Advanced Engineering Mathematics

Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture No. # 22

Laplace Transform and its Existence

Welcome to the lecture on Transform Calculus.

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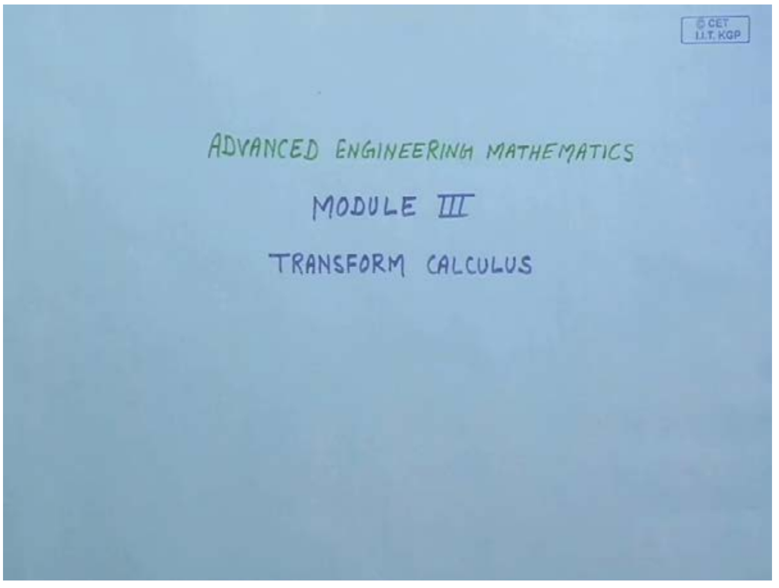

So, today we will discuss, various type of integral transforms (( )) transform and Laplace transform. So, first I will start with the general idea of integral transform.

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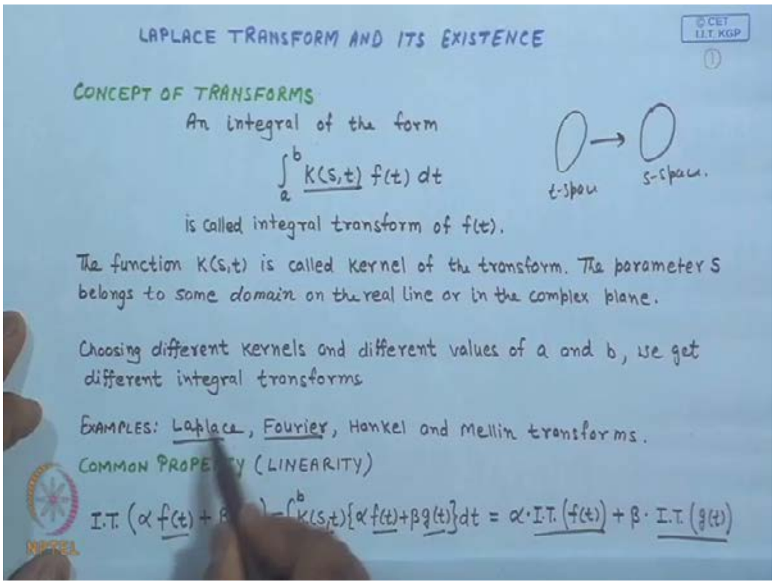

So, an integral of the form a to b k s t f t d t is called an integral transform of f t, so basically what we are doing here, the function f t here transform into another space s, so this is basically a transformation from t space to the space s with this integral. So, here the function k s t is called the kernel of the transform and the parameter s, which is independent of t, belongs to some to main on the real line or in the complex plane; you will come to to the detail of of the kernels and the range of s, while going for the particular transforms.

So, choosing different kernels and different values of a and b, we get different integral transforms, so for example, we have Laplace Fourier, Hankel and Merlin transforms depending on the kernels and the range of the integral. So in this lecture, we will mainly concentrate on the Laplace and Fourier transform. All these integral transforms enjoy common property on the axis linearity, because of this integral.

So, let us have a look on this, so if we have integral transform and we apply to a linear combination of two functions, so we can take here n functions. So, let us takes for simplicity these 2 functions f and t and if we applied the integral transform on the linear combinations of these two functions, that is alpha f t plus beta g t alpha and beta are some constants; so by the definition of this integral transform, we have limits from a to b k s t and the function that is alpha f t plus beta g t d t.

And then we split this integral into two integrals, that means the alpha is a constant we can take out, so alpha integulate to b k s t f t and that will be again the integral transform of f t, on the and the second integral we take this beta out and then we have integral a to b g t d t. So, that is the integral transform of g t, so all these transforms because of this integral they enjoy this linear linearity property, so we will Apply this today itself, while discussing the particular case of Laplace transform.

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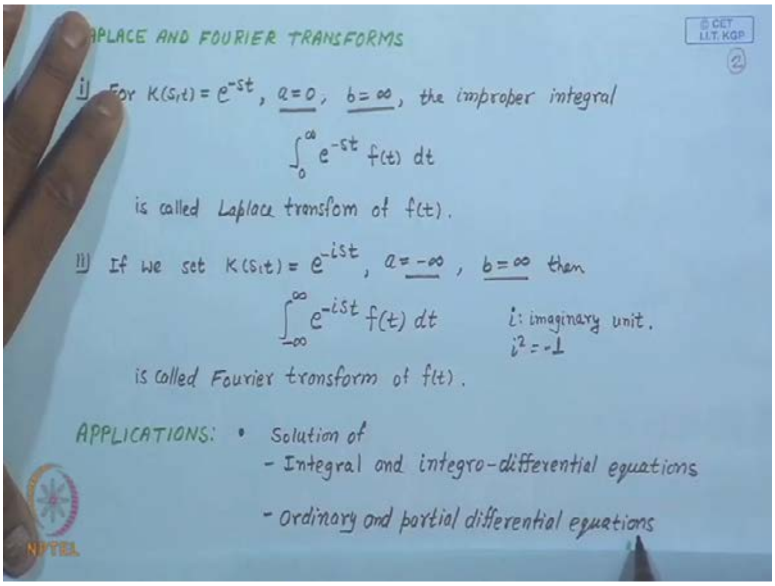

So now, as we have discussed that this Laplace and Fourier transforms will be discussing detail in this lecture, so for example, if we take this kernel k s t is equal to e power minus s t, the lower limit of the integral 0 and the upper limit infinity, then this improper integral that is integral transform 0 to infinity e power minus s t f t d t is called Laplace transform of f t. The second case for the Fourier transform, if we set this kernel k s t is equal to e power minus s t i s t and a the lower limit minus infinity on the upper limits plus infinity, then this integral transform minus infinity to plus infinity e power minus i s t f t d t is called Fourier transform.

So, this i clearly the imaginary unit, that means this i square we have minus 1, so this integral transform is called Fourier transform of f t, these transform have various applications for example, on the one of the most important application that is for solving the integral and integral differential equations also integral integral partial differential equations, the ordinary and partial differential equations they are also used for in some cases for evaluating some complicated integrals, so this basic idea of the transform for solving the integral in general the differential equations I will explain once again.

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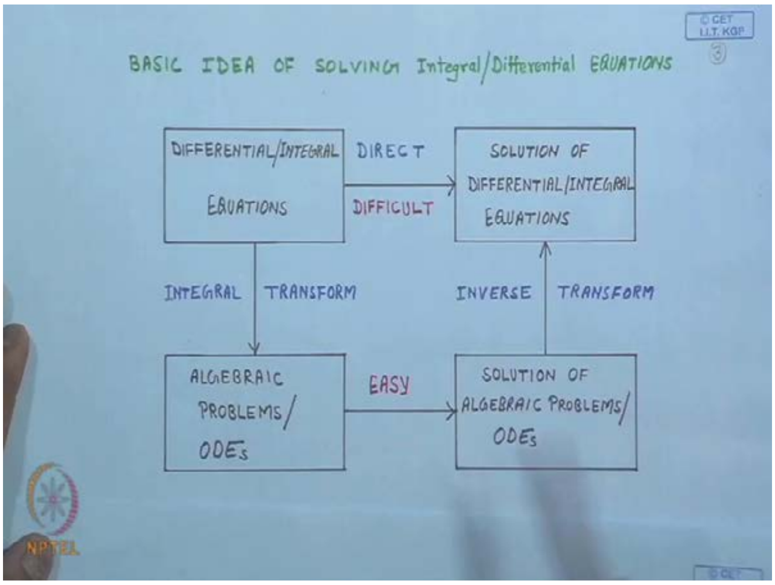

So, here we have for example, the the differential or integral equation and if we directly try to solve these equations it may be difficult to get the solution of these differential or integral equations. So, how these transform helps to get the solution this is as follows, so here if we take the integral transform of of these equations, then we get this algebraic problems or ODE's, so simply these complicated integral equations or partial differential equations, we can convert by the integral transform to algebraic problems or ordinary differential equations and they are easy to solve.

So, we can easily get the solution of this transform system or the solution of this algebraic or ordinary differential equations. So, here we get the solution of the transform system then, we need to go back to the solution of the original problem, then we have to take the inverse transform and in this view we get the solution of the original problem, so that was the the basic idea of the integral transform and now we will go to the particular case of Laplace transform.

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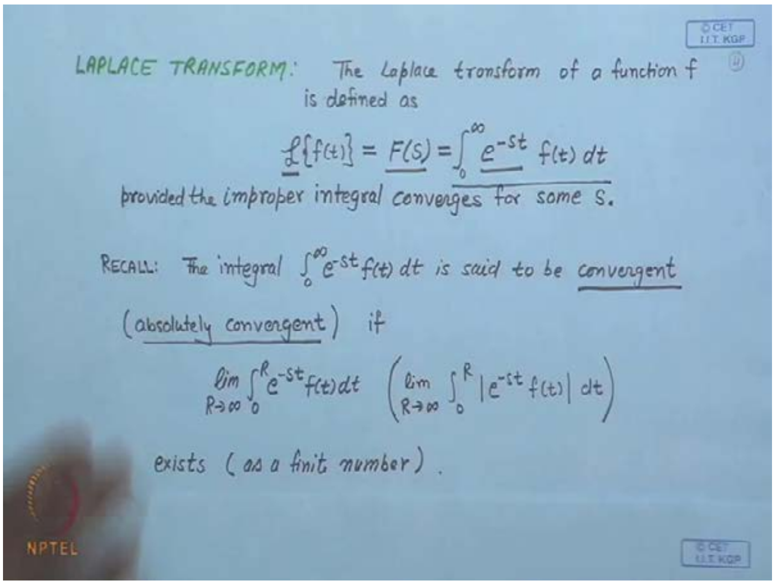

So in this case, we have already discussed this case, so the Laplace transform of a function just to to recall this is defined as the Laplace transform of f t, so use notation L for the Laplace transform or the Laplace operator on f t and we will denote this by this big F as a function of s and this is 0 to infinity the kernel is e power minus s t f t d t provided and that is very important, this will be called Laplace transform of course, if the proper integral converges for some s, what do we mean by convergence of the integral.

So, just to recall the integral 0 to infinity power minus t f t d t or any improper integral is said to convergent or absolutely convergent, if this limit limit R tending to infinity is 0 to R e power minus s t f t d t exists as a finite number or this integral is is said to be absolutely convergent, if this limits are tending to infinity 0 to R with absolute value of this integral of the d t exists as a finite number; so basically in this Laplace transform you will be discussing the convergence of this integral.

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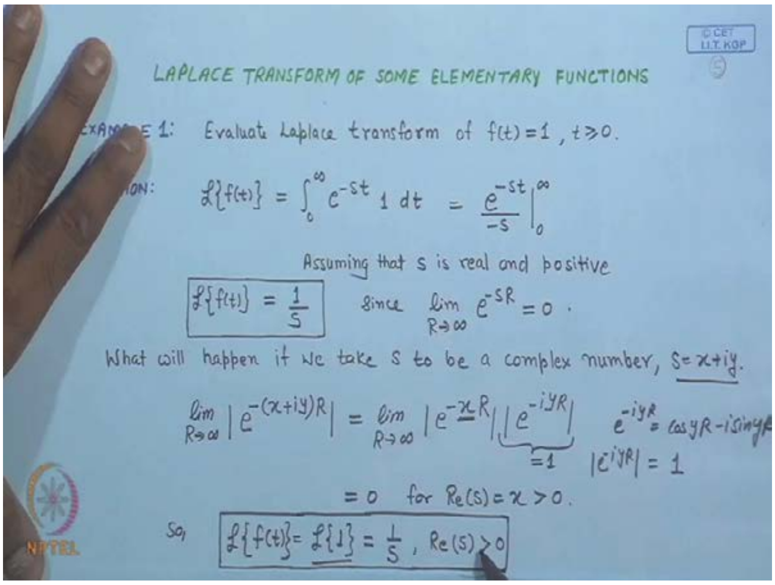

Now, we go to the Laplace transform of some elementary functions, so we take the first example and that is the simple functions its function of t is equal to 1, constant function 1 for t greater than or equal to 0. So, in this case this Laplace transform of f t as per the definition we have 0 to infinity e power minus s t and the function of t that is 1 in our case d t, we can integrate this easily, so we have e power minus s t over s and the limits 0 to infinity.

So, first the upper limit and the minus we set the lower limit, so we assume first that s is real and positive, then what will happen when t approaches to infinity this term will vanish and as t tending to 0, we will get here 1 over s, so simply the Laplace transform of function 1 is 1 over s, because the upper limit here is 0 what will happen if we take s to be a complex number that means the s we take is equal to x plus i y.

And in this case s well when we take s to be a complex number the Laplace transform of f t will be 1 over s and the reason is that, again in this case when we have s a complex number, the upper limit when we take when this t approaches to infinity will be again 0 and we can see that; so if we take this limit R to finite e power minus and for this s we right now x plus i y the complex form on this R.

So, what we have the R tending to infinity e power minus x R absolute value and the absolute value of e power minus i y R and this is this is 1 simply because e power minus i y R is nothing else cos y R minus i sin y R and if you take the modulus shear e power minus i y R on that will be sine cos square y R plus sine square y R and that will give us 1 so this term is 1 and as r tending to infinity this will again go to 0 if this x is positive.

So, we have the condition they are this limit, the upper limit is is 0 the term is 0 for real x positive, so finally, for the general case we have the result, that the Laplace transform of 1 is 1 over s for real s greater than 0.

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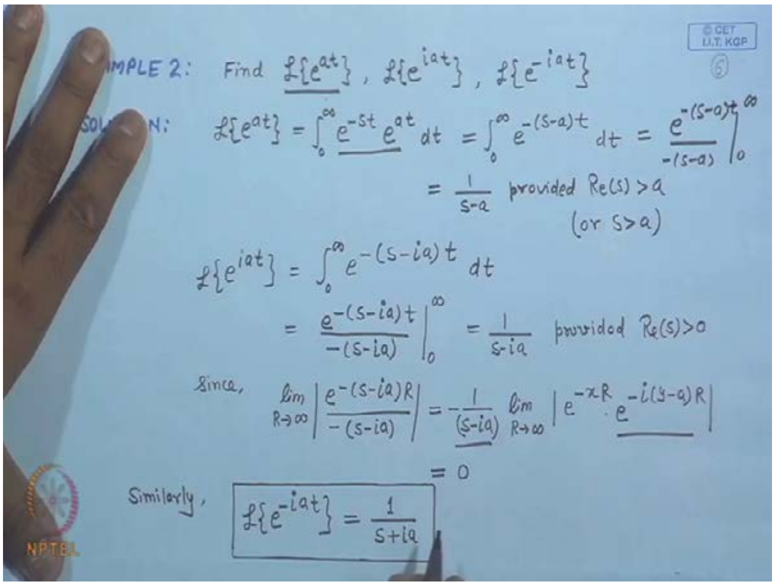

So let us take another example, on that is also very important so this is the Laplace transform of e power a t very similarly, we can get the Laplace transform of e power i a t and e power minus i a t. So we first take this Laplace transform of e a t and that is the Laplace transform of e power a t as per the definition 0 to infinity e power minus s t e power a t d t and then we can combine this two to get e power minus s minus a t d t 0 to infinity this integral.

And again its very similar to what we have done for finding Laplace transform of 1, so we have here e power minus s minus a, so again if we can integrate this and this is nothing else e power minus s minus a over s minus a and then the limits 0 to infinity. So again if this s minus a is positive, let us take s to be real, so if this is positive then again when t approaches to infinity this limit will be 0 and as t approaches to 0, so we will get simply 1 over s minus a again we can also think about the general case here, that real s is greater than a and the steps are very similar what we have done for the Laplace transform of 1.

So, we can assume this again as to be x plus i y and brake into the complex and the real part and again, we will see that while putting the upper limit here, t tends to infinity this term will vanish and then we have again 1 over s minus a in this case as well.

So let us, take 1 of the 2 functions here e power i a t or e power minus i a t, so we consider this e i a t Laplace transform and again as per the definition, we have 0 to infinity e power minus s minus a t, so e power minus s the kernel and then e power i a t we have the function and then this d t. So can we integrate this to get e power minus s minus i a over minus s minus i a t and limits 0 to infinity and again if we assume that this real s is positive this term will vanish as t approaches to 0 and we will remain again with 1 over s minus a as t approaches to 0 you can say this once again so the limits at to infinity e power minus s minus i a or n minus s minus a how this approaches to 0.

So this is anyway independent of R, so we have minus 1 over s minus a and then the limit are to infinity e power x we again assume x plus i y, so we here e power minus x r and then minus i y we put here, so we have e power minus i y and then, we have again this a i we have taken common and then R and this part modulus of this again will be 1 and then we have e power minus x r and we assume that x is positive than this will be go to 0. So similarly, we can get the Laplace transform of this e power minus i a t and that will be 1 over s plus i a.

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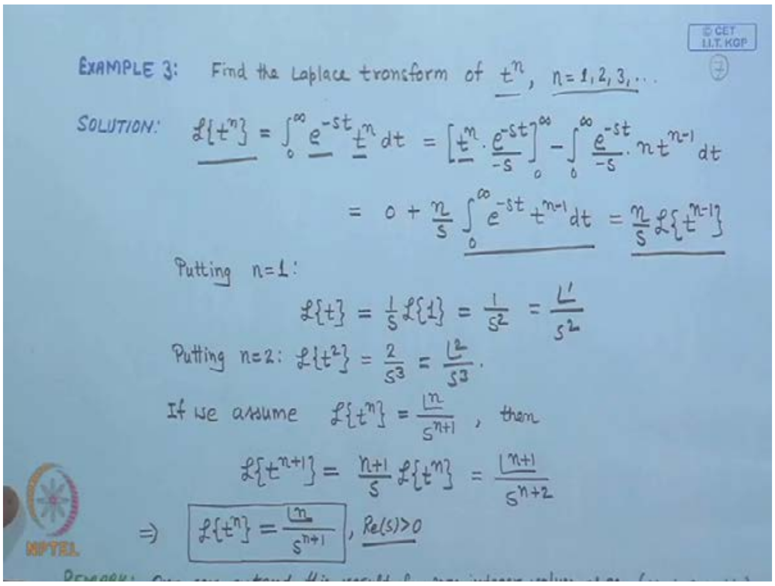

So, let us move to the next example and this is the Laplace transform of t power n.

So, let us move to the next example and this is the Laplace transform of t power n, n the positive integers, so 1 2 3 and so on, natural number, so very useful function and we will apply later on the lecture the Laplace transform of t power n. So, as per the definition we move, so we have t over t power n e power minus s t t power n d t and 0 to infinity, so we integrate this by parts, so t power n as it is the integral of e power minus s t, so e power minus s t over minus s on the limits 0 and infinity minus 0 to infinity can the integral of e power minus s t thus e power minus s t over minus s and the differentiation of this function that is n t n minus 1 d this will go to 0 as t approaches to infinity, because of e power minus s t and also it will go to 0, as this t approaches to 0, because of this t power n.

So, this term will vanish and then we have here n over s and 0 to infinity e minus s t t n minus 1, so what we see again this is nothing else but, the Laplace transform of t power n minus 1, so we got n over s in Laplace of t power n minus 1, so we got (( )) relation here that Laplace of t power n is n over s Laplace of t power n minus 1, so if you put n is equal to 1, so we have Laplace of t that is 1 over s and Laplace of 1, because n is 1, so its t power 0, so 1 so Laplace of 1 we know already that is 1 over s.

So, we have 1 over s and 1 over s, so 1 over s square also if we put n is equal to we will get the Laplace of t square and this is 2 over s and the Laplace of t Laplace of t is 1 over s square, so we will get 2 over s cube in fact for n is equal to 3, we will 3 over s from here and the 2 over s cube from Laplace of t square. So, we will get something 3 into 2 over s cube, so in general we can also write this is factorial 1 over s square and this is nothing else the factorial 2 over s cube based on this, now we can prove by mathematical reduction that the Laplace transform of t power n is factorial n over s n plus 1.

If we assume that the Laplace transform of t power n is factorial n over s n plus 1 and then, show that the Laplace transform of t n plus 1 is factorial n plus 1 over s n plus 2 then we have done. So, we show that the Laplace of t n plus 1 is again by this (( )) relation, so we have n plus 1 over s and the Laplace of t power n and Laplace of t power n, we assume the factorial n over s n plus 1, so we get this factorial n plus 1 over s n plus 2, so we have this formula to get the Laplace transform of t n Laplace of t n is factorial n over s n plus 1 and the real s, so real part of the the s is 0 greater than 0.

In fact we can also extend this result, because here what we have seen that this was for and positive integers, so we can extend this result for non integer values of n.

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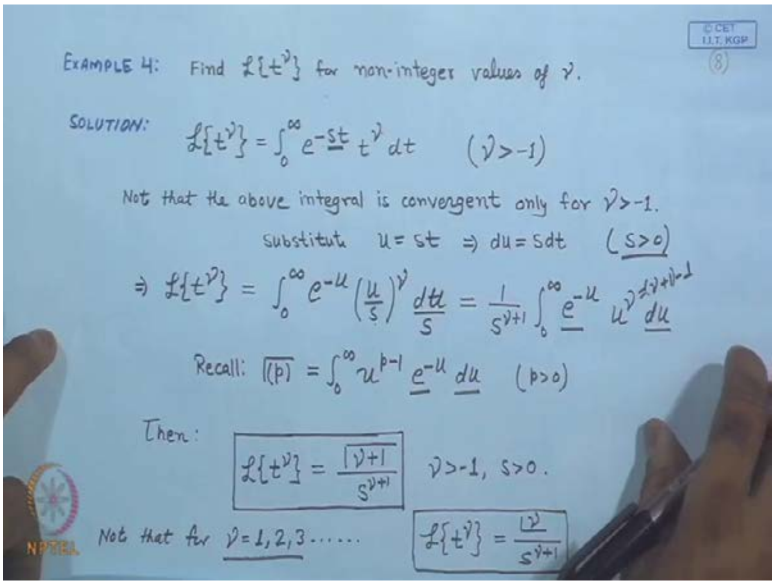

And thus see in next example, we will see, so in this case we take t power mu for non integer values of mu, so find this Laplace transform. We will go again by the definition, so 0 to infinity e power minus s t and t power mu d t and mu as greater than minus 1, this is very important because, only for these mu when mu is greater than minus this integral converges, so we should have this mu greater than minus 1, otherwise this does not make sense.

Now, we substitute this s t to a new variable u and then get this d u is equal to s d t and lets for simplicity we assume that s is real and and positive, so now the limits will remain 0 to infinity e power minus u and t is u over s power mu and d t is d u over s, so what we have s power mu and 1 s is sitting here, so we get 1 over s power mu plus 1 0 to infinity e power minus u and u power mu d u.

And now if we recall the definition of gamma function, so the gamma p is defined as the gamma p is equal to 0 to infinity u power p minus 1 e power minus u d u, so we have a similar integral there e power minus u d u and u power p minus 1 we have u power mu that is nothing as we can write this mu is equal to mu plus 1 and minus 1. So, we will exactly the same form for the gamma, so the p here is mu plus 1, so this Laplace of t mu will be the gamma mu plus 1 and this s power mu plus 1 for mu greater than minus 1 and we have taken this as positive.

So, we got the other general result t power mu is equal to gamma mu plus 1 s mu plus 1, and in fact 1 can see that if you just take this integers here mu to 1, 2, 3 and so on, then this exactly reduces throughout we got earlier, so the Laplace of t power mu is factorial mu over s power mu plus 1, so this is the other general formula for getting the Laplace transform of t power mu.

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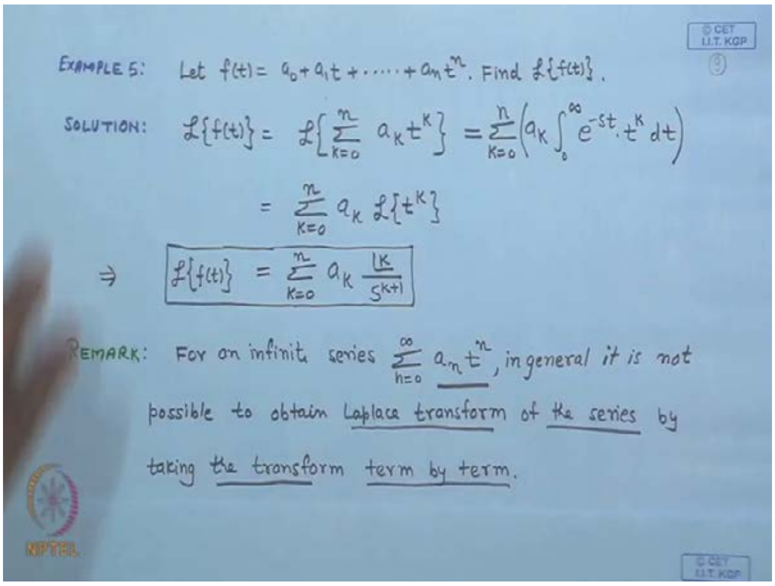

So, next example, we have let this f t is a 0 it is a linear combination of these t, so we have a 0 plus a 1 t plus and so on, a n t n, so it is a polynomial of of degree and in t, now we want find this Laplace of of this function. So, what we do as I have mentioned already that these transforms enjoy the linearity property, so here we can apply this Laplace to to each term.

So, we have Laplace transform f t is Laplace transform of of this sum here a k t k a k 0 to n and this is we can also (( )) you to linearity of this Laplace integral, we can write this 0 to n and a k that is a constant term, so 0 to infinity for minus s t and t k d t and this is exactly what we are talking about the linearity of the Laplace transform. So, we can have this k to n on this a k on the Laplace transform of t k and thus we know, so Laplace of f t is some k 0 to n a k factorial k over s k plus 1, because this is n we have assume to be integers, so we apply this formula for Laplace of t power k.

One remark about the series of a functions, so for an infinite series, so this was ah finite series, so this was the sum of of n terms but, if we take the infinite series n 0 to infinity a n t n, then in general it is not possible to obtain Laplace terms form of this series by taking the transform term by term. So, if we have finite term then we can apply the Laplace transform term by term to to the to this sum but, if we have infinite series, then this is not always possible to apply this Laplace transform term by term and we need to to take extra care to evaluate the Laplace transform in that case.

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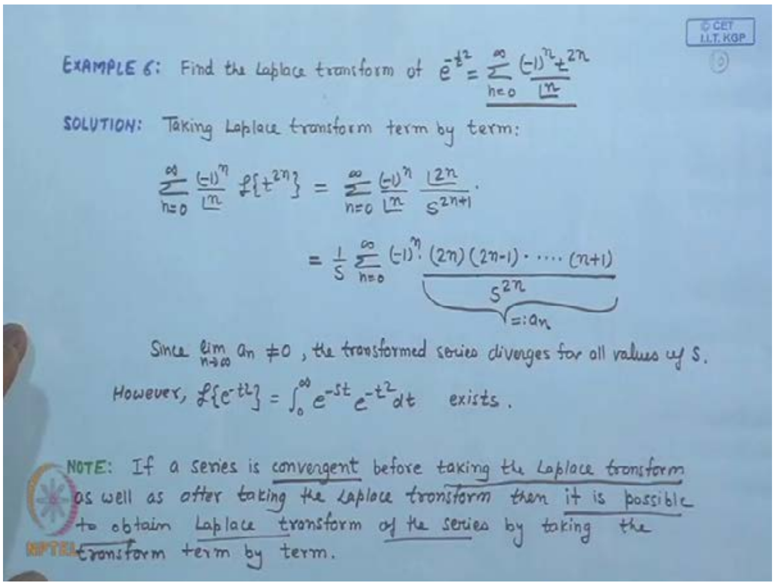

So, let us, just have a look on this example, So, you want to find the Laplace transform of e power minus t square and we can write this, when as a series and 0 to infinity minus 1 n and t its square power n, so t power 2 n and factorial n now let us see what will happen if we apply the Laplace transform to each of these terms.

So, we have summation and minus 1 power n over factorial n and the Laplace t power 2 n, so this sum and 0 to infinity minus 1 power n over factorial n and Laplace transform of t power 2 n, we know that is factorial 2 n over s power 2 n plus 1. And now let us simplify a bit more, so this is nothing 1 over s we have taken from here and to infinity minus 1 power n and then 2 n 2 n minus up to n plus 1, then you have factorial n that cancel with this factorial n, so we have this series. So, we just assumed that this is a and this is the alternative series and the interesting part here is now, that this is the a n as limit n approaches to infinity is not 0 because this a numerator goes first then the denominator and therefore, this terms form series what we got by applying this transform term by term does not converge for any value of s.

And however the Laplace transform of e power minus t square with is simply 0 to infinity power minus s t minus t square and and it is very clear that this integral exists because, this e power minus t square is is in fact bound by by 1, so this integral is is bounded by the Laplace of 1 that is 1 over s.

So, this is Laplace exists even though if we apply to the series here, the Laplace transform term by term, then the transform series does not converge, so we have to be very careful here, so does denote about this without going much into the detail. So, if a series is convergent before that means the original series is convergent the given series is convergent and the series after taking the Laplace transform is convergent; that means the both original and the transform series both are convergent, then it is possible to obtain the Laplace transform of the series by taking the transform term by term.

So, this point we need to be careful while working with the series we have to whether the transform series converges or not.

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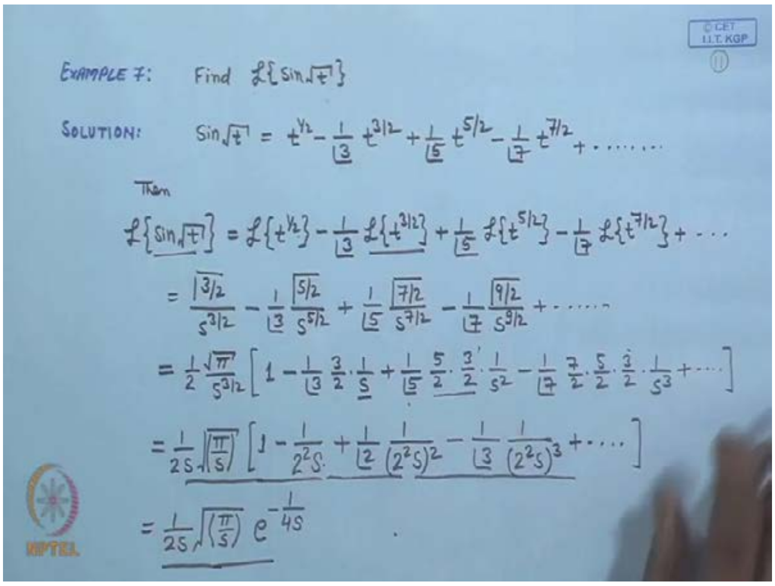

The next example that is sine square root t the Laplace of sine square root t and we have the sine square root t that is the series of this sine function, so we have t power half minus 1 over factorial 3 t power 3 by 2, is basically the cube of this square root t and plus 1 over factorial 5 t power 5 by 2 and minus 1 over factorial 7 t 7 by 2 and so on.

So obviously, this series is convergent then the value sine square root is so we do not have to worry about the convergence of this series but, if we take the Laplace transform term by term. So, we have Laplace of t half and then again the linearity of the Laplace transform tells that this constant you take out 1 over square factorial 3 the Laplace of t 3 by 2 plus 1 over factorial 5 and then the Laplace of t 5 by 2 minus 1 over factorial 7 and the Laplace of t 7 by 2 and so on.

So, we apply the formula to get this Laplace transform of t half that is nothing gamma half of plus 1, so t by 2 and s 3 by 2 minus 1 over 3 the Laplace transform of t 3 by 2, so gamma t by 2 plus 1 so 5 by 2 s power 5 by 2 and so on. Now let us take this common, so, what exactly this is the gamma 3 by 2 is half gamma half, so it is half square root pi, so half 0 and square root pi and s 3 by 2, so we are taking this term out of this series, so we have this first term as 1 minus 1 by 2 and square root pi, we have taken out s power 3 by 2.

So, we remain with s here and there we get 5, so 3 by 2 and this is factorial 3 then, here have 1 over factorial 5 from this gamma we get 5 by 2 3 by 2 and 1 by 2 root pi we have taken out and this s 3 by 2 we have taken out, so it is s square minus 1 over factorial 7 and from this 9 by 2 this gamma we have 7 by 2 5 by 2 3 by 2 and then half gamma half, so that we have taken and then s of cube.

So, let us simplify this, so we have this 1 over 2 s and then we can write this square root pi over s better organize form and then here 1 minus here the 3 factorial 3 we have 3 into 2, so 3 gets cancel we have 2 square s again here, the 5 will get cancel we get here 4 and the 3 also gets cancel. So, we have 2 square and then we have 2 square here, so 2 square s and the whole square similarly, here we get this 1 over factorial 3 and 1 over 2 power 2 s and cube.

And if you see here this is obviously convergent series and the value is the exponential minus 1 over 2 square s that is minus 1 over 4 s, so in this case this is the Laplace transform of of this sine square root t, because the series the transform series is is also convergent and in this case we get this transform taking the Laplace transform term by term. So, we have seen so far some basic examples of Laplace transform and the definition of of these Laplace transform, now we will be talking about about the existence of the Laplace transform, so the first question is whether the Laplace transform exists for for any function and obviously the answer is no, because of the convergence of that integral that integral will not converge for for any function, so there is a class of function for that that has only that that integral will converge and the Laplace transform will will exist.

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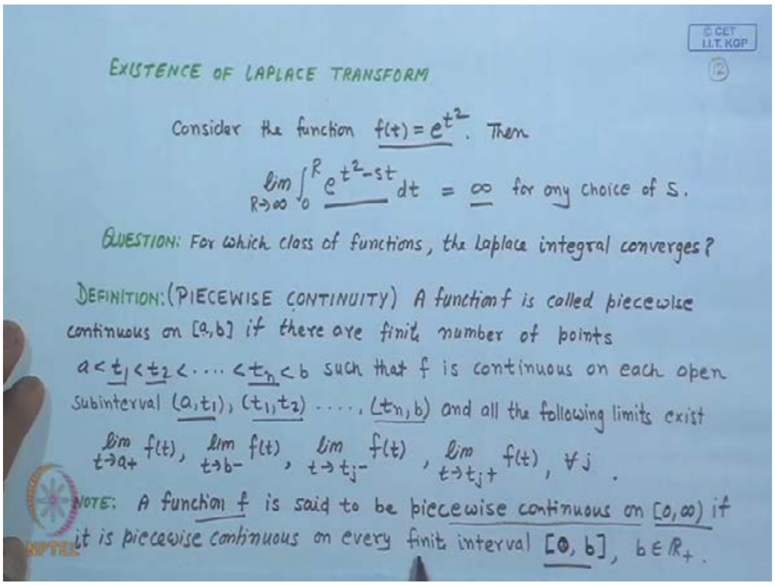

So, we will see just function here f t is equal to t square for example, for example, and then we try to get the Laplace transform by this limit at to infinity 0 to R e power t square minus s t d t thus you see this integrant goes without (( )), so this integral is the value infinity for any twice of s whatever s we take here, this integral will be infinity, so this integral is not convergent. So, the Laplace transform of e t square does not exist, then the question is that for which class of function the Laplace integral converges and to answer this question we will we have to give some definition and the one is piecewise continuity.

So, what is piecewise continuity, as by the name piecewise continuity functions are not continues but, they are continuous in pieces, so a function f is said or is called piecewise continuous continuous on the class interval a b if they are finite number of points t 1, t 2, t n such that, the function is continuous on each open subinterval, so a to t 1 function is continuous t 1 to t 2 is continuous, t n to t b is continuous. So, the function is basically continuous everywhere other than these points t 1, t 2, t n and so, in addition to that the following limits should exists what are these limits these are basically the left limit of the function s t approaches to way.

So, to this in and the right, so the right limit here as t approaches to a and the left limit as this t approaches to b of the function f t this should exist and that all these points t 1 t 2 t n where the function is not continuous the function the both the limits the left and right limits both should exist for all j's this is j 1 2 3. And so this is piecewise continuous function, so the function is basically continuous other than these points t 1, t 2, t n and if the following limits exist, then the function is said to piecewise continuous.

Just to add here a function is said to be piecewise continuous on on 0 to infinity if it is piecewise continuous on every finite integral 0 to b and b you can we can take any positive number, so in this case we call that this function is piecewise continuous on on 0 to open infinity here if it is piecewise continuous on every finite interval, we take from 0 to b.

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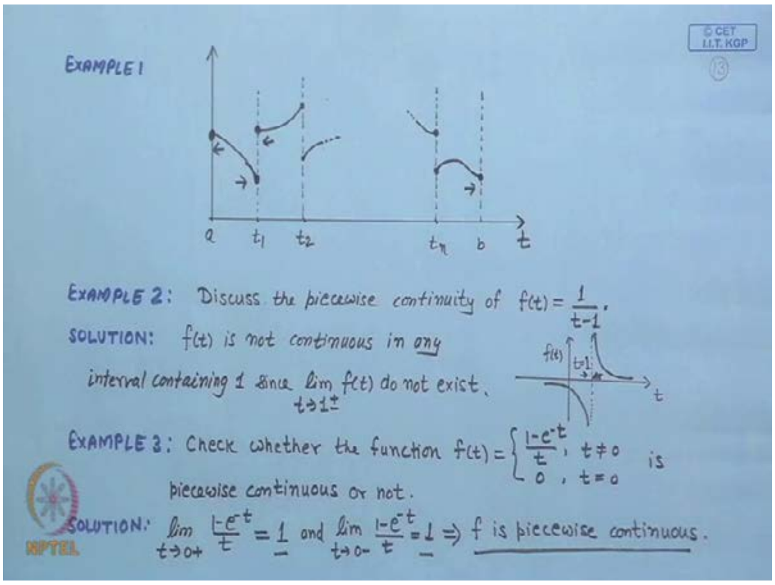

So, let us look some, look for some examples of this piecewise continuous functions, before we go the next next definition and this is a typical example of a piecewise continuous function. So, here the t as t approaches to a then this right limit exist and the function is not continuous at these points t 1, t 2, t n but, the limits as t approaches to this t 1 from this left side or t approaches to t 1 from this right side, the both should exist at all these points of this continuity and here the right limit as t approaches to b should also exist; so this is the typical graph of a piecewise continuous function, so we take 1 more example of this piecewise continuous.

So let us, discuss the piecewise continuity of the function f t is equal to 1 over t minus 1, so what will happen here, so the problem is f t is equal to 1, other than that the function is continuous and we do not have any problems we do not have to check anything else other than the limits s t tending to 1 from the right side as well as from the left side. So, if we take a look on the plots here, so as t approaches to 1 from the left side or from the right side, the limit does not exist, so then the the function this 1 over t minus 1 is not piecewise continuous in any interval which contains this 1, so this function is not piecewise continuous.

The another example, if we have this function f t 1 minus e minus t over t as t not equal to 0 and we have 0 1 t is equal to 0, so we have to check whether is continuous or not, so again we have problem at t is equal to 0. So, just just to note that I will go back again to this example choose a function in fact to written in this form is not defined as t is equal to 0, so we need to define this t is equal to 0, so we can set this function t is equal to 0 again 0 like like here and then discuss this piecewise continuity.

So let us, now come back to this example, so this t is not equal to 0 its 1 minus e minus t over t and t is equal to 0 0, so we have the problem at t is equal to 0 only otherwise this function is as a nice function, it is continuous there is no no problem, so at t is equal to 0 if we take the limit the left limit or or the right limit say if we take this this right limit 1 minus e minus t over t so it is getting 0 by 0 form, so we can apply this L'Hopital's rule.

So, the differentiation of the numerator will give us e power minus t and then this will be positive, so e power minus t over 1 and then t approaches to 0 we will get and also when t approaches to 0 from the left side, the both limits are 1, so the function is is piecewise continuous in this case.

So, very important consequence of this piecewise continuity is that the function is bounded basically, because at all these points when we have problem the limits exists and then we have actually the boundedness of the function.

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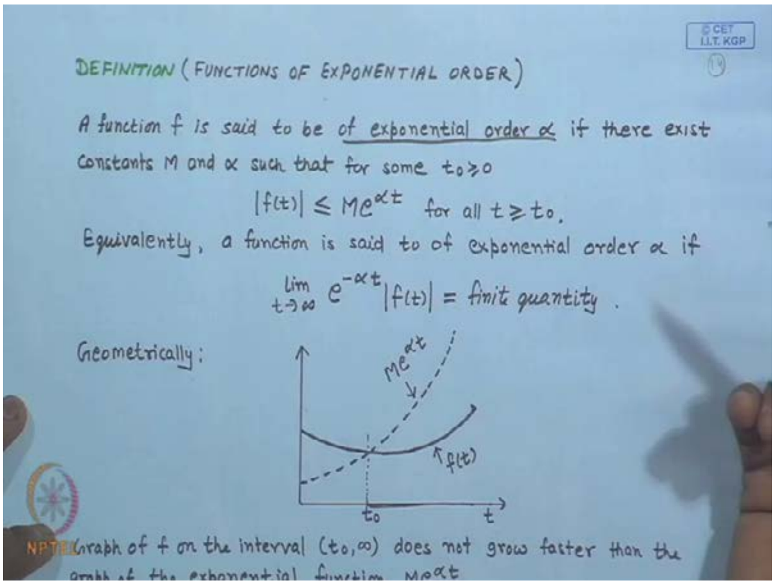

So, we go the next definition and thus function of exponential order, so what are these functions, so a function is said to be of exponential order alpha if there exist if there exists constants m and alpha such that, for some t naught greater than 0 (( )) holds that the absolute value of this function f t is bounded by this exponential function for all t greater than or equal to t naught.

So basically, the growth of the function is bounded by the exponential function, with this exponent alpha and then we call that the function is of exponential order, in practice this is difficult to check this inequality for checking whether, the function is of exponential order or not, so for that we have an alternative definition. So, a function is said to be exponential order alpha if this limit limit t tending to infinity and we take this f t with a absolute value on this e power minus alpha t; so if this limits exist of course, as a finite quantity then we say the other function if of exponential order.

Geometrically if we see this is basically the graph of a function f t and this is our exponential function with exponent alpha, so the graph of f on the interval t naught to t, t naught to infinity here does not grow faster than the graph of exponential function M e l alpha t, so does some meaning of this functions of exponential order.

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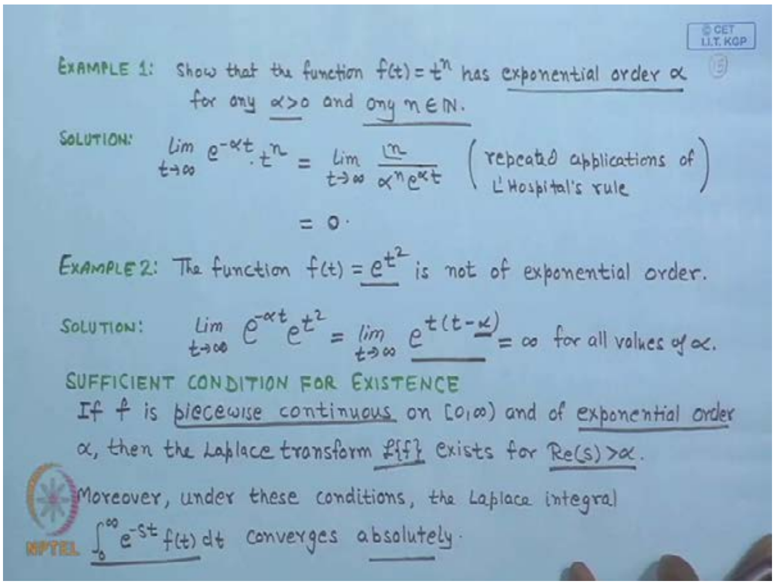

So let us, take some some example of these functions, first example show that the function f t t power n has exponential order alpha for any alpha and for any n, the set of this natural number, so whatever n we take here this is always the growth of this function is bounded by the exponential function of order alpha, for any alpha positive. So, it is very interesting to see now, we see the limit t tending to infinity e power minus alpha t and t power n on, so if we let this t tending to infinity is basically infinity and over if we write t power n over alpha t.

So, it is infinity by infinity form, so we can apply the L'Hopital's rule n times to get this factorial n out of these t power n when the difference here this n times and then e power alpha t here we will get alpha n e power alpha t and if now we let t tending to infinity we see the others is 0.

So, this function t power n is of exponential order for any n and for any alpha, so example two, so if we have function f t power t square and you will see that this is not of exponential order and the reason is clear; because if we take this limit t tending to infinity and e power minus alpha t e t square that is the limit t tending to infinity e power t and t minus alpha and whatever alpha we have as t tending to infinity this is going to be infinity; so for all values are alpha this limit does not exist, so this function e t square is not of exponential order.

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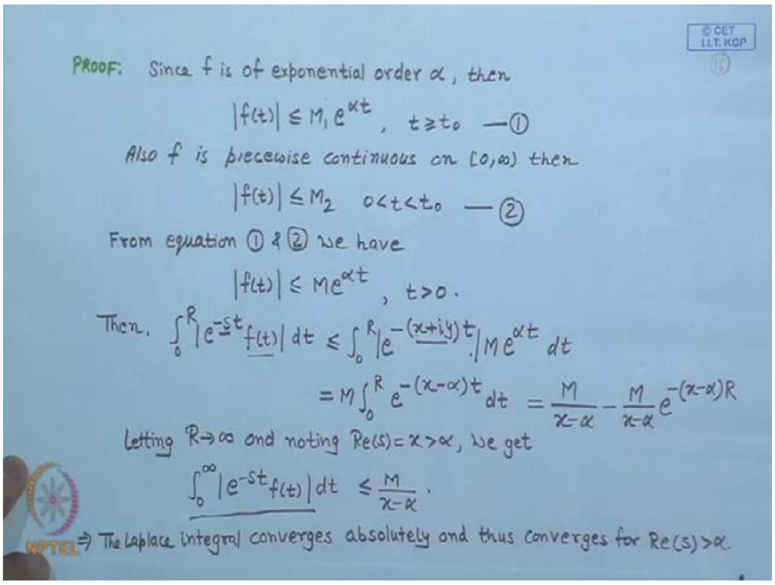

Now, we come to this sufficient condition for existence, so f is piecewise continuous on 0 infinity and of exponential order alpha, so the function is piecewise continuous and it is of exponential order alpha then the Laplace transform exists for real s greater than alpha. And in fact we have more stronger result that means that under these condition that the function is piecewise continuous and of exponential order the Laplace integral this 0 to infinity power minus s t f t d t converges absolutely.

So, we have other general results on these two conditions, that the function is piecewise continuous and of exponential order than this converges absolutely and we can quickly take a look at the proof. So, we have that the function is of exponential order that means the f t is bounded M 1 e power alpha t for certain t naught t greater than t naught and also the function is piecewise continuous then this is bounded from 0 to t naught; we can combine this two conditions to have 1 bound on the f t for the whole t, so this f t we can find M easily such that we have the absolute of f t bounded by M e alpha t for t positive

And in this case for example, alpha is positive we can simply take the maximum for M 1 M 2 to get this m, now let us take a look 0 r and the e power minus s t f t with the absolute value d t this bounded by 0 to R and e power s we take x plus i y and then t and then M e power alpha t d t for this f t and that is bounded by this M and e power minus x and this alpha, become combine on this i y t absolute or e power minus i y t with absolute value that is 1 and then this integral we have m over x minus alpha minus M over x minus alpha e power minus x minus alpha R.

And now we let R to infinity and note that the real part this s that is x is greater than alpha then we get basic value this will be 0 and this value is bounded by this, so we have seen that this integral in fact with the absolute value of this integrant, this is bounded by m over x minus alpha. So, the Laplace integral converges absolutely for real s greater than alpha and of course, this converges real s greater than alpha, so that was the sufficient condition for the existence.

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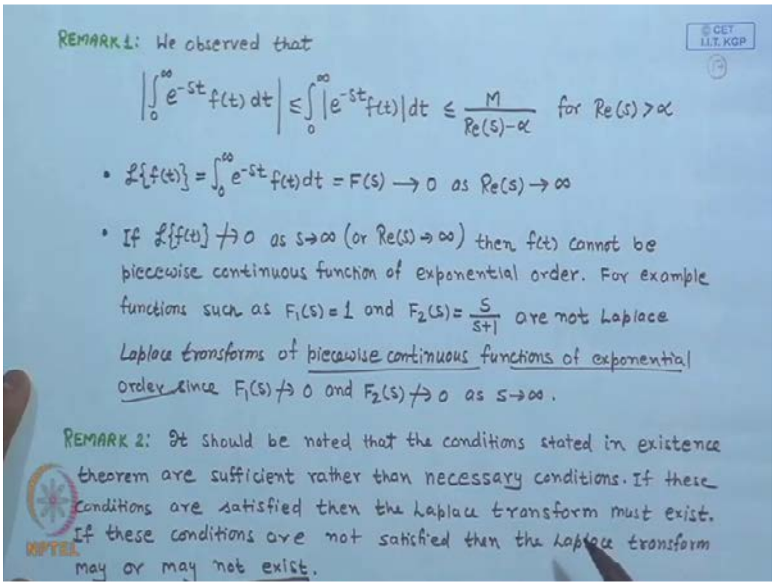

Now the quickly to remarks I am going to put the first in remark is that, we observe that the 0 to infinity power minus s t f t d t the absolute value of this is bounded by 0 to infinity and we take this absolute value inside e power minus s t f t d t on this we have seen that this is bounded by M over real s minus alpha, for real s greater than alpha. So, what is interesting to see here, that if we left this real s to infinity then what will happen this will go to 0 and we have the Laplace transform of any function which is of course, piecewise continuous and of exponential order.

So, e power minus s t f t d t or we denote this by f s will go to 0 thus this real s go to infinity because this term will go to 0, so what we can conclude from here, that if Laplace of f t does not go 0 s as tending to infinity or real s tends to infinity, then f t cannot be piecewise continuous function of exponential order for example, if we take we consider this function f 1 s is equal to 1 and f 2 s is equal to s over s plus 1.

So, they are not Laplace transform of piecewise continuous functions of exponential order, because this f 1 is does not goes to 0 and f 2 s does not go to 0 as s approaches to infinity but, this does not mean that they are not the Laplace transform of of of any function but, at least they are not Laplace transform of piecewise continuous of exponential order, that we can conclude from here.

The remark 2, it should be noted that the conditions is stated here in this existence theorem are sufficient rather than necessary conditions, that means if these conditions are satisfied, then the Laplace transform must exist, if these conditions are not satisfy then the Laplace transform may or may not exist. So, these are the sufficient condition what are the sufficient conditions, that function should be piecewise continuous and it should be of exponential order, in that case we are sure that the Laplace transform will exist but, if these conditions are not satisfied the Laplace transform may or may not exist, so to support this remark we have two examples.

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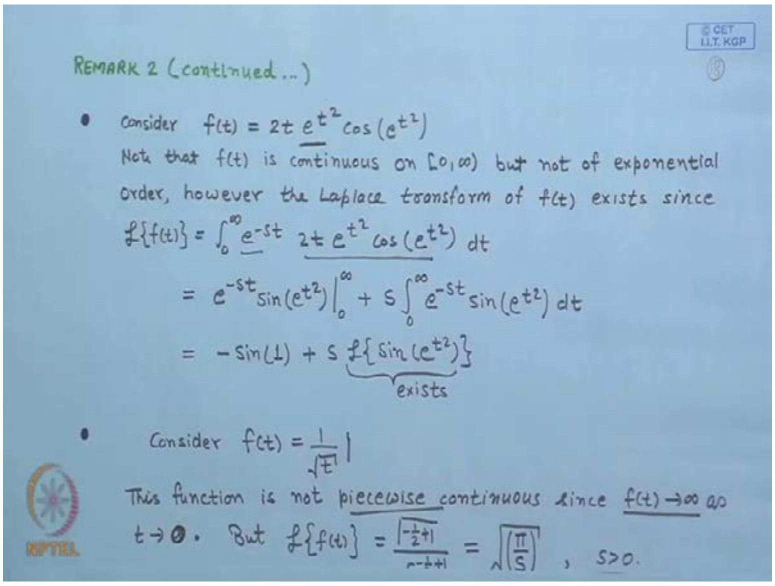

So, the first one is this function t is 2 t e t square cos e t square, so let us consider this function and note that this function is continuous on 0 to infinity but, not of exponential order, because e t square is sitting there, however the Laplace transform of this function exists because the Laplace transform as per the definition we have 0 to infinity power minus s t 2 t e power t square cos e t square d t and then integrate by parts.

So, we have e power minus s t and integral of this e sin e t square and than again here this differentiation of this will give us minus s we will get here plus s and then this term will go to to 0 as t tending to infinity and s t approaches to 0 this will be sine 1, so minus sine minus sine 1 and here we have the s and this is nothing a Laplace transform of sine e t square and this sine e t square this is of course, a continuous function and this is the boundary function. So of course, of exponential order, so the Laplace transform of sine e t square must exist, so this Laplace transform of f t exist, because this minus sine 1 plus s and the Laplace transform of sine e t square.

Lets take the another example, that is f t is 1 over square root t and this function is not piecewise continuous function, because as f t approaches to infinity s t approaches to 0 sorry 0. So, the function is not piecewise continuous continuous because f t approaches to infinity as to approaches to 0 but, the Laplace transform of this f t is minus half plus 1 and s minus half plus 1, so that square root pi over s for s positive.

So, we have these two functions 1 of them is not of exponential order the other 1 is is is not piecewise continuous but, the Laplace transform exists in both the cases. So, this supports the point at those conditions are sufficient conditions not the necessary conditions, so we have nor the sufficient conditions for the Laplace transform, that the functions should be piecewise continuous and it should be of exponential order, so that all for this lecture thank you.