

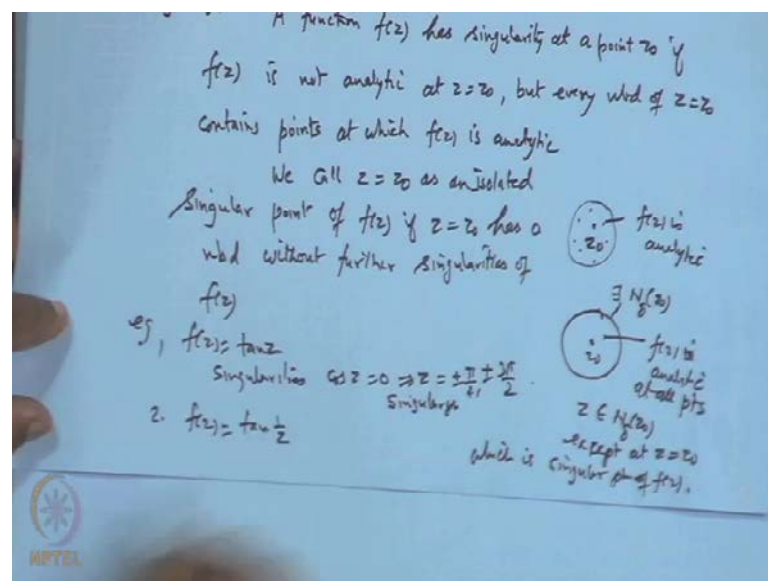
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Lecture No. # 21

Classification of Singularities, Residue and Residue Theorem

Classification of Singularities, so today, we will continue the same concept, the singularities and also give various type of singularities, concept of various type of singularities; and then go for the Residue Theorem, that will be used to evaluate the sum of the real integrals, which may be in the form of minus infinity infinity $f(x) dx$ and so on and so forth. So, today we will try to cover up this thing, these may be the last lecture today for this.

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So, as we have discussed the singularity, we have defined as, we call that a point z_0 is a singular point of a function $f(z)$ if $f(z)$ is not analytic at z_0 , but every neighbourhood of z_0 contains points at which f is analytic, that is the meaning is, if suppose we

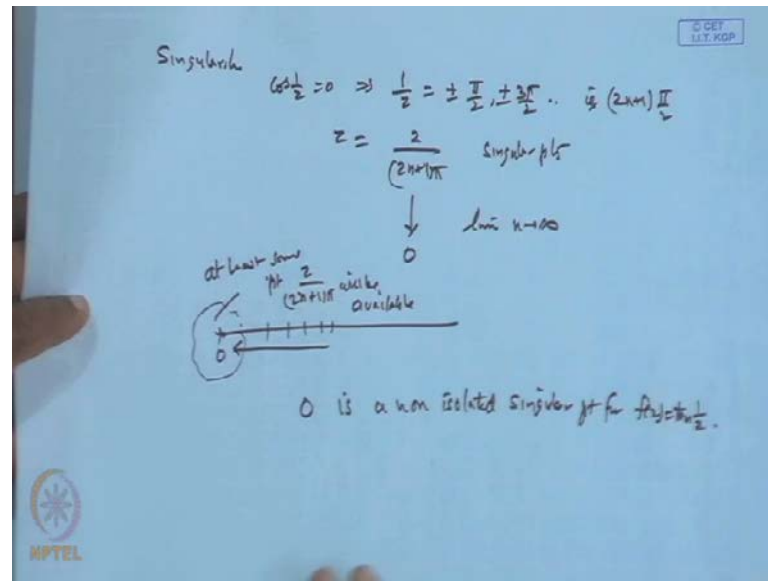
say z_0 is a singular point for a function $f(z)$. It means, at the point z_0 , the function seems to be analytic, either it is not defined or is not at all analytic. And but, if we draw any neighbourhood around the point z_0 , then this must include some of the point or may be all also, where the function should be analytic. If there exist a neighbourhood of the z_0 , which is free from any other singular point except z_0 itself, then such a point z_0 is called an isolated singular point.

So, we call the point z_0 as an isolated singular point of $f(z)$, if $z = z_0$ has a neighbourhood without further singularities of $f(z)$. It means, there exist a neighbourhood of a point z_0 with a suitable radius δ , such that this neighbourhood is free from the singular point of z , except z_0 means function is analytic everywhere in this neighbourhood except at z_0 .

So, at all other points f is analytic, f is analytic at all points, z belonging to Δ_{z_0} except at z_0 which is a singular point of $f(z)$, then z_0 is called an isolated singular point. An example we have seen that, if we take the function $f(z)$ which is $\tan z$, then obviously, this function has a singularity where $\cos z = 0$, it means z must be or multiple of $\pi/2$ like this.

So, these are the singular points of this and each one is an isolated singular point, because there is a sufficient gap between $\pi/2$ and $-\pi/2$, $\pi/2$, $3\pi/2$. So, one can obtain the neighbourhood with a suitable radius, if does not include the other singularity except the point z . However, if we look the function $f(z)$, which is $\tan(1/z)$.

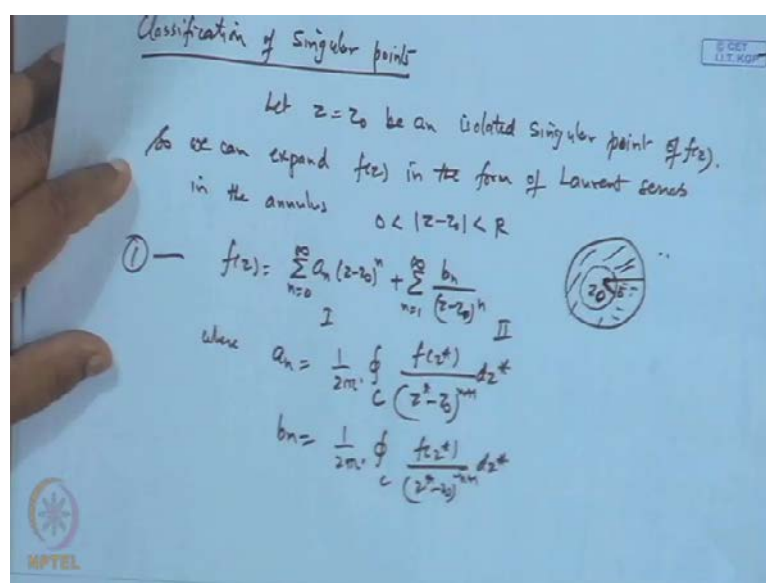
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Then, it has a singular point, the singularities of this will be or the singularities will be where \cos of $1/z$ is 0, it means $1/z$ must be all multiple of $\pi/2$, all these that is that is $2n+1$, all multiple of $\pi/2$, is it not? So, that is all n is 1, n is 0, n is 1, n is 2, n is 3 and so on. Now, z becomes what? z is $2/(2n+1)\pi$, so these are the singular points these are the singular points of this, now once you take the limit of this, so what is the limiting position of this? Limit of this as n tends to infinity, this limit is 0. Now, if we look the 0, this is our 0, here these are the sequence of the points these are the sequence of the points which are tending to 0.

So, if we draw any neighbourhood around the point 0, then we will get at least one of the point; at least some point of the type of this $2/(2n+1)\pi$ will be available will be available is it not, like this. So, will be available for this, it means 0 is not an isolated singular point; so, here 0 is a non isolated singular point, for the function $f(z)$ which is $\tan(1/z)$. So, this is the difference, we are interested in an isolated singular point. And if the function has an isolated singular point, then we can classify we can classify by means of the Laurent series.

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So, classification of the singular points **singular points**, let us suppose, let z equal to z_0 might be an isolated singular point of the function $f(z)$. So, **we can expand it so** we can expand this function $f(z)$ in the form of Laurent series in the neighbourhood in the annulus, $0 < |z - z_0| < R$, where R is the radius of this annulus, means outer circle where the singularity, all the singularity other than this z_0 will lie outside it. So, this is our z_0 point and here if I just remove it with a small pole on say δ , this is our δ , this radius, then we get this annulus $0 < |z - z_0| < r$; in this annulus, the function f is analytic.

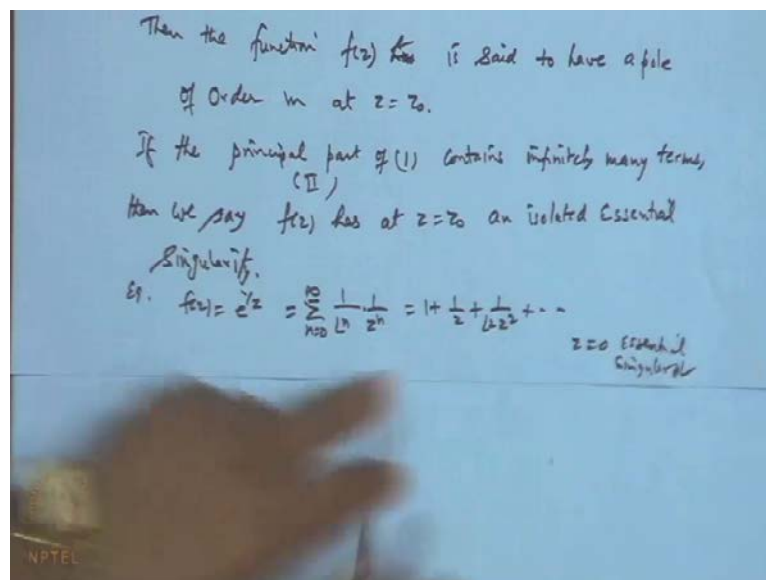
So, it can be expressed in the form of the Laurent series as follows. So, we get the $f(z)$ as $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$, where a_n 's and b_n 's, if you remember this can be defined by means of the formula as a_n 's or $\frac{1}{2\pi i} \oint_C f(z^*) (z^* - z_0)^{-n-1} dz^*$ and b_n 's are defined to be like b_n 's, we can write it minus $n + 1$ b_n 's we can write $\frac{1}{2\pi i} \oint_C f(z^*) (z^* - z_0)^{-n+1} dz^*$, that is single formula, we can write it in this fashion.

So, b_1 is nothing but what? When n is 1, it is nothing but $f(z^*) (z^* - z_0)^{-1}$ is 1, so b_1 will be integral of $f(z) dz$, so this is what we have. Now, if we look the series 1, then it consist of two parts, one is the **integral** positive integral powers of $z - z_0$

naught, other will the negative integral power of minus z naught. The first one will represent analytic function, this is a power series. The second one, it is known as the principal part of the function $f(z)$ of the series, now the singularity z equal to z naught is a singular point.

So, the characterization or classification of the singularities depends on the terms available in the second part. If there are finite number of terms available in the second part, then we say z naught has a pole and the order of the pole is the number of the terms available in the second part. If there are infinite number of the terms available in the second part of this expression, then z naught will be an essential singular point.

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So, we say we defined like this, if the principal part **principal part** that is second person, second part of it in 1 **in 1** contains only finitely **finitely** many terms **finitely many terms**, say b_1 over z minus z naught plus b_2 over z minus z naught whole square, n is 1, n is 2 and say b_m over z minus z naught to the power m , then the function $f(z)$ has the singularity, has function $f(z)$ which has a, then the function is said to have **is said to have** a pole of order m at the point z equal to z naught.

If this singular point will be a pole of order m . if the negative powers of z minus z naught in that series having only m terms, then it is of order m . Now, on the other hand, if the principal part of 1, part of the series 1, that is the second part of series 1 contains infinitely many terms **infinitely many terms**, then we say the function $f(z)$ has at z equal to

z naught and essential singularity, and an isolated essential singularities essential singularity **singularity**.

For example, if we just look the function $f(z)$ is $e^{1/z}$, now this function has an expansion which is 0 to infinity 1 by factorial n into 1 by z to the power n , that is it has a expression 1 plus 1 by z plus 1 by factorial 2 z square and so on like this, 1 by, so it does not have a positive powers. So, only one term is there, constant terms and star 0 . So, the negative powers contains infinitely many terms. So, z equal to 0 is an essential singular point for this, similarly when we get sine 1 by z , we get similar term.

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$$\begin{aligned}
 f(z) &= \frac{1}{z^3 - z^5} = \frac{1}{z^3(1 - z^2)} \\
 &= \frac{1}{z^3} (1 - z^2)^{-1} = \frac{1}{z^3} [1 + z^2 + z^4 + \dots] \text{ when } |z| < 1 \\
 &= \frac{1}{z^3} + \frac{1}{z} + z + \dots
 \end{aligned}$$

$z=0$ is a pole of order 3

If we take the function $f(z)$, say 1 by z cube minus z 5, say then in that case, it is equal to what 1 by z cube, then 1 minus z square and then when we expand it in the form of 1 by z cube 1 minus z square inverse, we get 1 by z cube 1 plus z square plus z 4 and so on, in the circle mode z when less than 1 and in that case, we are getting 1 by z cube plus 1 by z plus z and so on. So, basically these are only two terms, z equal to 0 will be a pole, but there three terms, because z square term is missing. So, z equal to 0 is a pole, because it is the first term, if we remember b 1 minus z naught, so that is available 1 by b 2 is missing b 2 is 0 b 3 is z cube. So, it is a pole of order 3 **pole of order 3** like this, similarly one can get.

Now, if we see here, the difference between the essential singularity and the pole is, that if z naught is an isolated singularity, you write down the expression on function in the

form of Laurent series, and that the negative part of it, that is the principal part it will decide about the nature of the singularity, whether it is a pole or it is an essential singularity. But someone may think like this, because in a pole also the terms are though it is finite. But $z - z_0$ lies in the denominator, so when z approaches z_0 , it goes to infinity, same case with here also, all the terms are lying in the denominator.

So, one can think that both are tending to infinity when z is tending to z_0 , the answer is not satisfactory, why? That behavior of the function in the neighbourhood of the pole and in the neighbourhood of the singularity is different. In fact, in the case of the pole, the modulus of $f(z)$ will always go to infinity when z approaches the singular pole z_0 . But in case of essential singularity, when z approaches z_0 , the value of the function $f(z)$, f of z may not definitely will not approach to a single limit, it will approach to different limit.

And in fact, if I take any number, any given complex number one can identify a path which approaches to z_0 such that limiting value of this function $f(z)$ when z approaches to z_0 and essential singular point, it will lead to the same value which we decide, except possibly 1. So, the characterization of this singular point, about in the neighbourhood of the pole and neighbourhood of the singularities differs, and that is given by in terms of the following (O).

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$$= \frac{1}{z^3} (1 - z^2)^{-1} = \frac{1}{z^3} [1 + z^2 + z^4 + \dots] \text{ where } |z| < 1$$

$$= \frac{1}{z^3} + \frac{1}{z} + z + \dots$$

$z=0$ is a pole of order 3

Characterization

Behavior of an analytic function in the neighborhood of a pole or essential singularity

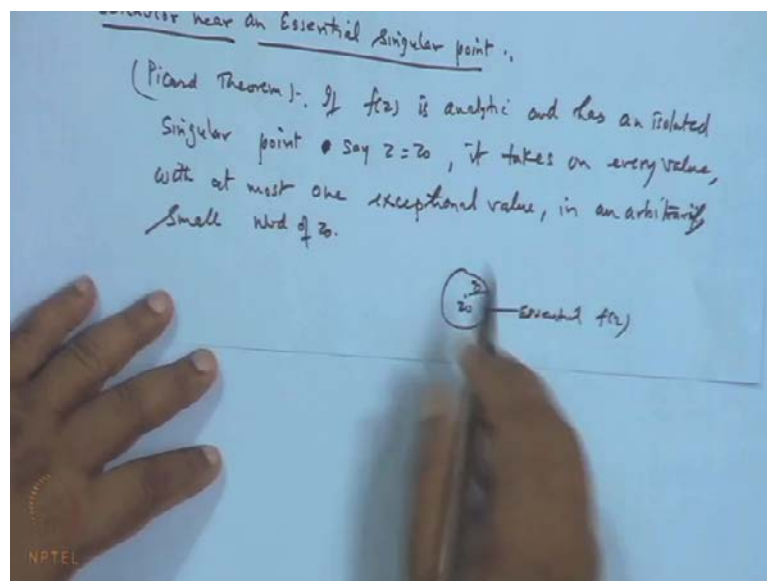
Behavior near a pole: If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$ in any manner.

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The first, the characterization, characterization of singularity, behavior or you can say behavior of the, do not say behavior **behavior of behavior** of an analytic function **analytic function** **behavior of an analytic function** in the neighbourhood of pole and essential singularity **essential singularity**.

So, first we have behavior near a pole **behavior near a pole behavior near a pole** the, it is given by like this, if the function $f(z)$ is analytic **is analytic** and has **and has** **is analytic and has** a pole **has a pole has a pole** at z equal to z_0 **has a pole at z equal to z_0** , then the absolute value of this, modulus of $f(z)$ will tends to infinity as z tends to z_0 **at z tends to z_0** in any manner **in any manner**. It means, whatever the path we choose with approach to z_0 , the absolute value of $f(z)$, the modulus of $f(z)$ will go to infinity that is if z_0 is a pole.

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For example, if we look the function $f(z)$ which is $1/z^2$, now it has a pole of order 2 at the point z equal to 0. Now, when we find the limit of this $f(z)$ when z tends to 0, this is the $1/\text{modulus of } z^2$ limit z tends to 0, and that nothing but what, $1/(x^2 + y^2)$ limit x tends to 0, and this we can establish by ϵ - δ definition also, or otherwise along any path we choose, **this limit will go to 0, this limit** **sorry** this will be go to infinity, because z is a, x is tending to 0, y is tending to 0. So, this will go to infinity, whatever the path we choose, will **(0)** keeps on increasing for that, so this goes to a **(0)**.

Similarly, the behavior near essential singular point **near essential singular point essential singular point** the function $f(z)$, this behavior is given by Picard Theorem, what is the Picard's theorem is, the Picard's theorem says, if the function $f(z)$ is analytic and has an isolated essential singular point **isolated singular point** isolated singularity singular point, say z equal to z_0 means it has a isolated singular point, then it takes on every value **it takes on every value takes on every value with at most** with at most one exceptional **exceptional** value in an **arbitrary small neighbourhood of z_0** arbitrary really small neighbourhood of z_0 .

So, what this theorem says is, Picard's theorem says, if suppose z_0 is an essential singularity for this function $f(z)$, then if I draw a neighbourhood around this point, in an arbitrary small neighbourhood of this point, the value of the function $f(z)$, when z approach to z_0 will take any value, all the value in fact, except possibly 1. So, there will be one **(())** value which may not be taken by the function when z approach to z_0 , but rest of the values can easily be achieved by choosing the suitable path which approach to z_0 .

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$\lim_{z \rightarrow 0} f(z) = \lim_{y \rightarrow 0} e^{1/iy} = \lim_{y \rightarrow 0} e^{-i/y} = \lim_{y \rightarrow 0} (\cos(1/y) - i \sin(1/y))$ does not exist
 $\lim_{z \rightarrow 0} f(z) = \lim_{x \rightarrow 0} e^{1/x}$
 $x > 0 \rightarrow \infty$
 $x < 0 \rightarrow 0$
 $z = re^{i\theta}$
 $e^{1/z} = e^{\frac{1}{re^{i\theta}}} = e^{\frac{1}{r} e^{-i\theta}} = e^{\frac{1}{r} \cos \theta} e^{-i \frac{1}{r} \sin \theta}$
 $r \rightarrow 0 \Rightarrow \frac{1}{r} \rightarrow \infty$
 $\theta \rightarrow 0 \Rightarrow \cos \theta \rightarrow 1, \sin \theta \rightarrow 0 \Rightarrow \infty$
 $\theta \rightarrow \pi \Rightarrow \cos \theta \rightarrow -1, \sin \theta \rightarrow 0 \Rightarrow 0$
 $\theta \rightarrow \frac{\pi}{2} \Rightarrow \cos \theta \rightarrow 0, \sin \theta \rightarrow 1 \Rightarrow 0$
 $\theta \rightarrow \frac{3\pi}{2} \Rightarrow \cos \theta \rightarrow 0, \sin \theta \rightarrow -1 \Rightarrow 0$

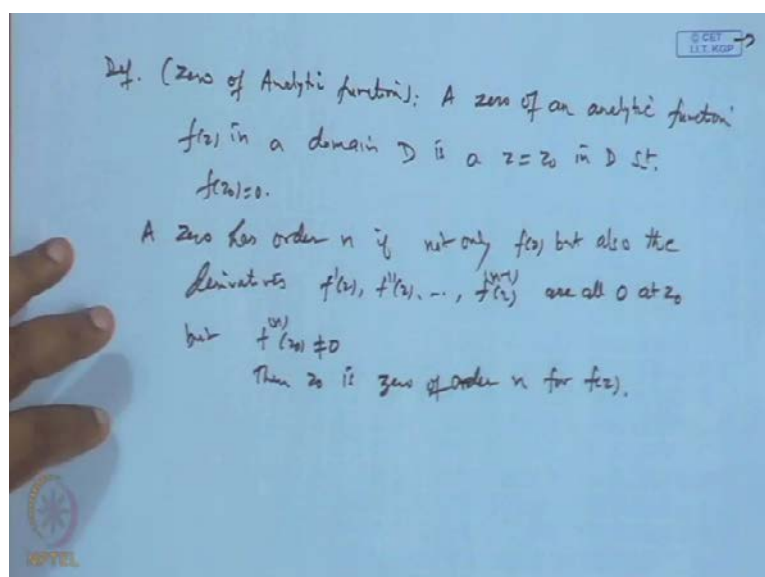
For example, if we take the function $f(z) = e^{1/z}$, now this function we have already seen that $z = 0$ is an essential singular point, the limit of this function $f(z)$ when z tends to 0. Now, if I take z to be iy , then what happened to this? This is the limit e to the power i/y means minus i square over i/y , is it not? And that will be the limit y

tends to 0 and once you take this, what happened to this, e to the power $\frac{1}{y}$ minus i over y , limit y tends to 0. So, this will be equal to $\cos 1$ by y minus $i \sin 1$ by y and limit y tends to 0, we know the limit of $\sin 1$ by y , limit of this will not exist, so does not exist.

However, if we choose the path, this is path 1, if we take the path 2, z equal to say x , then what is the limit of this function $f(z)$? When z tends to 0, it is e to the power $\frac{1}{x}$ as x tends to 0. Now, this limit will go to if x is positive, it goes to infinity; if x is negative, it will go to 0, so along different for the z different value. In fact, if I chose the function, suppose I **suppose I** want this value c which is equal to say c naught e to the power α different from 0, then one can identify the path which approach to 0, so that this one.

So, suppose there e to the power $\frac{1}{z}$, is suppose this value which is c naught e to the power $i\alpha$. So, if I compare it and find out the equation by we get from here is r^2 square is 1 by $1 - n$ c naught square plus α square where, θ comes out to be $\tan^{-1} \frac{\alpha}{c \text{ naught}}$, now what is α ? α is this (0) . So, α can be written as $\alpha + \text{minus } 2n\pi$ maximum no difference. So, all can be reduced to 0, it means along different value which is nearby 0, we can assume any value with a suitable path, so that gives the, so we are not going tell like.

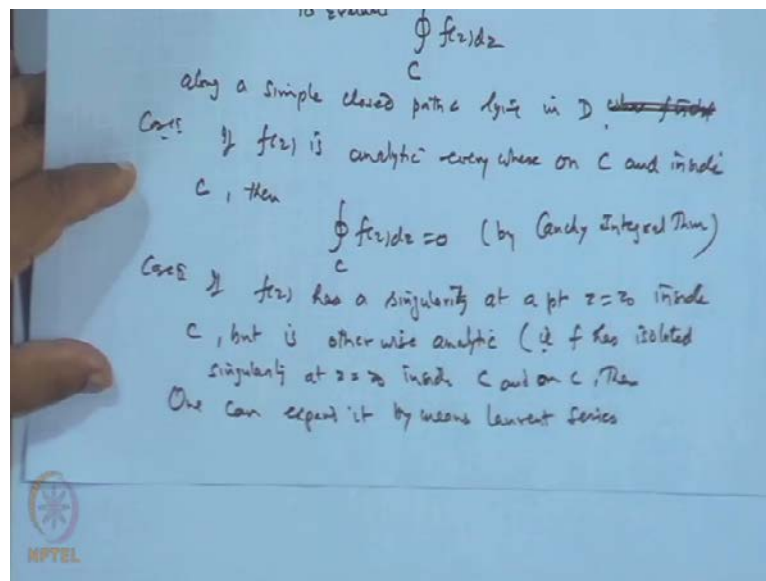
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The zero's of an analytic function and the definition which we required, the zero of analytic function **a zero of analytic function analytic function** $f(z)$ in a domain **in a domain** D is a point z equal to z naught in D such that such that f of z naught is 0. When the

function, when we say at the point, then the point said to be zero of the analytic function, a zero has an order n , if not only f , but also the derivatives f' , f'' , f''' , up to $f^{(n-1)}$, when it sees, when it sees 0, all 0 at z_0 or 0 at z_0 a 0 all 0, but the n th derivative of this function is different from 0, then we say z_0 is a 0 of order n then z_0 is a 0 of order n for the function $f(z)$ clear. So, this is also a concept which we required right.

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Now, let some to our main thing which is useful, the Residue Integration Method method achieve the purpose of the (()) residue integral method is to evaluate the integral $\oint_C f(z) dz$ along the path c , along the path along a simple closed path simple closed path c lying in the domain lying in the domain D where the f is defined, lying on domain D , that is all. Now, the integral of this $\oint_C f(z) dz$ in the two cases may be there if the function f is a, if the function $f(z)$ is analytic if the function $f(z)$ is analytic everywhere everywhere on C and inside C and inside C such an, then the value of this integral $\oint_C f(z) dz$ will be 0 by Cauchy integral theorem.

So, for evaluation of such an integral is very easy, just have to identify whether the given function is analytic through the domain domain contained inside c at every point inside c as well as only boundary of c . So, integration along the closed path c will be 0. However, if the function $f(z)$, if $f(z)$ has a singularity has a singularity at a point singularity at a point z equal to z_0 inside c inside c , but but is otherwise analytic, it means z

equal to z_0 is an isolated singularity, that is f has isolated singularity at z equal to z_0 in the inside C and on C inside C . So, there is now the point except z_0 and on C and on C then one can expand this function $f(z)$ by means of Laurent series, then one can expand it by means of Laurent series **Laurent series**.

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$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$
 This converges for all pts in some domain of the form $0 < |z - z_0| < R$.
 Here $b_1 = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{1+1}} dz^*$
 $\therefore b_1 = \frac{1}{2\pi i} \oint_C f(z) dz \Rightarrow \oint_C f(z) dz = 2\pi i b_1$
 b_1 is known as the Residue of $f(z)$ at $z = z_0$.
 \therefore Residue of $f(z)$ at an isolated simple pole z_0 is the coefft of $\frac{1}{z - z_0}$ in the expansion of Laurent series (1)

And the series will be, that is $f(z)$ will be $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ to the power n plus b_1 over $z - z_0$ plus b_2 over $(z - z_0)^2$ and so on and so forth. And this series converges **this series converges converges** for all points near to z equal to z_0 in the annulus in some domain **in some domain** of the form $0 < |z - z_0| < R$. Now, in this case the coefficient a_n b_n etcetera as I told you earlier, this was given by these formula coefficients, this was our formula for the a_n and b_n . So, if we look the formula for b_1 here, the b_1 is given by the formula $\frac{1}{2\pi i} \int_C f(z) dz$, that is b_1 becomes $\frac{1}{2\pi i} \int_C f(z) dz$, this is our integral.

So, from here, can you say integral of the function $f(z) dz$ along the curve C is $2\pi i$ times b_1 **$2\pi i$ times of b_1** , it means if I know that the coefficient of 1 upon $z - z_0$, then one can easily evaluate the integral $\int_C f(z) dz$ along the path C where the z_0 is a point of singular point and isolated singular point for the function $f(z)$, and this b_1 we

call it as, b_{-1} is known as, so b_{-1} has a very important role, this will as important role in evaluating the close integral along closed path which contains the singular point.

And so, b_{-1} we call it give a special name and the b_{-1} is known as the residue **as the residue** of the function $f(z)$ at z equal to z_0 . So, we get, we define the residue, the coefficients, the residue of the function $f(z)$ means, coefficient of that is residue of the function $f(z)$ at an isolated singular point z_0 is the coefficient of $1/(z - z_0)$ in the expansion of Laurent series, so we set to Laurent series 2. So, that is the one, means by using this result, we can easily get this, so let it be say here 3.

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Handwritten notes on a blue background showing the Laurent series expansion of $\frac{\sin z}{z^4}$ and the calculation of the residue at $z=0$.

$$\begin{aligned} \frac{\sin z}{z^4} &= \frac{1}{z^4} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right] \\ &= \frac{1}{z^3} - \frac{1}{3!} \cdot \frac{1}{z} + \frac{z}{5!} - \dots \quad \text{when } |z| > 0 \\ z=0 \text{ is an isolated singular pt} \\ b_{-1} &= -\frac{1}{3!} = \text{Res } f(z)/z=0 \\ \oint_{|z|=1} \frac{\sin z}{z^4} dz &= 2\pi i \left(-\frac{1}{3!} \right) = -\frac{\pi i}{3} \end{aligned}$$

So, suppose I have say this one, say suppose I want it to evaluate this integral $\oint \frac{\sin z}{z^4} dz$ along the unit circle, mode z equal to 1 in a counter clock wise direction **clock wise**. Now, obviously this function sine z by z for this is equal to $1/z^4$, sine z expression is z minus z^3 over factorial 3 plus z^5 by factorial 5 and so on.

So, when you divide by this, so you getting is $1/z^3$ minus $1/3!$ into $1/z$ plus $z/5!$ and so on, so forth. So, this is the expansion of the function sine z by z to the power 4 around the point 0. Obviously, z equal to 0 is an isolated singular point **singular point is an isolated singular point** and then it is of course, it is a pole of order 3, then what is our residue, coefficients b_{-1} here comes out to minus $1/3!$, that is the residue of the function $f(z)$ at the point z equal to 0.

So, if you are interested in this value, then we can get integral of this function $\sin z$ over z to the power 4 along the curve $|z| = 1$ which is close to $z = 0$ is an isolated singular point is nothing but by theorem, $2\pi i$ times into b_{-1} , b_{-1} is minus $\frac{1}{3!}$ and $\frac{1}{3!}$ by factorial 3 sorry $\frac{1}{3!}$ by factorial 3 and that comes out to be $-\frac{2\pi i}{3}$ say minus πi by 3. Here obviously, this result is valid and $|z|$ is greater than 0 mod z greater than 0. So, it is say annulus like this, so this way we can easily answer the questions for computing the integral for this.

Now, the residue which we are calling as a coefficient of this, then this is not only way to compute the residue, that is by expanding the function in the form of the Laurent series, and then try to find coefficient of $1/(z - z_0)$ and we say this is the residue of the function. Since, the Laurent series expansion of any function does not depend, does can not only be obtained with the help of the formula, we can also use many tricks, in many ways we can expand the function around the point z_0 and get the Laurent series expansion and this expansion is unique.

So, you are free to expand the function around the point z_0 and get the Laurent series expansion by either by using the formula this coefficient formula a_n/b_n or with some other method as we have discussed earlier. The main idea is once you expand, you find b_{-1} , but since the expansion when you expand the function $f(z)$ in the form of Laurent series, either there will be help of formula or with help of some other way, it will cost time and to evaluate the integral along the closed path $f(z)$ does not need expression at all of the function $f(z)$ in the form of Laurent series.

What does need is the coefficient 1 upon $z - z_0$, that is the residue of the function at a singular point $z = z_0$. So, in a state of going for the expression, if I know some formula which directly gives the residue of the function at the point z_0 , then it is very easy to compute the residue and then find out the value of the integral.

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Here $b_1 \neq 0$

$b_1 = \lim_{z \rightarrow z_0} (z - z_0) f(z)$ Residue of $f(z)$ at simple pole z_0

$\text{Res } f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$ ✓

simple pole

Ex $f(z) = \frac{9z/i}{z(z+1)}$, $z = \pm i, 0$ singular pts

$\text{Res } f(z) = \lim_{z \rightarrow i} (z - i) \frac{(9z/i)}{z(z+1)} = -5i$

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So, let us see the residue, certain formula to find the residue to find residue to find the residue at an isolated singular point at an singular point singular point z equal to z naught. So, I will give a different cases; case 1, if z naught is a pole, suppose z naught is a pole of order 1, pole of order 1 means that is simple pole, simple pole means only the function value f of z is not 0, but f prime z naught is not 0.

So, if it is a pole of order 1, then the Laurent series expansion will give only one term, the negative person principal part involve only one term, and residue theorem having the positive integral powers of z minus z naught. So, we can write down the expansion f z which may be, which will be in the form of b_1 over z minus z naught plus a_0 plus a_1 z minus z naught plus a_2 z minus z naught whole square and so on and so forth.

Here, b_1 will be different, here b_1 cannot 0, because it b_1 is 0, then z naught will not a simple pole, because it is simple pole b_1 will be different from 0. So, now you multiply by function f z by z minus z naught and take the limit f z approaches to z naught, what you get? The other term get cancelled, tends to 0. In fact, it will be 0 and we get only b_1 , so that is the residue of the function f z at simple pole simple pole z naught. So, this is the formula for this, that is the residue of the function f z at z equal to z naught which is a simple pole will be equal to limit of the function z tends to z naught, z minus z naught f z , so this will be the formula, this one.

Now, suppose for example, let us take if I take the function $f(z)$ say $9z + i$ over say $z^2 + 1$, now what are the singular point? Here if I look, the z equal to plus minus i will be the singular point, z equal to 0 will also be singular point. So, z equal to plus minus i , these are the singular points. So, I want the residue of function $f(z)$ at the point z equal to say i . So, what is this is, you just multiply by $z - i$, the whole thing $9z + i$ divided by $z^2 + 1$ can be written as $z - i$ and take the limit as z tends to i . So, if you compute this value, the value will come out to be minus 5 i that is the residue of this function at z equal to (i) , similarly, residue at minus i and 0 can be obtained by quickly.

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Handwritten notes on a blue background showing the residue of a function at a simple pole and a pole of order n .

Case I: $f(z) = \frac{p(z)}{q(z)}$ where z_0 is a simple pole, where $p(z_0) \neq 0$

So $q(z) = (z - z_0)g'(z) + \frac{(z - z_0)^2}{2!}g''(z) + \dots$

$\text{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) \frac{p(z)}{q(z)} = \frac{p(z_0)}{g'(z_0)}$

Case II: z_0 is a pole of order n

$\text{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \{ (z - z_0)^n f(z) \} \right]$

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Now, if it is a pole another method is, another rule or is since if suppose the function $f(z)$ is of the form $p(z)$ by $q(z)$ where z_0 is a simple pole, then obviously, when z_0 is a simple pole, the function $q(z)$ must have one of the factor $z - z_0$ involved. So, $q(z)$ will have **so $q(z)$ will have** the expansion at $z - z_0$ into $q'(z_0) + \frac{1}{2!}q''(z_0)(z - z_0)^2 + \dots$ because when z tends to z_0 , $q(z)$ must go to 0 plus $z - z_0$ whole square by factorial 2 $q''(z_0)$ and so on. So, and also be residue that $p(z_0) \neq 0$, where $p(z_0)$ is not 0, otherwise it will go identity function, zero function. So, residue of $f(z)$ at z equal to z_0 , you can just divide and take the limit multiply by this, we will get $p(z)$ by $q(z)$ and limit z tends to z_0 and what you get immediately, this will come out to be the $p(z_0)$ over $q'(z_0)$.

So, that is the another formula to compute the residue at a simple pole, but if the case second, if our z_0 is a pole of order m **is a pole of order m** , then the corresponding formula, I am not deriving the result, the corresponding formula for the residue of the function $f(z)$ when z equal to z_0 is a pole of order m , then this formula will be $\frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)$, so limit of this, that will give the residue of the function when m is a pole.

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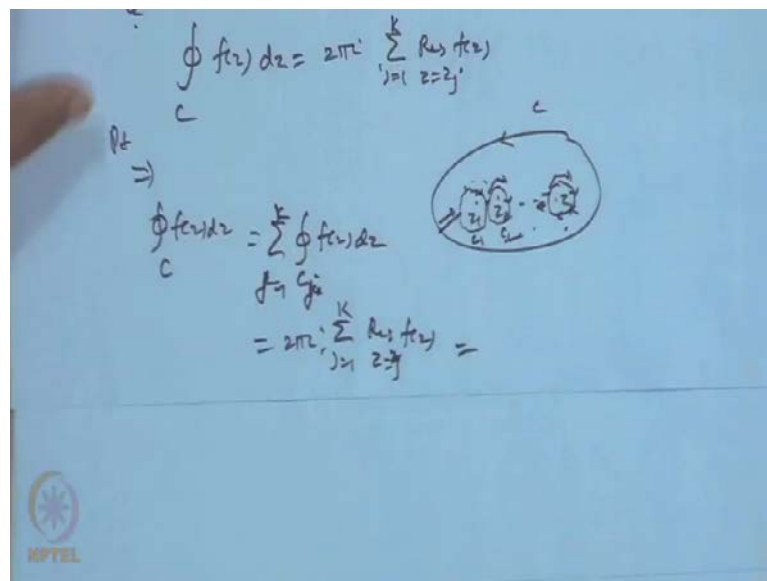
$f(z) = \frac{50z}{(z+4)(z-i)^2}$
 $z=i$ is a pole of order 2.
 $\text{Res } f(z)_{z=i} = \lim_{z \rightarrow i} \frac{d}{dz} \left\{ (z-i)^2 \cdot \frac{50z}{(z+4)(z-i)^2} \right\} = 8.$
 $\oint_C f(z) dz = 2\pi i b_1$
Residue Theorem: Let $f(z)$ be analytic inside a simple closed path C and on C , except for finitely many singular pts z_1, z_2, \dots, z_n inside C . Then the

So, just I will take one example and then come to our, say for example, if we take the function $f(z)$ which is say our $50z$ over z plus 4 z minus i whole square, suppose I take obviously, z equal to i is a pole, simple pole of order 2 , this is the second order. So, what is the residue of this function $f(z)$ at z equal to i , m is 2 . So, it is 1 over factorial m minus 1 becomes 1 the 1 m is 2 , so derivative d by $d z$ of this. So, simply it is a limit, z tends to i d by $d z$ and then z minus z_0 , z_0 is i . So, it is i z minus i m is 2 f of z f of z is $50z$ over z plus 4 z minus i whole square and then take this. So, first you this will can say and then differentiate it, after differentiating and substituting the value, you will get the answer is 8 , so that way we can find. So, if the function is given and it is asked to find the residue, it is easy to apply either this one or this one.

Hence, if integral of the $f(z)$ is required along the path C , along a close curve C which encloses the point z_0 as a singular point, then one can easily find out value of this as $2\pi i$ into b_1 as we have discussed earlier. However, if this curve C enclose is more than one points, here the function is not defined, function as a singularity at more than one point; then in that case, this formula will not help much, it means everywhere first we have to identify the region where the function is analytic, apply the Cauchy theorem to find out the integrals and then only we can get the something about the value of this integral.

So, the case is given by residue theorem, this result is given by residue theorem, this is known as the residue theorem. What the residue theorem says, let $f(z)$ be analytic, $f(z)$ be an analytic inside a simple **inside a simple** close path **close path** C and on C , except for finitely many **for finitely many** singular points **finitely many singular points** z_1, z_2, \dots, z_k inside C .

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Handwritten notes on a blue background showing the residue theorem formula and a diagram of a contour C enclosing multiple singular points.

The formula shown is:

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f(z)$$

Below this, the formula is repeated with a summation over j :

$$\oint_C f(z) dz = \sum_{j=1}^k \oint_{C_j} f(z) dz$$

Then, it is shown that:

$$= 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f(z)$$

To the right of the equations is a diagram of a closed curve C (an oval) with an arrow indicating a counter-clockwise direction. Inside the curve, there are three points labeled z_1 , z_2 , and z_k , each enclosed by a small circle, representing singular points.

Then, the integral **integral** of the function $f(z)$ taking counter clock wise **clock wise** direction around C **counter clock wise around C** equals $2\pi i$ times **$2\pi i$ times** the sum of the residue **residue** of $f(z)$ at this point z_1, z_2, \dots, z_k , that is the meaning is this, that integral of function $f(z)$ along the path C where C is close curve where the function is analytic everywhere, except possibly these k points which has a singular point. Then the

value of integral, this will be $2\pi i$ times sum of the residue $\sum_{j=1}^k$ residue of the function $f(z)$ at z_j .

The proof is very simple, what we do is, we replace this, remove this z_1, z_2, \dots, z_n by means enclose the z_1, z_2, \dots, z_n by a circle and let the direction of this, we suppose a clock wise **clock wise**, then if I go along the path C , then what happened that when we move along this curve along this direction, and then come back again, this direction like this. So, what we get it, **we are getting a region** we are getting a region where the function is analytic.

So, integral of this curve C outer boundary plus this integral must be 0, but here the integration is taken in clock wise, here it is in the anti clock wise. So, we get from here is that integral of the function $f(z)$ along the outer boundary will be sum of the integral of the function $f(z) dz$ along the inner boundary say C_1, C_2, \dots, C_k , this is the sum, k equal to 1 to say j equal to 1 to k .

Now, this is integral is taken along a curve which contains only one singular point, hence by the previous result, the integral of the function $f(z)$ along a curve C which encloses is only singular point is $2\pi i$ times the residue of the function at this point. So, this is equal to $\sum_{j=1}^k$ **1 to k** residue of the function $f(z)$, $f(z)$ equal to $\frac{4-3z}{z(z-1)}$ and $2\pi i$, so this proves the result.

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$$\oint_C \frac{4-3z}{z(z-1)} dz$$

$$f(z) = \frac{4-3z}{z(z-1)}$$

Singular
 $z=0, 1$

$$= 2\pi i \left(\text{Res}_{z=0} + \text{Res}_{z=1} \right)$$

$\begin{matrix} \text{Res}_{z=0} & \text{Res}_{z=1} \\ \circ & \circ \end{matrix}$

$\rightarrow 2\pi i \text{ Res}_{z=0}$

$\rightarrow 0$

$\begin{matrix} \text{Res}_{z=0} & \text{Res}_{z=1} \\ \circ & \circ \end{matrix}$

Just one example I will give and then finish it, suppose I take this integral, $\frac{4 - 3z}{z^2 - z} dz$. So, for this integral along the path C where C is such enclose is the point 0 and 1 and so on. If we look the function $f(z) = \frac{4 - 3z}{z^2 - z}$, it has a singularity z is 0 and 1 these are the two singular point at this, z equal to 0 and z equal to 1, both are the simple pole only.

Suppose, the C encloses both the point 0 and 1, then the value of integral will be $2\pi i$ times the sum of the residue at the residue at z equal to 0 plus residue at z equal to 1. If it encloses only one point and 1 is outside, then the value will be equal to $2\pi i$ times residue at z equal to 0, 1 is here and third case, suppose it both the point 0 and 1 is outside, then integral along this curve will be 0, because the function becomes analytic.

So, only thing is we have to compute the residue at 0 and 1, and residue at 0 means multiply this by z , take the limit tends to 0, we can find the residue at 0, when z is 1 multiply by $z - 1$, take the limit z tends to 1, you get the residue of them, hence we get them. So, what we conclude that, the integral when you find the integral of the function around the close path, we have to see that the function $f(z)$, what are the singular point of the function and whether they lie, where they lie, if they lie inside, then accordingly the formula will be there; if there lies outside, accordingly the value will come, or 1 lies inside, or 1 lies outside, then also are called, then means every, in all cases we are able to apply the residue theorem to get the value of the integral. Thank you very much, thanks.