over L instead of one over pi we have one over L and minus L to L, f x and these functions k pi x over L d x or we can also integrate from a to a plus 2 L that integral value will be same. So, for the b k we have one over L minus L to L to L over from 0 to 2 two L a to a plus 2 L; this f x sin k pi x over L d x. So, this is the general Fourier series where a function is of period 2 L.

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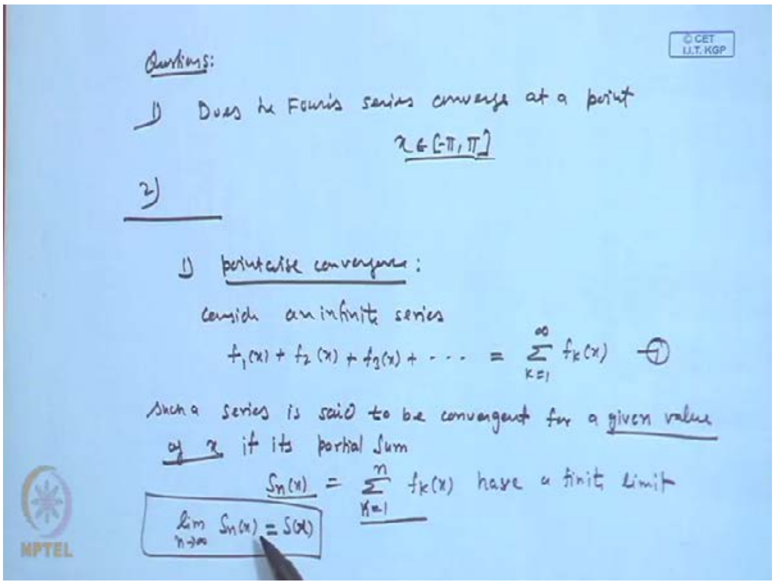

Now, we have the following questions to be answered. So, does the Fourier series converge at a point x minus pi to pi or if this series converges at x is the sum function equal to f x. So, the answer to these questions are; in negative and there are functions define an minus pi to pi whose Fourier series divergence everywhere and there are functions. In fact, continues function Fourier series diverges at countable number of points. But, we are not going into the detail of all these convergence and and the conditions for the different kind of convergence. But, because this is your other introductory lectures and we should go to these Fourier transform later on. So, what we do, but, let let me just mention that we definitely, I mean have some additional conditions on this f2 sure that this series convergence and that we will see in eminent, but, before let me just mention that they are several notion of conversions and the one is the point wise conversions which will be talking about, point wise convergence. So, in other informal manner if we define here; so, consider an infinite infinite series an infinite series of 1 x plus f 2 x plus f 3 x and. So, on or we can write in summation form k one to infinity and f k x and such a series is said to be convergent for a given value of x if its partial sum as s n x which is n or k is equal to 1 to n and f k x has or sequence of partial sums have of finite limit. That is the limit n tending to infinity s n x exists. Let us call it this is equal to s x. So, if this is the case that for a given value of x; this sequence of these partial sums is an x converges to sum s x then, we call that we have here the point wise conversion because depending on a particular value of x we have a stronger version of conversion and that is called that is called uniform convergence uniform convergence

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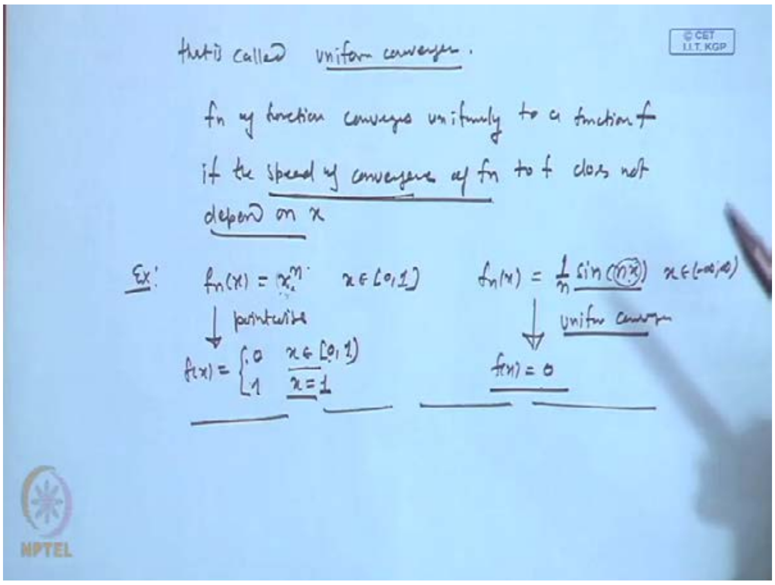

And in the uniform convergence is sequence this f n of functions converges uniformly. So, this is the stronger an version of conversion than the pointer wise conversions uniformly to a to a function, limiting function f if the speed of convergence of this f n x to f does not depend on x. So, what does that mean the speed of convergence that little bit with the help of examples we will see. So, if we just take an example f n x is equal to x power n is very standard example and x is between 0 and one. So, this function will converse point wise to a function f x which is 0 and 1 0 and x is 0 and 1 the open interval and when x is equal to one. So, let us see what is happening here.

So, if we take any x between 0 and one this x power n has n approaches to infinity; will always go to 0 and if we take x is equal to one. So, then in fact it is one. So, we have this one which constant sequence in that case. So, now, each we were talking about the speed of convergence that for the uniform convergence this speed of convergence of this f n x does not depend on x. So, what is happening here that if x is close to 1 for example, if x is close to 1 then this convergence is very slow because we have very close to one power n and then this sequence will go slowly to the 0 and if we have x close to one this will go or this will converge very fast to this 0.

So, what we see here that this convergence this speed of convergence where is with this x. But, for example, if we take a function sequence of function one over n and sin n x and for basically any x, we can talk about and this will always go to the function f x is equal to 0 and here we have uniform convergence and of course, the point wise convergence because this is stronger version of of the convergence. So, in this case whatever x we take, one over n will go to 0 and it it basically does not depend on x, but, here it was different if x close to, x is close to one then we have a very slow convergence to 0, but, in this case we have one over n here and whatever x we take and we can never the case of 0 then is a constant sequence. So, there this of course, zero, but, in general and this is not depending on on x yeah. So, we are not going in to very much, in to the detail of of these two notions of convergence, but, I hope that I have given at least some idea of the uniform convergence and also of for the point wise convergence.

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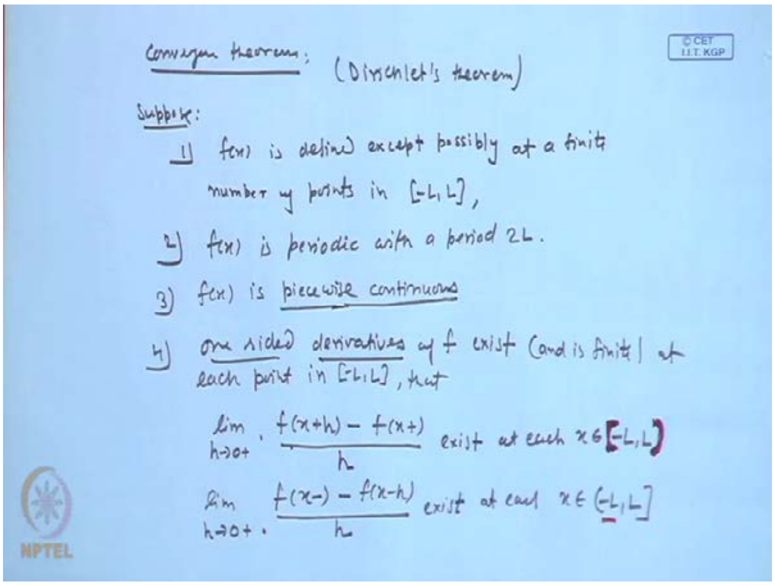

So, now, can talk about the convergence theorem convergence theorem or this is also called Dirichlet's theorem and these are the sufficient conditions for the convergence. So, what we have? Suppose that the f x is defined except possibly at finite number of points in elastic general interval. So, minus L to L or we can take an open interval minus L to L and the f x is periodic with a period 2 L and f x is piece wise continuous. So, we have already defined this piece wise continuity. So, the function is continuous other than some at finitely many points and at those points write in the left limit exist. And the one more condition here we need that one sided derivative one sided derivatives of f exist and is finite is finite at each point in this minus L to L.

So, what does that mean one sided derivative? So, we have the limit h to 0 from the positive side. So, f x plus h this is right derivative minus f x the right limit of this function that x the function name not be defined at x. So, we have divide by h if this exist and is finite of course, at each x minus L to L. L here the closed interval and the left limit h to 0 minus over 0 plus sorry. So, f x left limit minus f x minus h over h this exist at each at each x minus L to L here we have. So, here we are talking about the right limit. So, when we are here that f x plus. So, basically if this here x should be the open interval the right side because here we take we exclude the case because at this point L we do not require this right derivative to exist and here at this left point minus L also we do not require that left derivative to exist. So, at all other points the both the right and the left limit should exist at the extreme left end. Only the right limit the right derivative and at extreme left only, the left limit or left derivative should exist. So, what we have these conditions that if f x is defined except possibly a definite number of points and the conditions that f x is periodic, piece wise continuous and these one sided derivatives of f exists at each point in minus L to L. If these these conditions are satisfied and these are the sufficient conditions for the convergence.

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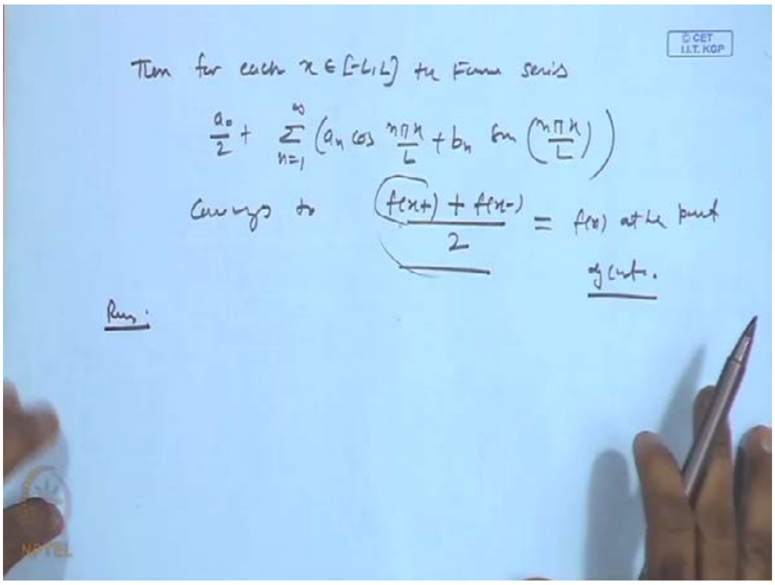

Then for each x minus L to L, the Fourier series a naught by 2 plus n 1 to infinity a n cos n pi x over L and this p n sin n pi x over L converges to f x plus minus f x minus divide by two. So, to the average value. And note that so, in that case this is equal to this basically and at the point of continuity because this right limit they are equal to f x of simplest this average value. So, in at the point of continuity this f x plus is equal to f x and this f x minus is also equal to f x. So, in that case we have this equal to f x at the point of continuity. Otherwise, we need to take the average value to make this equal and the last remark what we have that in addition in addition to all these assumptions what we have if a function is is continuous then we have even stronger version of this convergence and that is the uniform convergence. So, but we are not going in to the detail of all these convergence.

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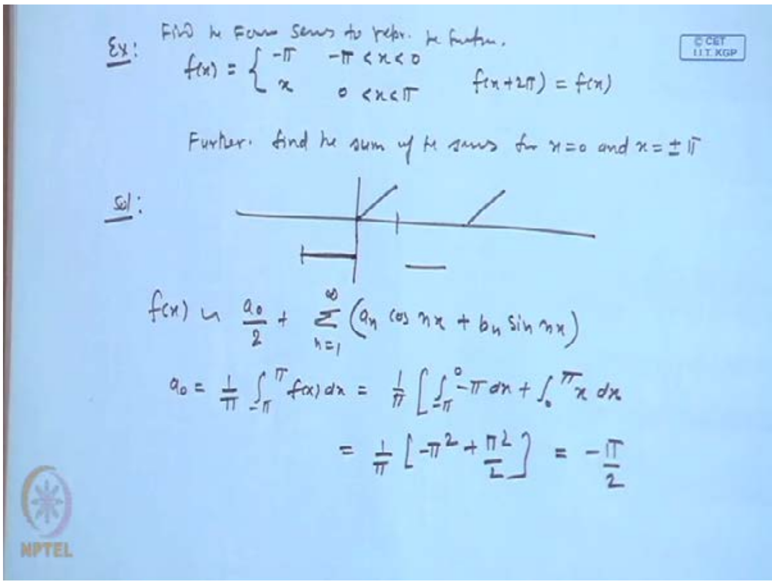

So, let us just quickly go through the one example. So, find the Fourier series of the function minus pi x and we have minus pi when x is minus pi to 0 and x is 0 to 2 pi and the function is periodic. So, f x plus 2 pi is equal to f x and further. So, we have find the Fourier series Fourier series to represent to represent the function which is defined by this and the further find the sum of the series for x is equal to 0 and x is equal to plus minus pi. So, this function if we just take a look; its x when 0 to pi on its minus pi 0 to minus pi and then its its periodic. So, then we have again we will on to go like this. So, the function is periodic.

So, this and all other conditions have of course, satisfy that the piecewise continuous and the derivative yet all these points left and right derivative exist. So, it would not have to check that. So, this is the Fourier series cos n x plus b n sin n x and we calculate this a 0 first that is one over pi and we have minus pi to pi f x d x. So, f x d x, so, you can break this from minus pi to 0 we have minus pi d x plus 0 to pi; we have x d x and this is one over pi and minus pi square because minus pi and then we have x here. So, we get again minus pi. So, this is minus pi square and plus will get pi square by 2 and this is nothing else but pi by minus pi by 2.

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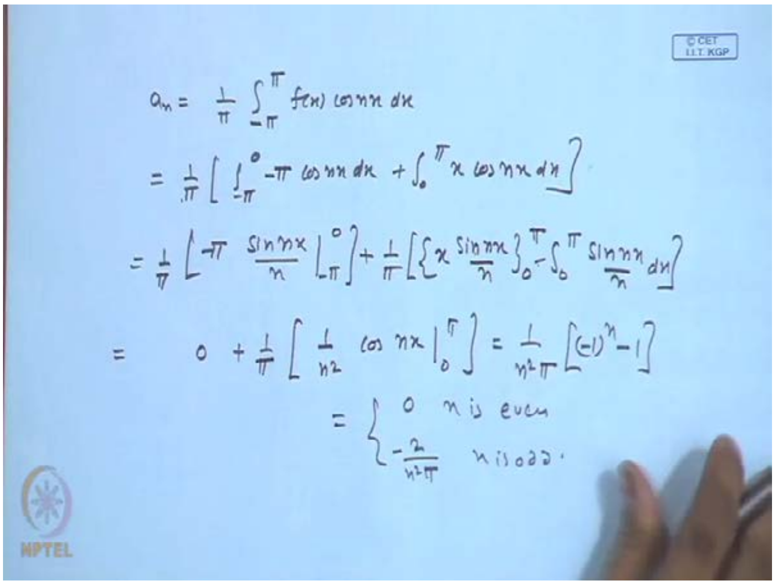

Now you calculate the a n for this function. a n 1 over pi minus pi to pi and f x cos n x d x we have one over pi minus pi to 0 minus pi cos n x d x plus 0 to pi in the function is f x cos n x d x. So, 1 over pi here we take minus pi and this is sin n x over n minus pi to 0 plus one over pi. This one over pi and we have the we have to integrate by parts. So, we have x and sin n x over n 0 to pi minus 0 to pi and sin n x over n d x. So, here when we could 0 sin 0 is 0 and then sin this minus pi is also 0. So, this 0 and then we have one over pi here again, we have this from 0 and we will get this one over n square and cos n x with this minus this minus will be cos and we have 0 to pi. So, we have n square pi and then the cos pi that is minus 1 n and minus 1. So, this is 0 when n is even and minus 2 over n square pi if n is odd.

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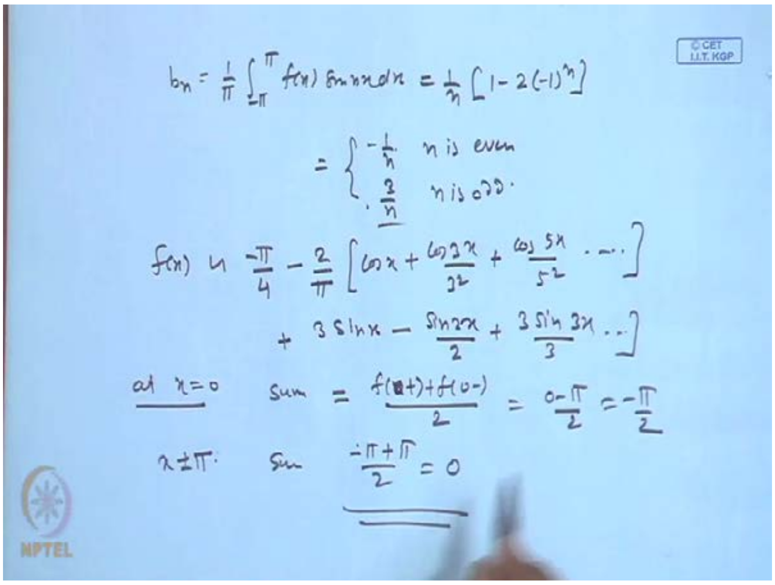

So, similarly, we can get this b n which is 1 over pi and minus pi to pi f x sin n x d x. This is one over n and I skip all these calculations. So, we will get 2 n minus 1 over n. So, in this case we have minus 1 over n and n is even and 3 over n if n is odd. So, our Fourier series representation for this function a 0 by 2 was minus pi by 2. So, we have minus pi by 4 minus and that this a n and b n. So, we will get finally, this cos x plus cos 3 x 3 square plus cos 5 x over 5 square and. So, on plus for the sin you will get three over and sin x. So, 3 over 3 and sin x we will get an the next term will be sin 2 x over 2 then 3 from here 3 sin 3 x over 3 and so on. Now, at x is equal to 0 what will be the sum of the series. So, the sum of the series at x is equal to 0. So, if we just see here at x is equal to 0 we have to take the average of these two values and that is 0 and minus pi and divide by 2. So we have to take the x and at f zero plus and f zero minus, divide by 2 so we have 0 minus pi by 2. So we have minus pi by 2. And same at plus minus pi, where that plus minus pi, we have again the pi and we have the minus pi. So at these two points we have the plus pi and minus pi by 2. So, it will be zero. Okay so in this lecture we have seen the Fourier series and the conditions for the convergence and that was the piece wise continuity and the different feasibility of the function, left and right derivative and more we will continue in the next lecture. Till then, bye Thank you. .