Advanced Engineering Mathematics

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Lecture No. # 26
Application of Laplace Transform to PDSEs

Welcome back to series of lecture on Laplace transform calculus and in the last lecture we have seen how to apply this idea of Laplace transform to solve different kind of initial value problems, boundary value problems, integral equations. So, we mainly discussed in the last lecture solution of ordinary differential equation where we had only one dependent independent variables. Today, we will extend idea solving differential equation using Laplace transform to partial differential equation in general partial differential equations are difficult to solve, but, using Laplace transform some of the partial differential equation can easily be solved.

So, today we will continue with this discussion one application of Laplace transform to partial differential equations. So, before we go for solving partial differential equations, I will briefly introduce the partial differential equations.

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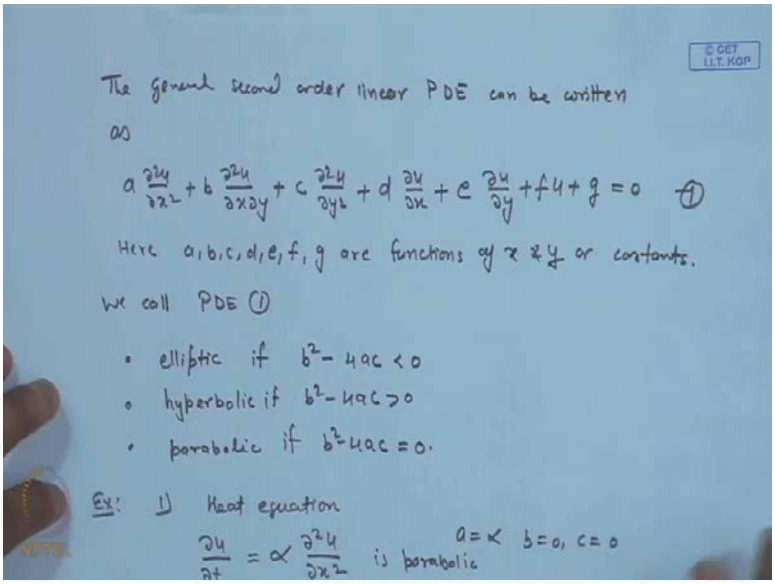

So, the general second order linear partial differential equation PDE, linear partial differential equation can be written as a del d 2U over del x square plus b del 2 U over del x del y plus c del 2 U over del y square plus d del U over del x plus e del U over del y plus f U plus g is equal to 0. So, here a, b, c, d, e, f and g are functions of x and y or they are constants or they are simply constants. So, we call this, there are classifications for the partial differential equation and we call this PDE one elliptic if b square minus 4 a c is negative and we call the PDE hyperbolic if b square minus 4 a c is greater than 0 and parabolic if b square minus 4 a c is equal to 0. So, this is standard examples for this. So, one is the heat equation the heat equation and this equation model of phenomenon of conduction of heat in a solid. So, this equation is del U over del t and some constant which is the material constant del 2U over del x square and if see here that a is a is alpha and we have b 0 and c 0.

So, b square minus 4 a c will be. So, b square minus 4ac is 0. So, this equation is parabolic is parabolic is parabolic.

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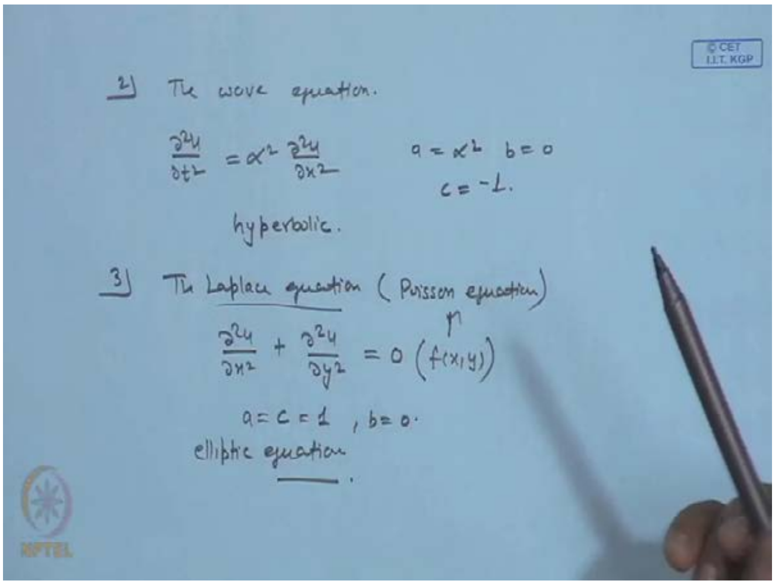

So, this was example of parabolic equation. We have also the wave equation wave equation and in this case the equation is del 2 U over del t square is equal to alpha square del 2 U over del x square and in this case this a is alpha square b is 0 and then c is minus one. So, b square minus 4 a c is positive. So, we have this hyperbolic equation. This is also called this hyperbolic equation. Number three, third example we have the Laplace equation and also different variation of the Poisson equation poisson equation and this equation are del 2 U over del x square plus del 2 U over del y square is 0 then this is call the Laplace equation and if this is a function of x y then this is called Poisson equation. And in this case we have the a is equal to c and this is one b is 0. So, the equation is elliptic equation because b square minus 4 a c is negative. So, this equation is elliptic equation. So, in, we have this according to different classification we have different equations and in order to obtain unique solution of the partial differential equation we have to supply some initial and boundary conditions to get with this partial differential equation. So, that part you consider while discussing these examples.

So, now, we come to the working steps and working steps in this case also the same working we have seen in working by solving differential and ordinary differential equation. So, we will apply the Laplace transform to both side of equation and then we will get a rather simple ordinary differential equation that can be solved and again we take the inverse Laplace transform to get the solution of the given partial differential equation.

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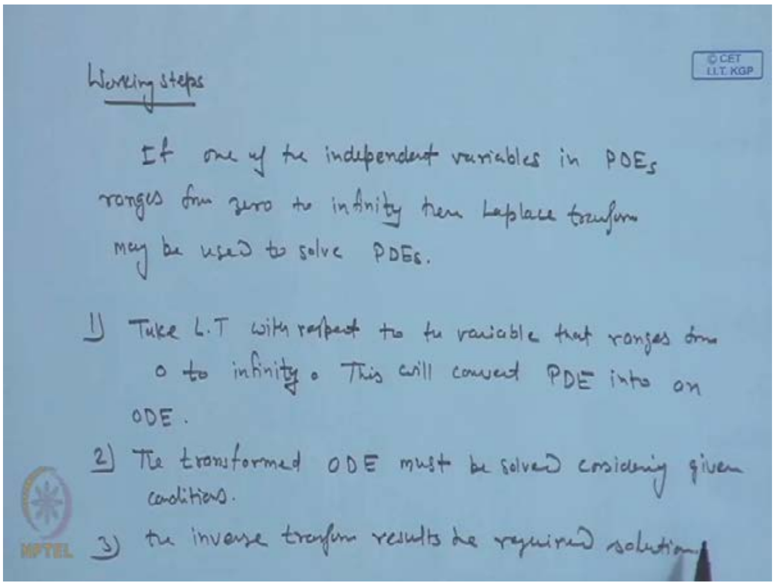

So, the working steps; so, if one of the independent variables in partial differential equation ranges from 0 to infinity then Laplace transform may be used to solve partial differential equations. So, mainly solved heat equation heat and the for example, heat equation we have this heat variables which ranges from 0 to infinity and also in the case of this wave equation and this wave equation models of phenomenon of the waves, c waves or waves in a string. So, in this case also t starts from 0 and we can go up to up to infinity. So, here also we can apply Laplace transform. So, we mainly apply this wave equation and heat equation. So, the steps are again the same. So, we takes the Laplace transform with respect to the variable that ranges from 0 to infinity and this will convert convert PDE into an ordinary differential equation and the second step; the transformed ODE must be solved considering given conditions. And the last step; we take the inverse transform and that results the required solution the required solution. So, before we move to demonstrate the application of Laplace transform to a variety of partial differential equation we need to address some points.

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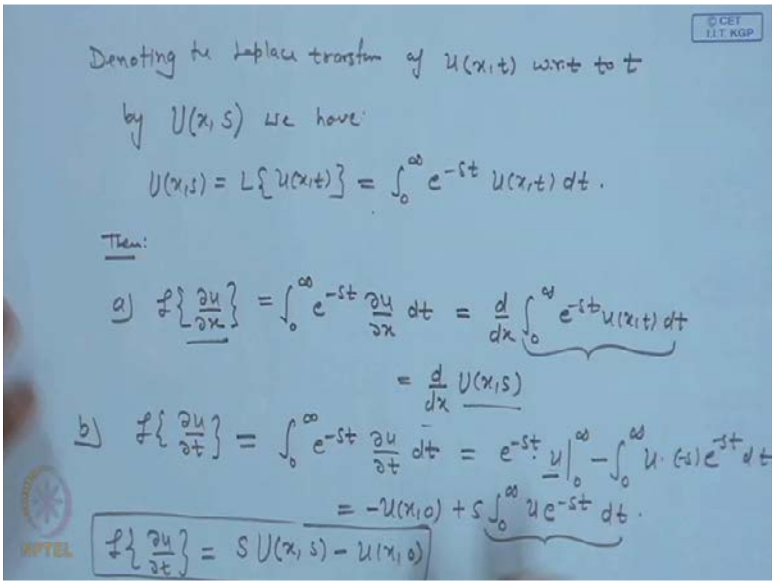

So, these points are basically so we denote, denoting the Laplace transform Laplace transform of the unknown variable U x t. So, we have to two dependent independent variables x and t. So, we have denote Laplace transform and with respect to t by U x underage and with respect to t. So, here this transformed variable x will come and we have by definition that is U x s is Laplace transform of U x t and this is 0 to infinity e minus s t and U x t d t. Using this definition, we have the following results that will be useful solving U n solving partial differential equation. So, we have the Laplace of del U over del x 0 to infinity e minus s t del U over del x d t and we assume that this differentiation and integral sign is valid here. So, we have d over this is equal to d over d x 0 to infinity e minus s t and U x t d t. So, this is nothing else d over d x and U of x s this is the Laplace transform of U x s t as for the definition. So, what we see here the Laplace transform of the derivatives if the derivative of the Laplace transform.

We have next result that Laplace transform of del U over del t and in this case we have here we have here time variable and this t variable and we are taking Laplace transform with respect to t variable. So, we have then 0 to infinity e power minus s t and del U over del t. This by parts we can integrate. So, e minus s t U and limits then we have minus 0 to infinity minus 0 to infinity U and minus s e minus s t and then d t. So, if you assume that U is of exponential order then this will go to 0 as t approaches to infinity and then we have minus in this a s t goes to 0 we have U x 0 with minus sign and then minus, minus plus s 0 to infinity and we have U e minus s t and d t and this is Laplace transform of u. So, we have the result that Laplace transform of del U over del t is simply s and Laplace transform of U x t, we denote U x s and minus U x 0.

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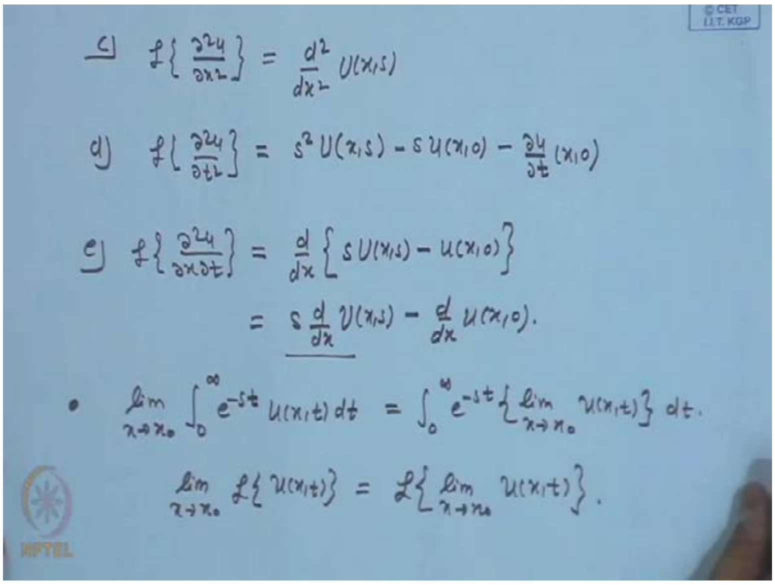

So, its again the derivative theorem what we have in case of this one variable. Now we meet again this for second derivative. So, we have the Laplace transform of del 2 U u over del x square and here we do not have proof because we need already for single variable. So, the Laplace of this double derivative will be the double derivative and Laplace of u. So, that is U x s. The next we have the Laplace del 2 U over del t square, now here we have t. So, we need to go for the derivative theorem and similar to one dimensional case or ordinary differential case we have here x s minus s and U x 0 and we have the derivative of U with respect to t and at x 0. This is for second derivative and Laplace for this del 2 U over del x del t for next derivative. So, for the x will not touch, So, will keep it is d over d x and then Laplace transform of del U over del t and that is as for the definition we have U x s and minus U x 0. So, this is s is constant with respect to s d over d x and U x s minus d over d x U x 0. So, this is for next derivative. So, in addition that what assumption what we have in this case is that the derivative can take out of integral, we need one more assumption. So, that assumption is that limit x tending to x 0 and Laplace transform of this U x t d t. So, assumption is that the limit we can take inside the integral as limit x tense to x 0 U x t and d t. That means, the limit of x tending to x 0 of the Laplace transform of U x t will be equal to the Laplace transform of the limit. So, limit x tending to x 0 U x t.

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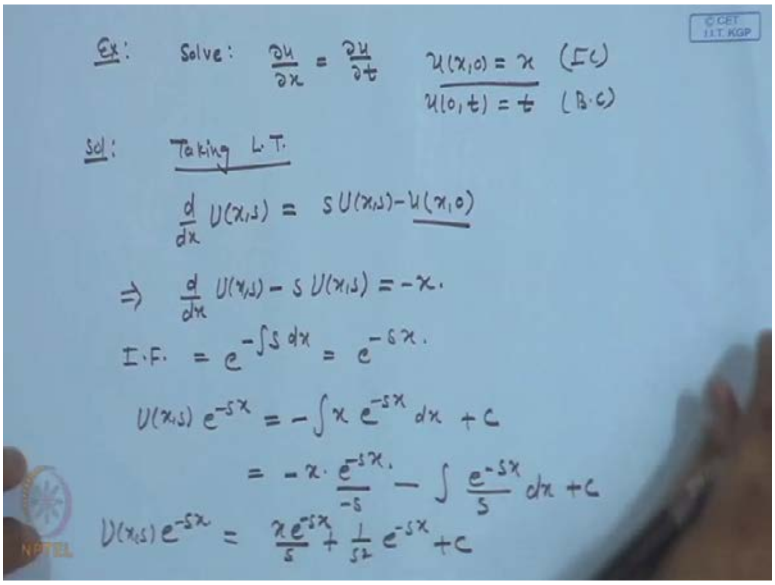

So, with this information we can now continue for solving different differential equations and we take the very simple example first. So you solve solve del U over del x del U over del t and we have U x 0 is x and U 0 t is t. So, this is the initial condition at t equal to 0. So, and this is the boundary condition. So, at x equal to 0 is given. So, we take the Laplace transform, taking Laplace transform both side get here d over d x and Laplace transform of U we can denote U x s and we here the derivative theorem s U x s minus U x and 0. So, this U x 0 is given. So, we have d over d x of U x s minus s U x s and is equal to minus x y a 1 more point here one should mentioned that in this case we have both variables here from 0 to infinity. So, t starts from 0 and we do not have upper bound for t as well as for x we, it starts from 0 we do not have the upper bound. So, we can take the basically Laplace transform with respect to x s well. So, this is minus x and then we can solve this linear differential equation and integrating factor is e power minus this s and d x is e power minus x and therefore the solution will be U x s and e minus s x then the right hand side here we have x integrating factor and then constant . So, this we can take we can again apply the integration by parts technique. So, we have minus here x and e power minus s x over minus s and minus. So, differentiation of this x is one and e power minus s x over s. So, here we have then the plus. So, we can anyway one minus was here. So, we get the minus and that minus will be plus. So, we have minus here and then s and d x and a constant. So, what we have x e minus s x over s and once again we differentiate this e power minus s x minus s. So, we get plus one over s square e minus s x and plus c.

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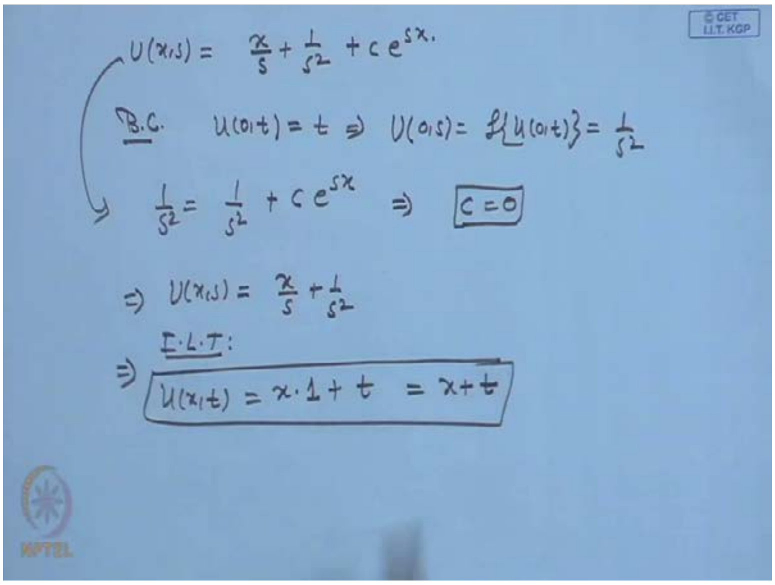

So, this is our U minus x s e minus s x. Now, we get U x s. U x s will be x over s plus 1 over s square and plus c e s x. The boundary conditions we have to use now. So U at t x equal to 0 t was t and if we take Laplace transform both sides. So, what we get? U 0 s that the Laplace Laplace transforms this U 0 t and this t so, we have one over s square. So, if go back to this equation put x equal to 0. So, we have one over s, x equal to 0 we have again one over s square sorry and then we have one over s square and plus c e s x. So, what we see from here that c must be 0. We get this constant c should be 0 and then our solution is U x s is x over s plus 1 over s square and now we take the inverse Laplace transform to get the solution. So, we have e U x t and this x Laplace of Laplace inverse of one over s that is one and Laplace inverse of one over s square that is t. So, solution is x plus t. So, we go for another case and now we complicated partial differential equation.

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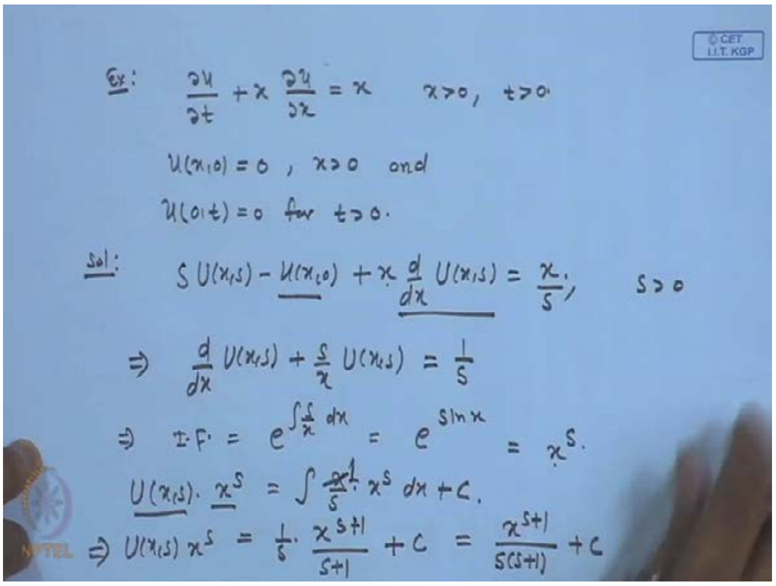

So, this example we have del U over del t plus x del U over del x is equal to x for x positive and also t positive and with initial and boundary conditions. So, we have this conditions x 0 is 0 for x positive and U at 0 t is 0 for t positive. Now, the similar steps so, we take the Laplace transform and we have in this case s U x s minus U x 0 and plus this x d over d x the Laplace of U will be U x s and right hand side x lap place transform of 1 that is 1 over s. So, s positive and this will lead to the differentiation of d over d x of U x s and then we have here plus s over x and U x s U x 0 is 0. So, this will disappear and then we have is equal to dividing by x we have just one over s. This is again linear differential equation and in this case the integrating factor to solve this will be e s over x d x. So, this is e and we have s l n x. So, this will be just x for s in the solution then U x s x s will be integral of this x over s and x power s d x plus c. This is x power s plus 1 which obtain this integration will give x s plus. So, we have here one no sorry sorry. So, we have here U x s multiplied by this integrating factor, the right hand side we have only one over s. So, 1 over s x s d s plus c. So, in this case one over s and integral will be x s plus 1 over s plus 1 and then the constant. So, what we have then here, x power s plus 1 over s and s plus 1 and a constant. So, this is here U x s and x power s. Now, U x s or now let us consider with equation itself and get this constant c.

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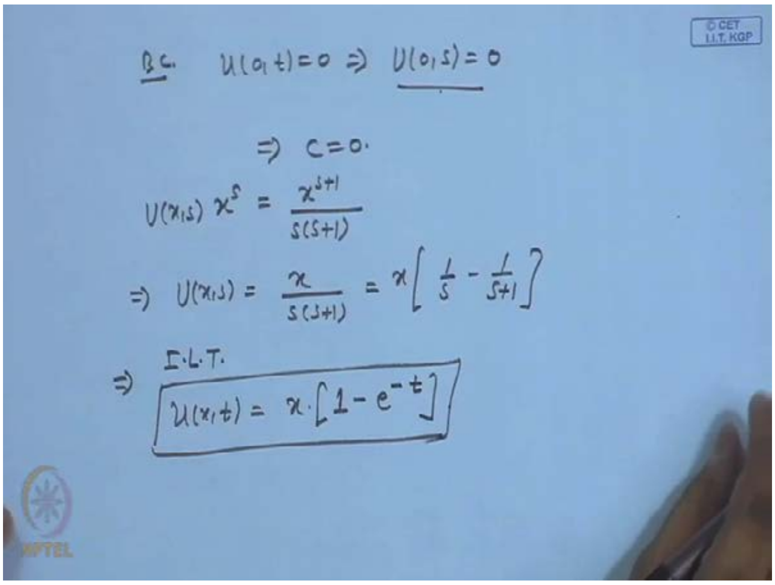

So, boundary condition was and the second condition was U 0 t is 0 and this will k was that U 0 s is 0. So, now, we use this condition here. So, this is 0 and x is 0 this is 0 and this will apply that our c is 0. So, U x s x s is x power s plus 1 over s and s plus one or we have U x s is x over s and s plus 1 and then we do partial fraction. So, we have 1 over s minus 1 over s plus 1. Now, taking the inverse Laplace transform we get, U x t is x and Laplace inverse of one over s that is 1 minus Laplace inverse of 1 over s plus 1. We have e power minus t. So, this is the solution.

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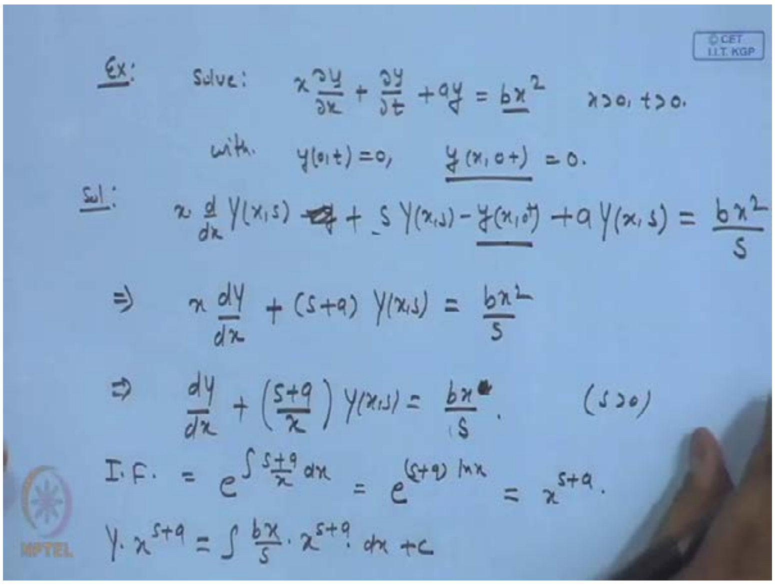

So, we consider one more example of similar kind of first order partial differentiations and then we would move to second order partial differentiation. So, here we solve the x del y over del x plus del y over del t plus a y and b x square, x positive and we have t positive and with given the initial conditions that y 0 t is 0 and we have y x and 0 is 0 take the Laplace transform. So, we have x and d over d x Laplace transform of y we take y x s and minus y this is for this and del y over del t. So, we have plus and then s y x s minus y x 0 or 0 plus because initial condition as given at 0 plus. So, we have in the derivative theorem 0 plus and then plus a y x s. Right hand side we have b x square as constant be true it and then Laplace of one will be one over s. So, we can simplify now. So, this term will vanish because this is given as 0.

So, we have simply x d y over d x plus s plus a and y x s and right hand side b x square over s or we have d y over d x plus s plus a over x y x s and then b x square over s s positive. So, we do not have problem in the division. In any case if some other stop appear here for some example s minus 2 we can take s sufficient be large there is no point of having any problem while dividing by this s any other term which contains s. So, we have integrating factor in this case again then e power s plus a over x d x. So, this will be e s plus a and l n x. So, this will give x s plus a and then the solution y x s plus a integral the right hand side we have b x square. So, we have d y by x. So, we have here b x over s. So, we have b x over s and x s plus a and we have d x and we have a constant. So, here we have x power s plus a plus one. So, we integrate this.

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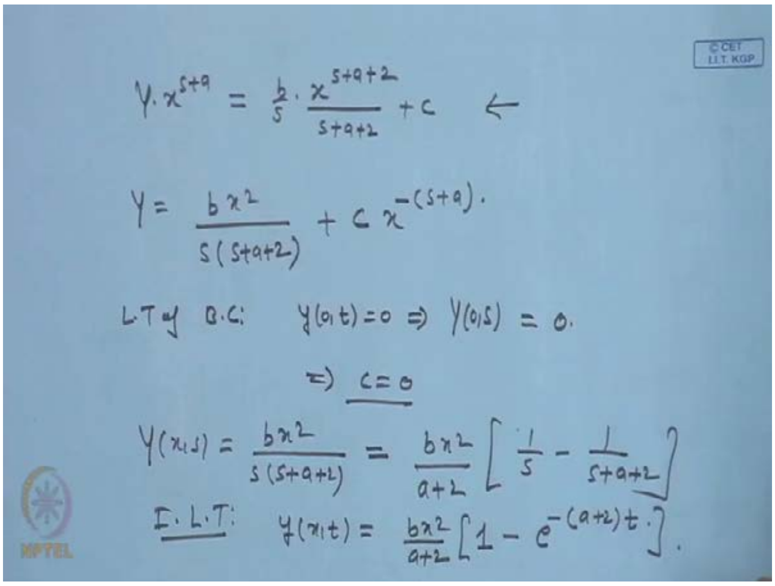

So, we get y x s plus a is b over s and integral will be x plus a plus 1 plus 1 so 2 and then we have s plus a plus 2 and constant. Now, what we do here y we can take. So, b over b x square over. So, x plus a, here we have x plus a. So, this square term will leave. So, b over s and s plus a plus 2 and then we have c term and x power minus s plus a. Now, we take the Laplace transform of the boundary condition. Boundary condition was y 0 t is 0. So, here we get y 0 s is again 0 and then we see that x equal to 0 this is 0 and in fact, we can just easily see from the equation x equal to 0 this term is 0, this is 0 c will be 0. This implies c equal to 0. So, y x s we have b x square over s and s plus a plus 2 and this we need again this partial fraction. So, we have b x square and for this two terms we have one over s we have one over s plus a plus 2 plus will get s plus a plus 2 minus s. So, we get a plus 2 extra there. So, we have a plus 2 we have to divide. These are the partial fraction and now if we take the inverse Laplace transform. So, we will get y x t y x t will be b x square over a plus 2 and then Laplace inverse of one over s here. We have 1 minus the Laplace inverse of this function this is e power minus a plus 2 t. Now, we move to the order partial differential equation second order and we take the following heat equation.

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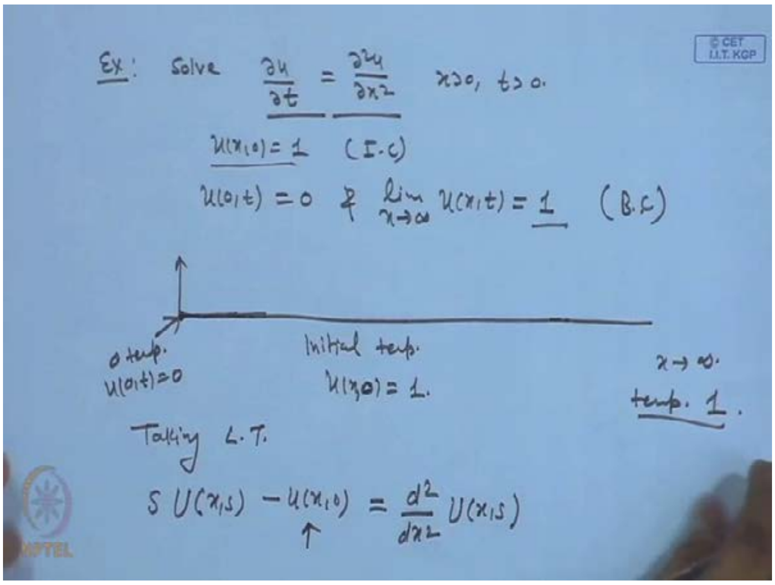

So, here we have del U over del t is equal to del U over del x square, x positive t positive and conditions are given; x 0 is one initial condition at time t equal to 0 and boundary condition at x equal to 0 is 0 and limit as x approaches to infinity U x t and also given. So, these are the boundary conditions and now these models as I discussed already here phenomenon of heat conduction. So if you consider this is infinitely long metal bar. So, in that case what we see here then initial if we I will see this is U if we think this U as temperature was initial temperature that is U x 0, x 0 was 1 and this end of this part kept at 0 temperature 0 temperature because given U 0 t is 0 and this end is x approaches to infinity, that temperature is again one. As initial temperature because we put this end x is 0. So, here the conduction will take place and as x approaches to infinity to the temperature is one. So, now, we are interested to finite temperature that is the solution now this heat equation. So, we take the inverse Laplace also Laplace transform taking Laplace transform both side what we get? So, we have s and lap place transform of U we denote x s minus U x 0 and right hand side d 2 over d x square Laplace of x s and this is given. So, U x 0 is 1.

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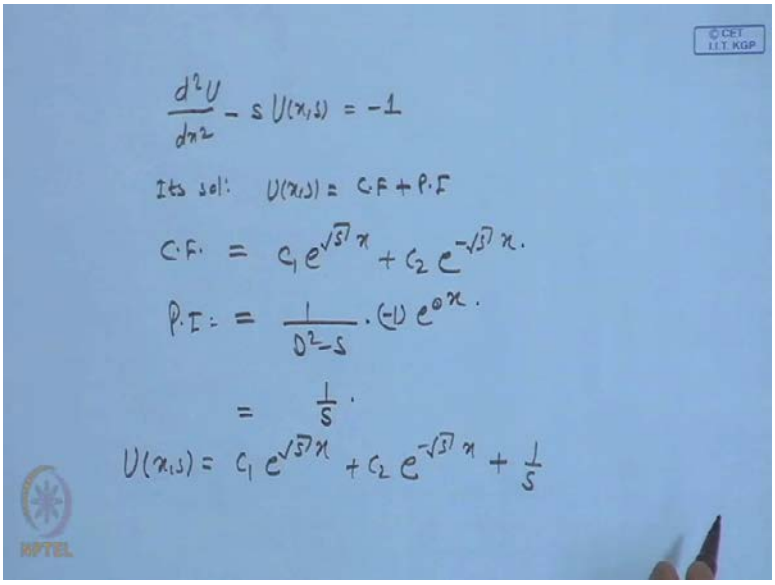

So, what we have differential equation? d 2 over d 2 U over d x square and minus s and U x s, this is minus 1. This we can solve, this is the second order differential equation will constant coefficients. So, its solution will be U x s the complimentary function and plus this particular integral and complimentary function will be c1 because this characteristic equation will be m square minus s. So, U s will be square root s plus minus c 1 e square root s x plus c 2 e minus square root s x. So, this particular integral you will get d Square minus s and minus one here we think e power 0 x and then we get this one over s 0 the 0 we can put it here will get this. So, we have 1 over s 0 we can put it here will get minus s 1 over s. So, the solution U x s is c 1 e square root s x plus c 2 e minus square root s x and plus one over s that complimentary function.

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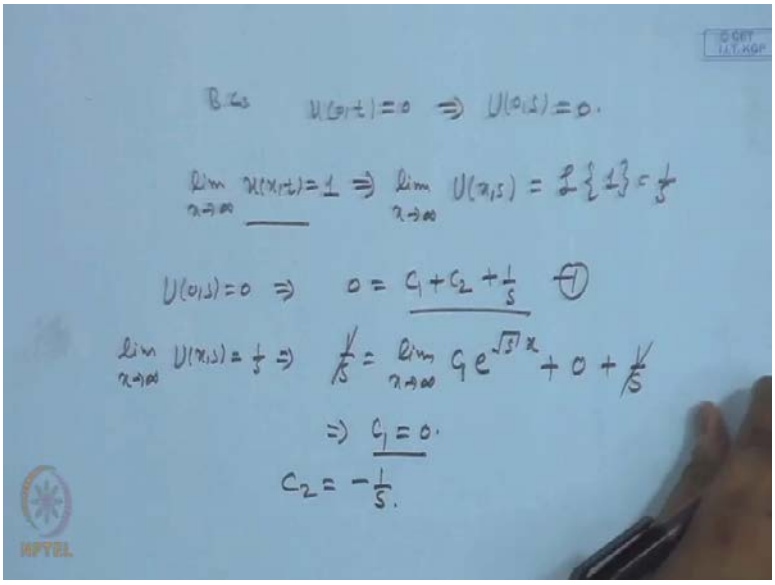

So, now, we consider the boundary condition and boundary condition was boundary condition were U 0 t is 0 and from here we will get U 0 s is 0 and we have the second boundary condition that as x approaches to infinity U x t is 1 and this implies limit x approaches to infinity U x s and this is the Laplace transform of one and this will be one over s. So, we have taking Laplace transform both the side of both the differential equation and now we used to get this constant with first condition what we have which is U 0 s is equal to 0 will give us 0 is equal to c 1 plus c 2 and plus 1 over s. So, this is one equation and then be used second one that is the limit x approaches to infinity U x s is 1 over s. This will give us 1 over s and limit as x approaches to infinity the first term c 1 e square root s x plus the second term was c 2 e power minus square root s x. So, as x approaches to infinity s x positive. So, we will get term 0 and we have one over s. So, this one over s, this one over s and then this limit. So, this is possible when c 1 is 0. So, c one is 0 and c one is 0 we can get c two. So, c 2 is minus 1 over s.

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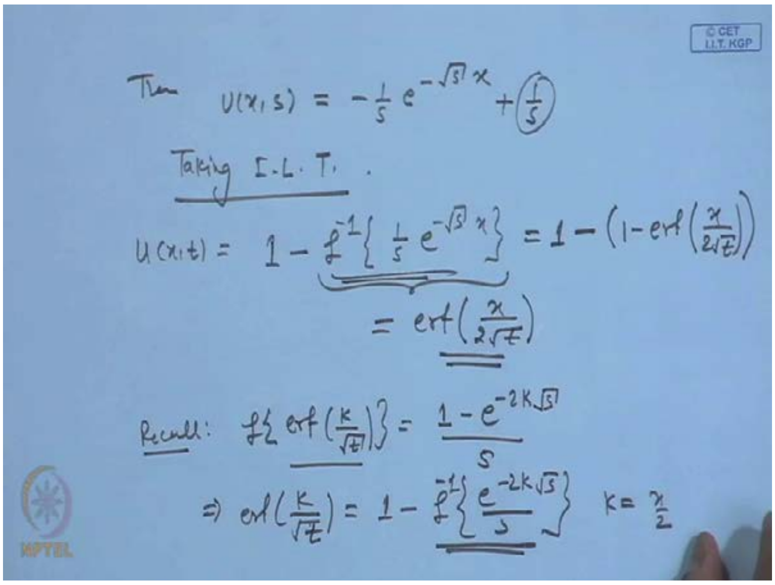

So, the solution is then U x s U x s minus one over s e minus square root s x plus 1 over s. This is the solution. Now, taking Laplace inverse Laplace transform and this will be U x t is. So, 1 over s 1 then we have minus and Laplace inverse of 1 over s e minus square root s x and now this one if you remember. So, we recall that the Laplace of error function of k over square root t was 1 minus e minus 2 k square root s over s. So, the error function we can see here error function k over square root t will be the Laplace inverse of this that is one minus and Laplace inverse of e minus 2 k square root s over s and this is exactly what we need here Laplace inverse one over s and minus square root s. So, this k we can substitute x by 2 to get exactly the result. So, what we have then? One minus and this will get one minus error function. So, one minus error function x by 2 square root t; one will get cancel with this one and we have error function of x over 2 square root t and this is the solution of that heat equation.

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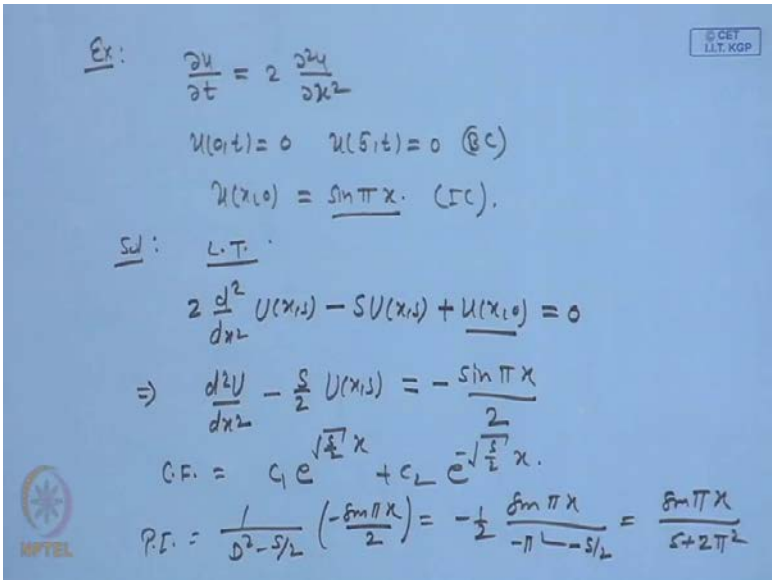

So, next example that is again example of heat equation 2 del 2 U over del x square subject to the conditions; U 0 t is 0 and U five t is 0. These are the boundary conditions given and we have initial condition at t equal to 0 there is sin pi x initial condition. Now, we solve take the Laplace transform both side of equation here we have let first take for right hand side. So, we have 2 del 2 over del x 2 and U x s and then we have minus for this here s U x s and then minus, minus plus U x 0 and this equal to 0. So, this is sin pi x. So, we have d 2 U over d x square minus this s by 2 U x s and this we take to the right hand side here sin pi x over 2. So, again in this case the complimentary function would be constant e power is square root s by 2 x and plus another constant e power minus s by 2 x. The particular integral we get one over this t square minus s by 2 and we have here minus sin pi x by 2. So, we have minus half and this is sin pi x and here we have minus pi square d will be d square will be replace by minus pi square and minus s by 2. So, this is sin pi x over s plus s plus 2 pi square.

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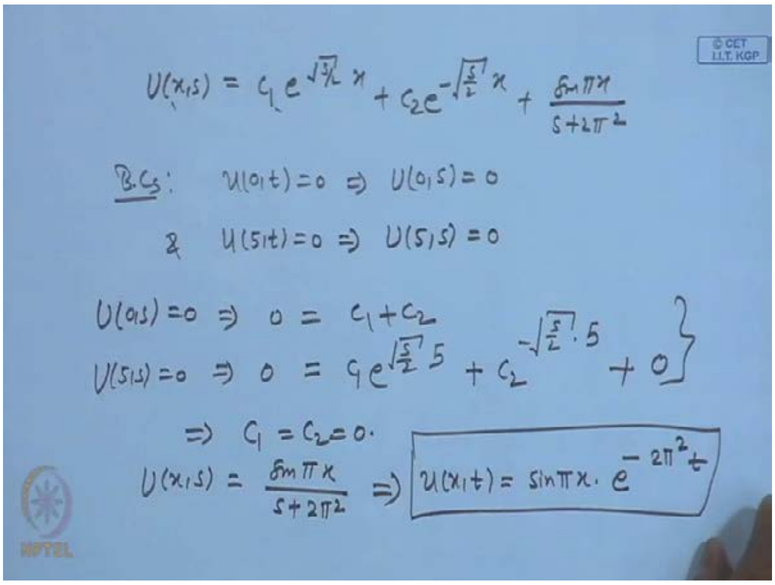

So, now, solution is U x s is c one e square root s by 2, x plus c 2 e minus square root s by 2 x plus sin pi x over s plus 2 pi square. Now the boundary conditions; boundary conditions are U 0 t is 0. So, this will give us U 0 s is 0 and we have U 5 t is 0 and this will give us U 5 s is 0. So, the first boundary condition U 0 s is equal to 0 will give us 0, c one plus c 2 and this is when x this is 0 is 0. So, this second condition 5 s is equal to 0; we will get 0 is c one e s by 2 in to 5 and plus c 2 minus s by 2 in to 5 and plus 5 by sin 5 pi is 0. So, from this 2 equation what we see that c 1 and c 2 both are 0 c 1 is equal to c 2 are 0. We have U x s is sin pi x over s plus 2 pi square and now take the inverse Laplace transform we get U x t sin pi x as it is because the Laplace inverse was with respect to t and we have one over s plus 2 pi square. So, this is nothing, but, e power minus 2 pi square t. So, this is the solution of the series equation.

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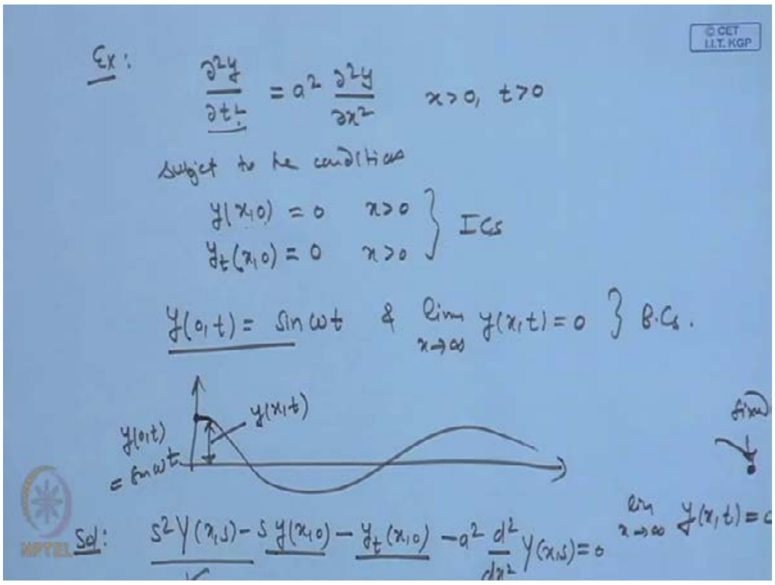

Now we consider the example of one dimensional wave equation that is del 2 y over del t square is equal to a square del 2 y over del x square where x is positive and t is positive. So, in this case we have subject to the conditions that y at t equal to 0 is 0 and also y t. So, we have the double derivative with respect to t. So, we need two initial conditions. So, partial derivative of y with respect to t at x is 0 is also 0. So, these are for x positive these are all initial conditions and then the boundary conditions we have at 0 t. At x equal to 0 we have the function sin omega t and the limit x approaches to infinity y x t is 0. So, these are the boundary conditions. So, if we just look at the physical phenomenon, this first equation models.

So, we have a string here and this end as we see the boundary condition. This is the displacement or solution y x t. So, this end of this string we have moving with this function y 0 t is equal to sin omega t and y x y 0 t and x approaches to infinity. So, this limit s approaches to infinity y x t 0. So, this is fixed, this end is fixed and then we have the initial condition that y x 0 at t equal to 0 it is the string is 0 y x 0 and also the velocity y t x 0 is also 0. So, the string was in rest and then this end we have move to according to sin omega t as s t varies here. So, now, the solution of this wave equation we again take the Laplace transform of that, so we have s square y x s minus s y x 0 minus y t. So, the derivative with respect t x 0, this is the derivative theorem for this and then we have minus a square the second derivative d 2 over d x square and we have y x s is equal to 0.

What we see then y x 0 is 0 and y t is also 0. So, this is two term term when is and then we have x square y x s and this term. So, just see right again.

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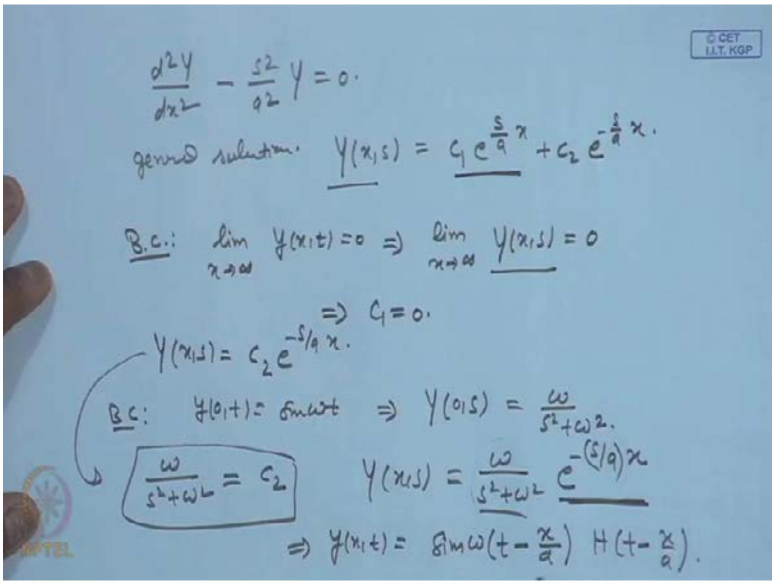

So, we have d 2 y over d x square minus s square over a square and y is equal to 0 and now general solution it is easy. We have only the complementary functions. So, we have y x s y x s c 1 e s over a. So, the roots are s over a plus minus s over a. So, s over a x plus c 2 e minus s over a x again the boundary conditions. So, one boundary condition is x approaches to infinity y x t is 0. So, from here we will get that the limit x approaches to infinity y x s is 0 as x approaches to infinity this y is 0. So, from here we can see that our c 1 is 0. So, this implies that c 1 is 0 because here we have e power s over x. So, this will grow abundantly. So, we have to have c 1 0 to get this 0. So, then our solution is with other constant minus s over a x and then we can apply the second boundary condition which was y 0 t as sin omega t and we take the Laplace transform here. We will get y 0 s omega over s square plus omega square and then we apply to this equation.

So, we will get omega over s square plus omega square is c 2 is c 2 and this x is 0 we have just c 2 is constant. So, we get y x s is omega over s square plus omega square e minus s over a x and now we can apply this second shifting theorem to get this inverse. So, this is simply y x t because this is sin omega t. So, we have sin omega and due to this shift we have t minus x over a x over a and we have heavy side function t minus x over a this is solution. Now come to the last example of this lecture that is again wave equation problem.

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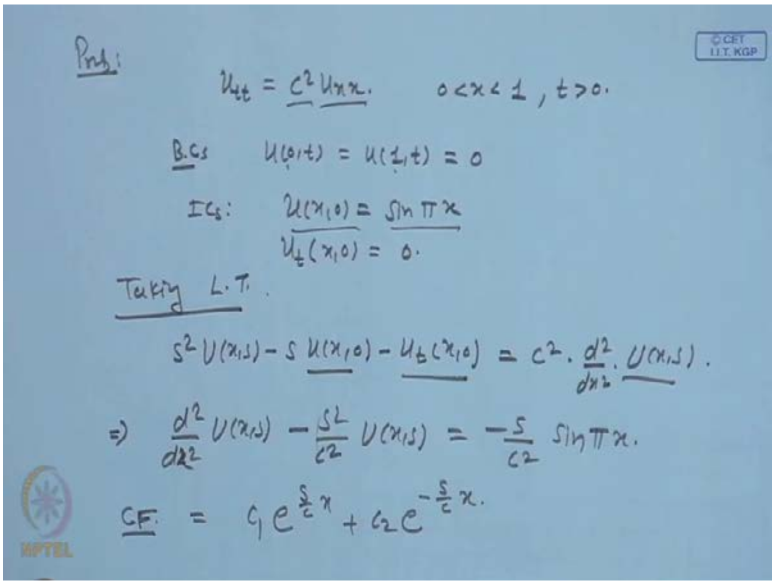

So, we have the equation U t t is c square U x x, x is between 0 one and t positive. The boundary conditions are U 0 t U 1 t. So, if this end to end 0 and the initial conditions are U x 0 sin pi x and we have U t x 0 is 0. So, we take Laplace transform. Taking Laplace transform of the equation we will get s square U x s minus s U x 0 minus U t x 0 is equal to we have c square c square and Laplace of x s d 2 over d x square and this U x s. So, this will give us U x 0 is sin pi x and U t x 0 is 0. So, here we get the 0 term and so, we have d 2 over d x square our this U x s minus this s square over c square U x s and this minus s over c square and we have sin pi x. The initial condition now again this general solution the complimentary function and the particular integral for the complimentary function, we have c one e the roots are plus minus square root s over c s square c square. So, this is s over c x plus and c 2 e minus s over c x and the particular integral in this case.

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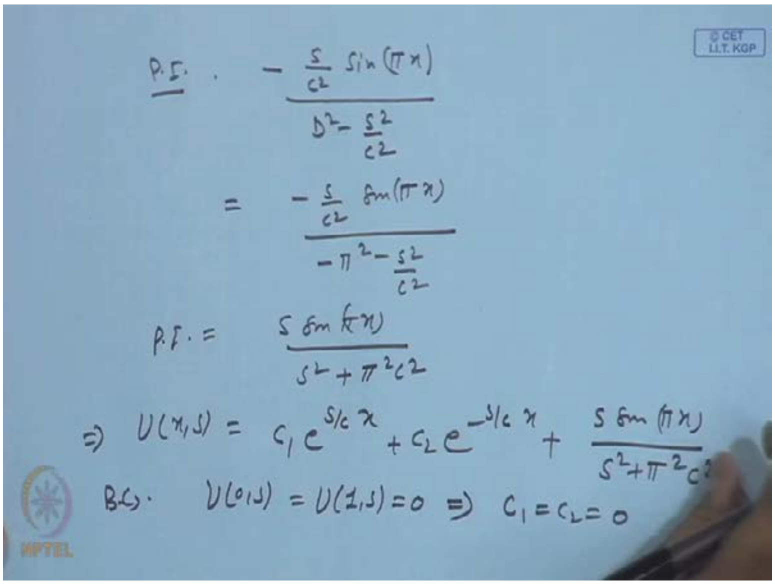

So, we integral will be minus s over c square sin pi x and we had here d square minus s square over c square. So, this will be minus pi square. So, we have s over c square and sin pi x over minus pi square minus s square over c square. This c square gets cancelled. So, here minus sin also s sin, pi x over s square plus pi square c square. This is the particular integral. So, also the equation is U x s c 1 e s over c x plus c 2 e minus s over c x and plus this s sin pi x over s square pi square c square and boundary conditions are U 0 s and also at one both are 0. This boundary conditions we will get this c 1 c 2 both will be 0.

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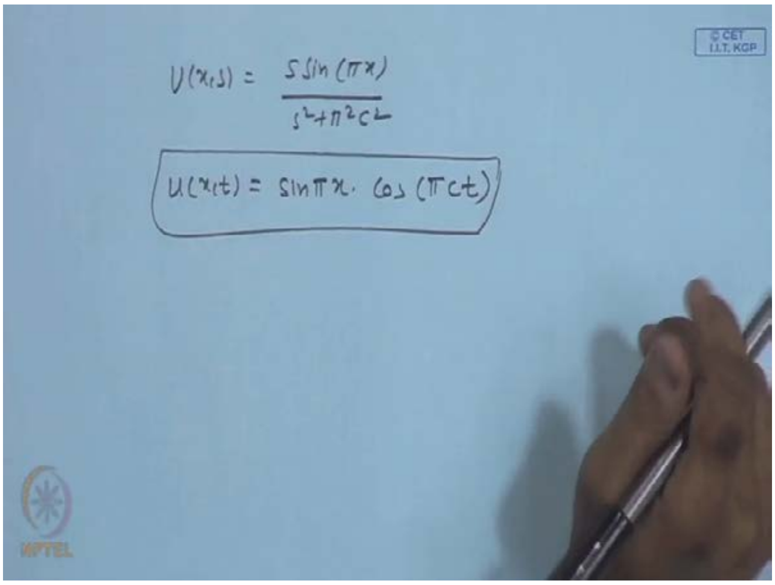

So, our solution will be then U x s is s sin pi x over s square plus pi square c square and now take the Laplace transform again you have U x t this is sin pi x independent of t independent of s and then we have s over s square plus pi square c square this is cos pi c t. So, this is the solution of the given equation. At this point now we wind up this series of lecture on Laplace transform. So, we have introduced the concept of Laplace transform by solving various well-chosen differential equations, different kind of boundary value problems, initial value problems as well as integral equations and the other applications of Laplace transform is beyond this scope of this lecture. This was well rather an introducing in this lecture. So, next lecture we will continue with a Fourier series and then Fourier transform. So, that is all for today. Thank you. Good bye.

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