Advanced Engineering Mathematics

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Lecture No. # 20

Tayor's, Laurent Series of f(z) & Singularities

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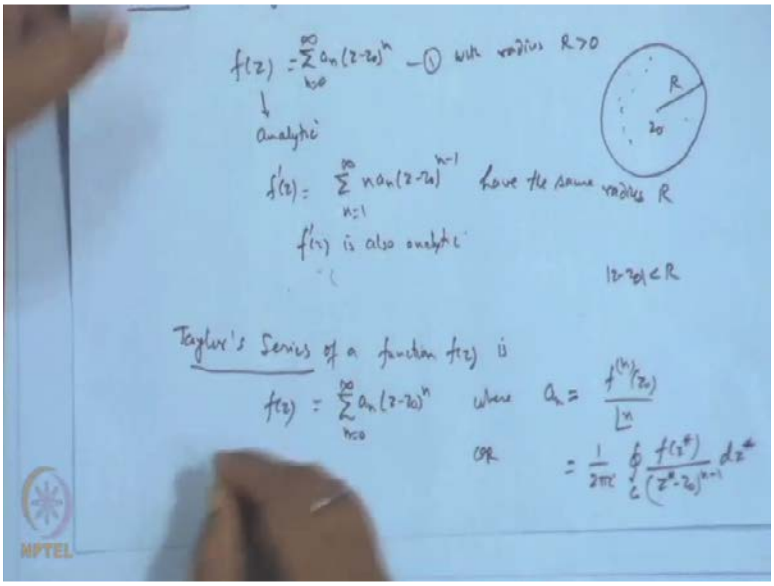

So, in the last lecture we have discussing about the power series and what we have seen power series with non radius of convergence a n z minus z naught to the power n, n is say 0 to infinity, this is the power series with a non radius of convergence with radius say R, which is greater than 0.

So, it has centre at z naught radius R, so a power series with a non 0 radius of convergence will always represents of function f z, which is analytic at each point inside this circle of convergence at each point inside the circle of convergence. Now even also if we differentiate this term and we find the derivative of function then it becomes it comes out to be a power series again and that power series will be sigma n a n z minus z naught to the power n minus 1 and then n is 1 to infinity. Now this power series will also have the same radius have the same radius R as the original one and the function f will be analytic function f prime z is also analytic, so in this way if I keep on differentiating the power original power series one then continuously we are getting the successive derivative and each derivative will exist will be a analytic function inside the region of convergence mod z minus z naught is less than R.

So, this the converse of this that, if a function f is given to be say analytic then also one can find out the corresponding the power series, if the function is analytic then one can also find the corresponding power series and that is given by the with help of Taylor's formula of the function. The Taylor formula says Taylor's series a series of the form Taylor series of a function f z is basically a series sigma a n z minus z naught to the power n is a power series, where the coefficients a n can be computed with the help of this threshold. The derivative of n at the point z naught divide by factorial n or equivalently we can say a n is equal to 1 by 2 pi i integral along the curve along the curve c f star z over z star minus z naught to the power n minus 1 dz star, where the integration is taking the counter clock direction along the path c where this is our f z.

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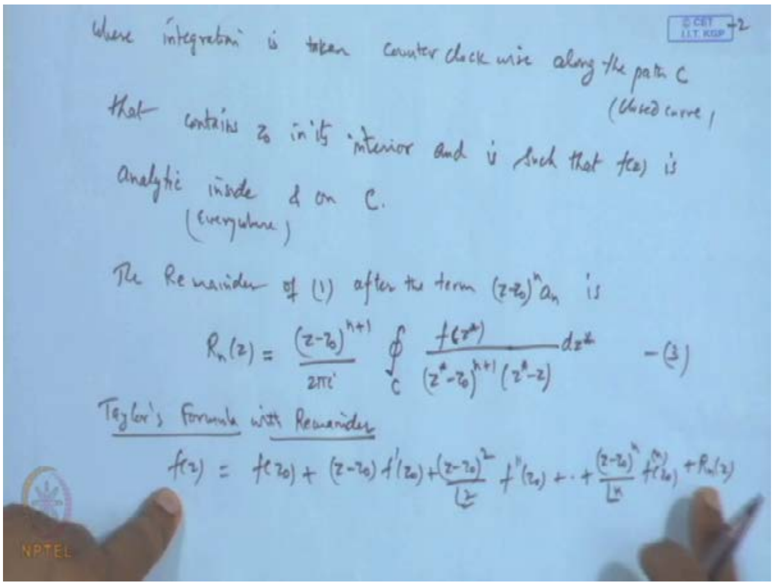

Where the integration is taken counter clock wise along the curve along the path C across curve C in along the path C that contains this closed curve; that contains the point z naught in its interior in its interior and is such and is such that f z is analytic inside and on inside and on C inside everywhere and on C, so this is given by the Taylor series is basically a power series Taylor series is function is a power series, where the coefficient n can be computed with the help of this thresholds or f 1 z equivalent to this one f 1 z factorial n is nothing but, one upon this 2pi integral c upon this, where integration is taken along this close path C contain this; which contains the point z naught in its interior and the function f z is analytic inside and outside the remainder of this remainder of this series one this is one said remainder of this series one.

Taylor series remainder of series one after the term z minus z naught to the power n into a; n is denoted by R n z and basically is equal to z minus z naught to the power n plus 1 over 2pi i integral along the path C f z star divide by z star minus z naught to the power say n plus 1 z star minus z into dz star, let it be this equation a n this is given said two and this one is given by 3 say now when we write this equation this function f z in the form of this series first few and terms plus the remainder then this form is known as Taylor's formula with remainder.

So, we get the Taylor 's formula with remainder be mean a function f z whose Taylor formula with remainder f z is nothing but f z naught plus z minus z naught f prime z naught plus z minus z naught whole square over factorial 2 f double prime z naught up to say term z minus z naught to the power n over factorial n f n the diagonal derivative of this function n at a point z naught plus the remainder term R n z, where R in z defined by three and this is called the Taylor's this known as the Taylor's formula of the function f z, which is analytic with in remainder R n z.

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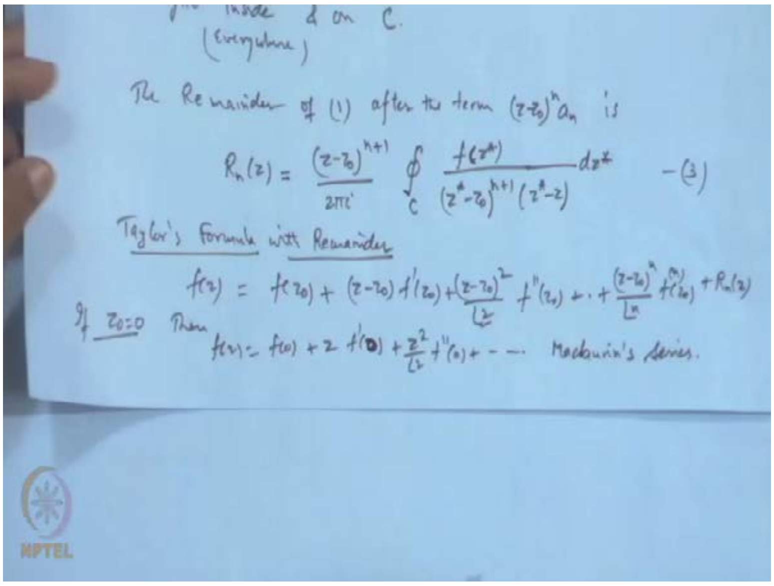

Now, particular case, if z naught equal to 0 then this known as the Taylor's series with remainder z naught is 0; if z naught is 0 then the equation one then the expression f z, which is f 0 plus z f prime 0 plus z square by factorial 2 f double dash 0 and so on is known as the Maclaunin's Maclaunins series, so Maclaunin's series is particular case of the Taylor series when the centre is taking to be 0 like this. Now every analytic function can be represented by Taylor series.

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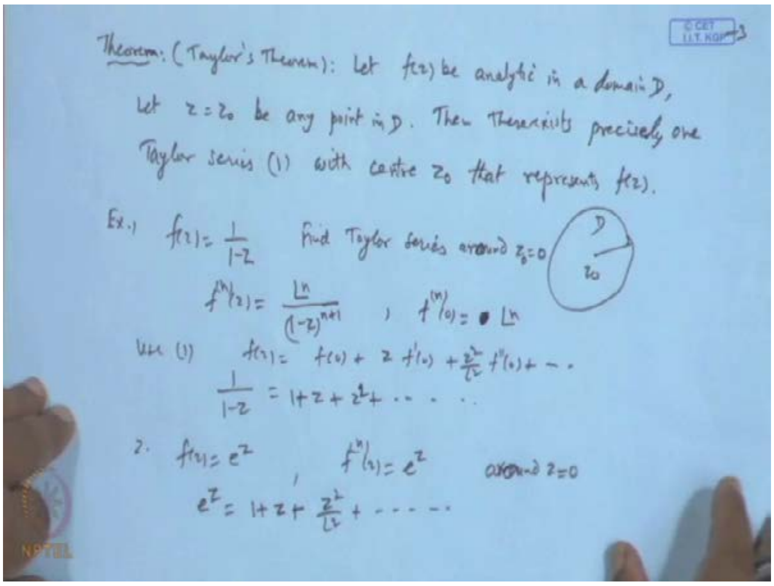

This is given, so we claimed that threshold, which is converge of previous one what threshold says; which is also known as the Taylor's series theorem threshold is let f z be analytic analytic in a domain D in a domain D and let z equal to z naught be any point in D and function is analytic in domain D.

So, obviously it is so analytic at point z naught in D then there exist the threshold says then there exist precisely there exist precisely prescisely one Taylor series of the form one this is equation, one which given with centre z naught with centre z naught that represents the function f z; the function f z this representation is valid in the largest represents be centre z naught in which efficient rating, this representation of course, what is the idea this z naught here this is the domain D the function is analytic of this domain D.

So, we can explain this function; we can corresponding to this function f z; We can always get Taylor series expression in equally precisely want Taylor series corresponding to represents f z and the domain will be in the largest circle, where the function is analytic with centre z naught the remainder of this R n z is defined by 3 and so on, the proof we are not given for this so proof we (( )) , but how do find out the Taylor series that see the various example, where to construct the Taylor series with a for the given function f z.

So, let see the example, suppose the function f z 1 over 1 minus z, we want the Taylor series expression around the 0.0. Find Taylor series at around z equal to 0 z naught equal to 0 or about the point z equal to 0, so this function f z 0 according to the Taylor series, if we go to the formula; the formula says the Taylor series of the function f z is of the form a n z minus z naught to the power n, where the coefficient a n is given by f z over factorial n, if I differentiate this function n times.

Then threshold we get the value as factorial n over 1 minus z to the power n plus 1 n plus 1 say n and then the value of this Taylor series around z naught will be 0, so f n 0 will be 0 f n 0 will be 0 and then we know f n 0 will be factorial n because this z is 0. So use the one and once you use for then f z equal to f 0 that is f z equal to f 0 plus z f prime 0 z square by factorial 2 f double prime 0 and so on. So if you substitute this value, we are getting this is 1 plus z plus z square and so on, this is the Taylor series expansion of the function 1 over 1 minus z around z equal to 0. Similarly, if we take this function f z is e to the power z, then we know the derivative of this f n e to the power z is the e to the power z whatever this and corresponding expression around the point around z equal to 0 around around the point z equal to 0 will be 1 plus z plus z square over factorial 2 z cube over factorial 3 and so on.

So, this will be the means, we can apply the derivative formula the Taylor's series expression around that point but, sometimes we need not to apply we need not to apply this formula to get Taylor's series expression of the function just by using the knowledge of binomial theorem or may be some other method, we can also write the Taylor's series expression of given function f z, because the threshold says that for a given function f z there will be uniquely after side, we want Taylor's series expression around the point z not in the circle of convergence.

So, we are feed to take up any method to express the function f z in the form of Taylor's series, so let see the few threshold example, where we are not apply, where not applied that formula just finding the derivative coefficients a n with the help of this formula, simply using some binomial form expression or some other way one can write the expression quickly.

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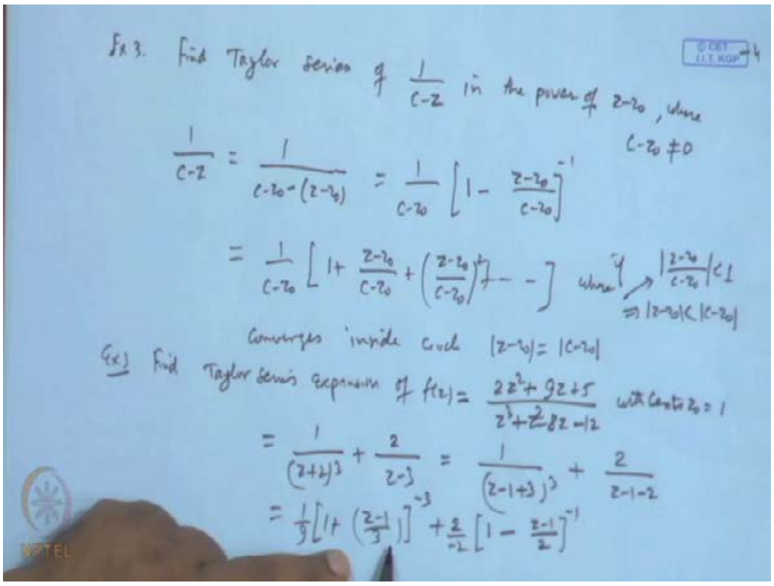

For example, suppose I take I say right now, the expression find Taylor's series expression Taylor's expression of the function one over c minus z in the power of in the power of c z minus z naught, where c minus z naught is different from 0. So what we do is, we write it first c minus z, we write this expression such way so one of the term comes out to be z minus z naught. So if we rewrite this think it can writ as z minus c minus z naught minus z minus z naught. Now, we want in the power of z minus z naught for this integral power.

So, if I take c minus z naught outside because this is different from 0, so what we get is 1 minus z minus z naught over c minus z naught inverse, now if we impose the restriction, if mod of z minus z naught over c minus z naught is restrictedly less than 1 then one can apply the binomial expression and we get easily the expansion as 1 over c minus c naught c minus z naught 1 plus z minus z naught c minus z naught plus is square of this z minus z naught c minus z naught square and so on.

And so forth, where z minus z naught less than 1 it means, this series converges inside the circle mod z minus z naught equal to c minus z naught that is region of convergence is mod z minus z naught is restrictedly less than c minus z naught. So it is this one implies mod z minus z naught is restrictedly less than c minus z naught this will be the radius of convergence of the series like this. Similarly, suppose we get another example, where one can also use it say find Taylor series expression expansion of the function of the function f z, which is say 2 z square plus 9 z plus 5 divide by z cube plus z square minus 8 z minus 12 with centre z naught equal to 1, it means we want it to find the expansion of the function in the power of z minus 1.

So, what we do is first we will write the partial fraction, so if will write the partial fraction of this number then partial fraction will come out to be 1 over z plus 2 whole cube plus 2 over z minus 3. Now since we want in the power of z minus 1, so rewrite this again in the form of z minus 1 plus 3 whole cube plus 2 over z minus 1 minus 2, now if I take 3 outside from here, because we want the power z minus 1 positive power, so 1 by 9 1 plus z minus 1 by 3 and then base to the power minus 3 then from here, if we take since yeah if I take minus 2 outside if I take minus 2 outside, then what happened this minus 2, and then we get 1 minus z minus 1 by 2 power minus 1, so we are getting this way is this 1 plus z minus 1 by 3, and then 1 minus z minus 1 by 2 now apply the binomial again.

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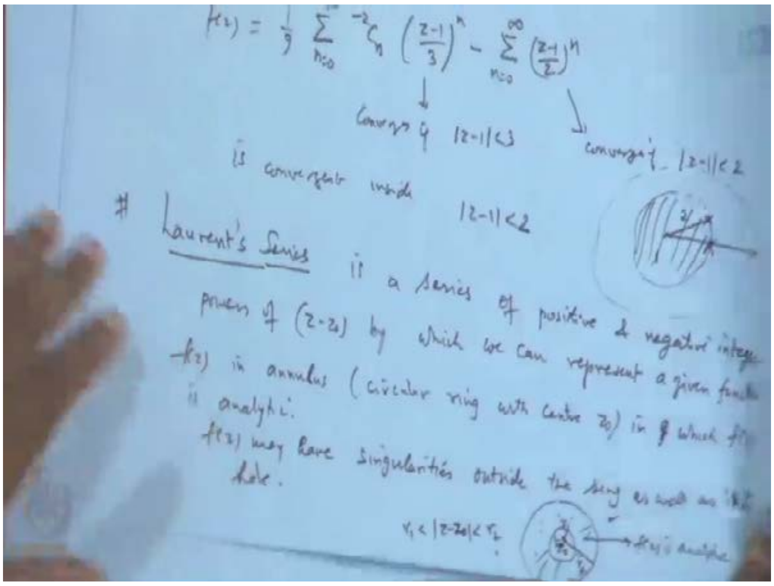

So when's you apply the binomial the value will come out to be this is the same as 1 by 9 and the series will be sigma n is 0 to infinity minus 2 c n z minus 1 by 3 power one and this will be sigma n equal to 0 to infinity z minus 1 by 2 base to the power n that is all. Now the question is what will be the reason of convergence, now this series converges, if mod z minus 1 by restrictedly less than 3; this series converges, if mod z minus 1 restrictedly less than 2.

So, it means centre one with a radius 2 with a radius 3, so if I take this circle then at every point inside the circle the function is analytic, but it has similarities at this point is it know, so we get here at this point. And similarly, when we go this think it has singularities at this point so as soon as the first singularities the region of conversion remains up to here only, so this series one, the expansion we can write it f z is convergent inside this region mod z minus 1 restrictedly less than 2; and that will be the region of this convergence, so this way one can find out the series power series of the given function.

Now in this power series when we say the function is given then what is the condition for this function to get a power series, is the function must be analytic inside the domain D, where the point z naught like this, then only we can expand the function f z positive power of z minus z naught. If function f is not analytic at the point z naught about this, we want it to expand the function then obviously our Taylor series does not help.

So, in that case we have to look some other series, which will be responsible to have representation of that function around the point z naught, where f is not analytic and that leads to concept of Laurent series, so the Laurent series is basically is a series of positive and negative positive and negative integral powers of integer powers of integer powers of z minus z naught, which can be represent, which can represent z minus z naught by which we can represent a given function; given function f z given function f z in annulus annulus in annulus in which circular ring with centre z naught annulus circular ring with centre centre z naught in which in which f z is analytic f z is analytic f z may have singularities may have singularities outside the ring outside the ring as well as as well as in in it is holes, what is the meaning of this, what is says, suppose a function is given and it is not analytic at a point z naught then we cannot express it in the form of power series.

So, what we do is, we remove this z naught and let us find out the region, which is in the form of annulus, where thought this region f is analytic f is analytic, so basically this region is nothing but like this r 1 is nothing but less than mod z minus z naught less than r 2 here, this is say r 1 and here is say r 2; this one so it is annulus it is annulus at centre at z naught, so function is analytic inside this annulus. Now this function may have singularities outside may have also, some singularities here at this we are not both we are both ring about only the point, where z naught, where the function is analytic or not.

So, if function is not analytic, it means this is to be analytic at point, so that point is called as singular point. So once the function have a singular point, then one can expand it in the form of Laurent series in the annulus, which is this of form and in which the function will be analytic, if the series will represent function in this. So let see the threshold is given by the Laurent's and which is known as the Laurent's theorem.

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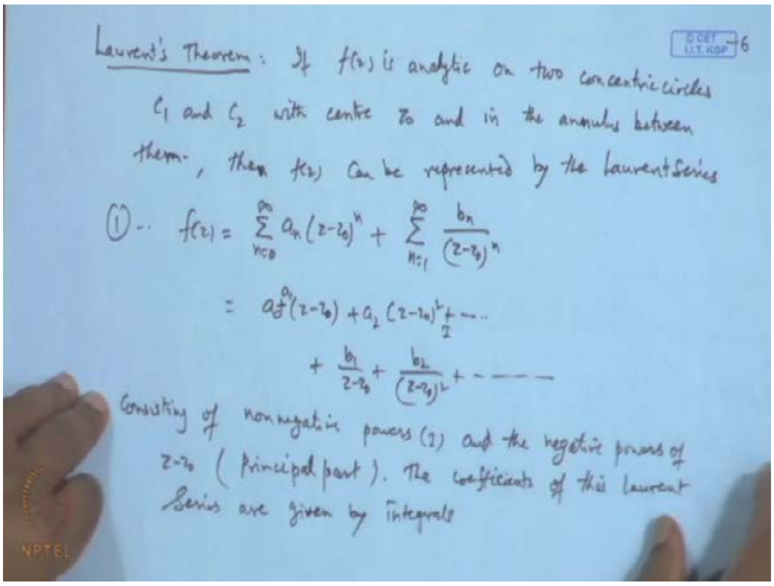

So, what this Laurent's theorem says the Laurent's theorem says, if f z is analytic is analytic on two concentric circles circles C 1 and C 2 with centre with centre z naught and in the annulus between them annulus between them so function not only analytic inside them, but also with in this C 1 and C 2, it is analytic between then f z can be then f z can be represented represented by the Laurent series by the Laurent series, which is of the form sigma n is 0 to infinity a n z minus z naught to the power n plus sigma n is 0 to infinity b n over z minus z naught to the power n or if we rewrite again then we get a not plus a 1 z minus z naught plus a 2 z minus z naught whole square and so on like this plus b 1 over z minus z naught plus b 2 over z minus z naught whole square and so on. This now this series consists of let it be equation one, so f z be represented by means of the Laurent series consisting of non negative powers that is first part first one and the negative powers and the negative powers negative powers of z minus z not that is known as the principal part principal part principal part of this function f z principal part the coefficients.

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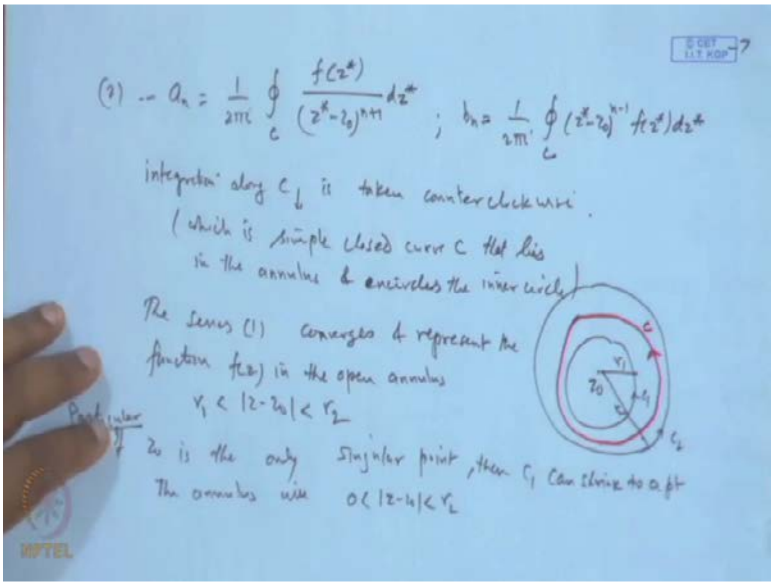

The coefficients of this Laurent series are given by integrals or given by integrals a n ‘s are 1 by 2 pi i integral along the path C f z star divide by z star minus z naught base to the power n plus 1 the z star b n ‘s are given by 1 by 2 pi i integral along the close path C z star minus z naught base to the power n minus 1 into f of z star d z star, let it be two this equation is two the integration is taken, the integration along C is taken counter clock wise counter clock wise C is smooth along C, which is which is smooth simple, which is a simple close path C, which is simple close curve C approx C that lies that lies in the annulus and in circles and in circles inner circle in circle in circle the inner circle inner circle. The counter clock wise direction, where C is a single close curve lies in the annulus and inner circle curve.

That is if suppose this is point z naught, where the function is not analytic, we have the two curves one is C 1, other one is C 2, C 1 and C 2 the C is this curve C is this curve this is C which is which lies inside the annulus and in circles that inner circle C 1 like that, so function a n f z can be expressed in the form of the positive and negative integral powers of z minus z naught, which we call it as a Laurent series expression of this function. Now this power series converges, the series one converges converges and represent the function the function f z represented by the open annulus in the open annulus. So here I will take r 1, suppose and this is say r 2 then open annulus r 1 less than mod z minus z naught less than r 2, where r 2 will be as large till the function at till the function is analytic as soon as it case a first singular point over C 2 then be stop it similarly, when you shrink the C 1 and we can string at the point, so that the function becomes is analytic and as soon it reaches to a some singular point then be stop it, so that is the largest region by shrinking C 1 or extending C 2, the function annulus can be obtained and where the function is analytic.

So that is what now z naught is only singular point inside C can be shrink to a now there can be see some other threshold, so if z naught is particular case, if z naught is the only singular point say then C 1 can shrink to a point then C 1 can shrink to a point and in that case the annulus the annulus so obtained the annulus will be 0 less than z minus z naught less than r 2, this will be annulus and this will be now again we see the proof we are stopping because is time is not permitting for me, but we can goes through proof in the (( )) books we are (( )) so see the proof of this (( )) but, the problems to find out Laurent series expression of this function as I told I have seen in the Taylor series expression.

Then two base I there we can use formula and write down Taylor series expression similarly, here also we can use the formula a n b n computed and then write down expression of the Laurent series as given in the form of and from here, one can find out region convergence or without using the formula two, one can also expand the function f z around the point z naught, where the function is not analytic in the form of Laurent series, so let see the few examples, where we can (( )) use the formula or sometimes without using the formula one can get it.

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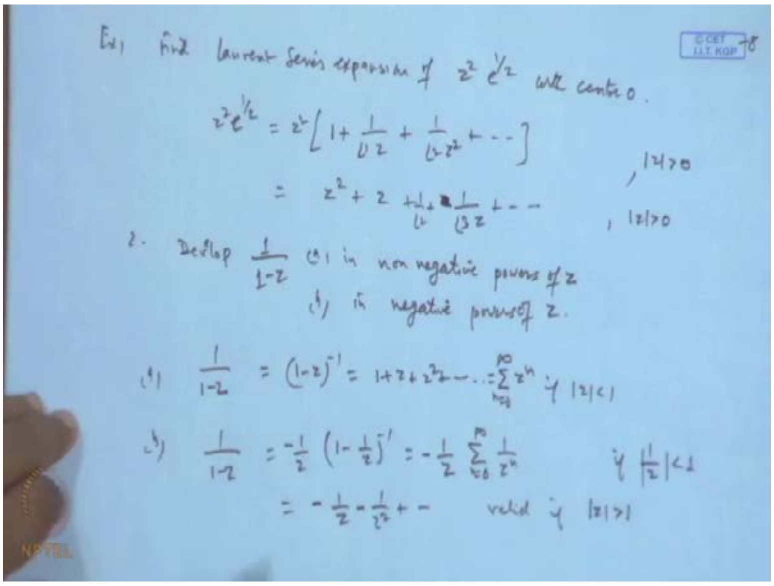

Let see the examples how to computed, find the Laurent series expression find the Laurent series expression Laurent series expression of the function say I say z square e to the power 1 by z with centre z 0, now obviously the function z square e to the power 1 by z at z equal to 0 is not analytic exist to be analytic 0 becomes singular point, so what we do is, we want the expression of this Laurent series expression in the around the 0.0, so it will have both positive and negative power of z.

So, z square already given in positive power, we will not touch it, what we do is, we will write to the e to the power 1 by z, but in case of the Taylor series expression for e to the power z they already seen e to the power z is 1 plus z and so on say, if we reply z by 1 by z then one can easily write now this expression without much problem, so just reply z by 1 by z here, so here we are getting 1 plus 1 over factorial 1 z plus 1 over factorial 2 z square and so on with mod z greater than 0, this is annulus, so what expansion will be z square plus z plus z 1 by factorial 2 plus say 1 by factorial 3 z and so on, this will be the expression valid in the region mod z greater than 0.

So, this will be the expression, now another example, let us see develop develop 1 over 1 minus z 1 in non negative power of z and second part is in negative power of z powers of z m, so let see 1 over 1 minus z, we want the non negative powers of z, it means if I assume mod z less than 1 then obviously this can written as 1 minus z inverse, if mod z is less than 1 then we can expand this and expand this form of the series, which is 1 plus z plus z square plus and so on.

This is the expression of the series, if mod z less than 1, second part is we want in negative powers of this so negative power of z means we want to take z outside, so if I take z outside then what we get it now again this will be z outside 1 by z, so take the minus sign outside we get 1 minus 1 by z inverse. And if I put in restriction that 1 over z with mod sign is less than 1 that is the mod z is greater than 1, then this expression will be valid as minus 1 by z and here also this is the series like sigma n is 0 to infinity. Earlier series is, if we remember z n, now here what is the difference z is represents 1 by z, so sigma n is 0 to infinity 1 by z to the power n and valid is it means this expression will be minus 1 by z minus 1 by z square and so on and valid if one mod z is greater than 1, so this way we can there, and then let it be a b now use this partial fraction.

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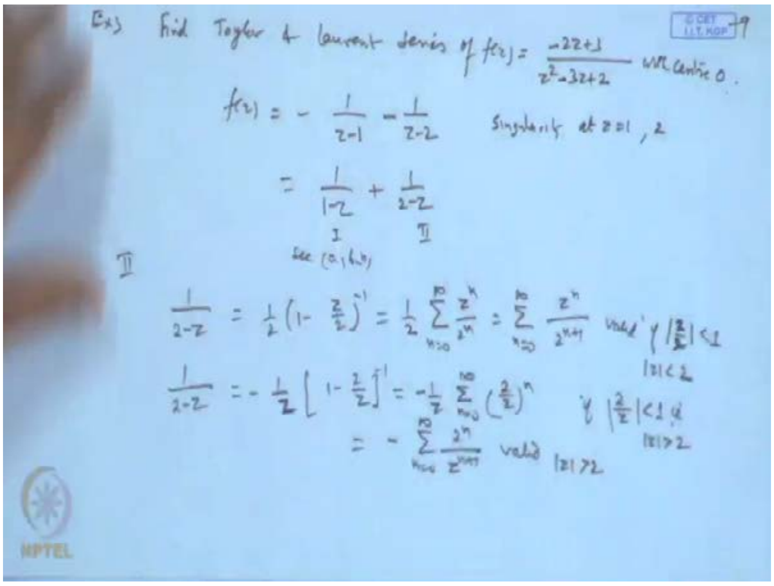

Suppose another example says write down the Taylor series expression of the function find find Taylor and Laurent series expression of the function f z, which is minus 2z plus 3 divide by z square minus 3z plus 2 with centre with centre 0 original, so what we do is because this is a fractional function. So first we will write down the partial fraction of this and then we apply our deal, so if we write the partial fraction of this the partial fraction of this comes out to be 1 over z minus 1 with minus sign minus 1 over z minus 2, it means this expression around the point say 0 with positive and negative powers of z equal to 0, which already we have discussed, because this is same as 1 over 1 minus z plus 1 over 2 minus z with 1 over 1 minus z. We have already discuss this part a and b if mod z less than 1, it has a positive power when mod z greater than 1, it has a negative power of this is it not.

So, we can get it for expression however for the second part of this, so second part for some first part is done see a and b and second part for the second part of this if we write 1 over 2 minus z in the positive part of the problem take two outside, so we get 1 minus z by 2 inverse. Now if we make the restriction mod z by 2 is restrictedly less than 1, then it can be expand it; and we can write now expression again in the form 1 by 2 sigma z to the power n by 2 to the power n that is equal to sigma n is 0 to infinity z to the power n over 2 to the power n plus 1, if valid if mod z is less than 2 this is mod and if I take it say z outside then we can say minus 1 by z outside, then we get 1 minus 2 by z inverse.

And if we put the restriction that if 2 by z mod z less than 1 that is mod is greater than 2 then again we can apply the binomial expression and we get sigma n is 0 to infinity 2 z to the power n and that comes out to be series, which is minus sigma 2 to the power n by z n plus 1 n is 0 to infinity valid for mod z greater than 2. Now if we take now, what we are interesting is we are interested find Taylor series and Laurent series expression. So for this mod z less than 1 and for mod z lying between one and two, because there are two singularities; this function if I look the function f z, here it has a singularity at z equal to 1 and z equal to 2, these are singular point.

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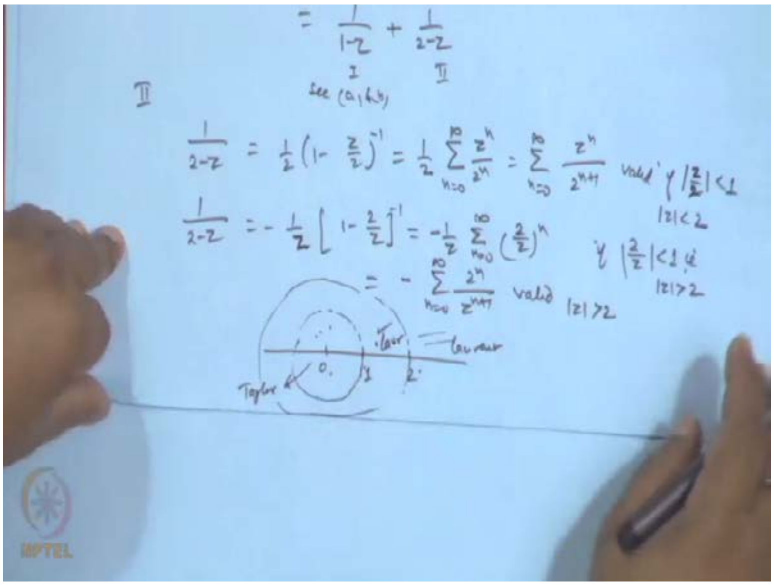

So, what we this point this is 0, this one is 1, this is 2. The function has a singularity, so from this this mod z less than 1, there is no singularity in it, so we can write down the Taylor series expression, because no singularity, but from 1 to 2 1 to 2 if I look, then this region function is analytic, but it is annulus, so we can write down the Taylor series expression of this. Again when two is greater than 2 again this is annulus, so it we can write down the Laurent series sorry in this case, we can write Laurent series expression here also we can write the Laurent series expression. So basically three reason; one mod z less than 1, we can write in mod z less than 1.

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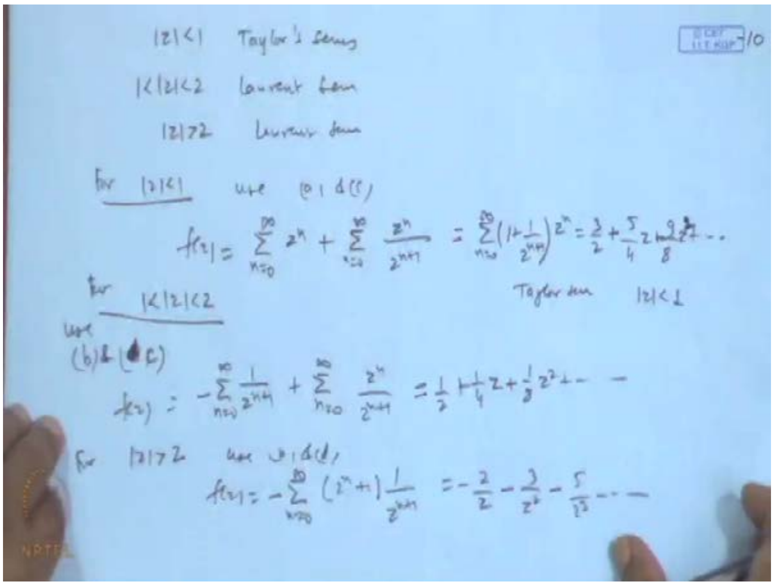

So, what we get in mod z less than 1 Taylor's expression Taylor's series, because there is no singular inside it then for 1 less than mod z less than 2 Laurent series expression and then mod z greater than 2, it is also be an annulus, so we can also write another one series expression.

So, let us see all the three cases, what we get it, so in the first case mod z less than 1 mod z less than 1, we have already discuss d in the part, let it be c and d this part is c; and this part is d. So when we take mod z less than 1, the a part if I look mod z less than 1, this expression is sigma z to the power n, but for mod z less than 1 also available here in c; so in c the for expression is this, so basically for this expression can be given from a for this expression can be consider for c.

So, if I take the a and c so for mod z less than 1, use a and c; and if you use a and c, the expression of the f z will be sigma n is 0 to infinity z to the power n the first one is a shows what this is our a yeah this is a sigma z to the power n for this and then second part will be for this mod z less than 2, which is also mod z less than 1. So sigma of n 0 to infinity z n over 2 to the power n plus 1 and this is valid for this, so we get the series as sigma n is 0 to infinity 1 plus 1 by 2 n plus 1 z to the power n and if you expand it this will come out 3 by 2 plus 5 by 4 z plus 9 by 8 z cube z square and so on and so forth.

So this is the Taylor series expression Taylor series expression inside the circle mod z less than 1, because this function f z is consist of two parts; one is this, other one corresponding to this we get this expression corresponding to this we have this expression inside the region now for the region 1 less than mod z less than 2 when one more less than less than it means this when one is mod z is restrictedly greater than 1 the series 1 over 1 minus z has this expression that is this part this part has the expression minus 1, so we can get from a from d and b b and d use this. So from b we are getting the series b and sorry b and c, because mod z less than 2, so c is there so b gives you sigma this is b b gives the series sigma minus sigma n is 0 to infinity z n plus 1 this is our f z minus this part and then c this c will give c it gives mod z less than 2, it gives sigma z n over 2 n 2 minus z, where is that function function is two minus z.

So, it will give the c sigma n is 0 to infinity z n over 2 n plus 1, so basically we are getting n plus 1 z n over 2 n plus 1 and so on, if you expand it we are getting half plus 1 by 4 z plus 1 by 8 z square and so on and so forth. This will be the expression for this then third case for mod z greater than 2. So mod z greater than 2; what are the possibilities? The d is available and also here b is available, because mod z greater than 1, so use b and d, so if I use b and d f z is written as minus sigma n is 0 to infinity 2 to the power n plus 1 1 by z n plus 1 and that this will be minus 2 by z 3 over z square and minus minus 5 by z cube and so on and so forth. So this way we can find out the Laurent series expression of the function.

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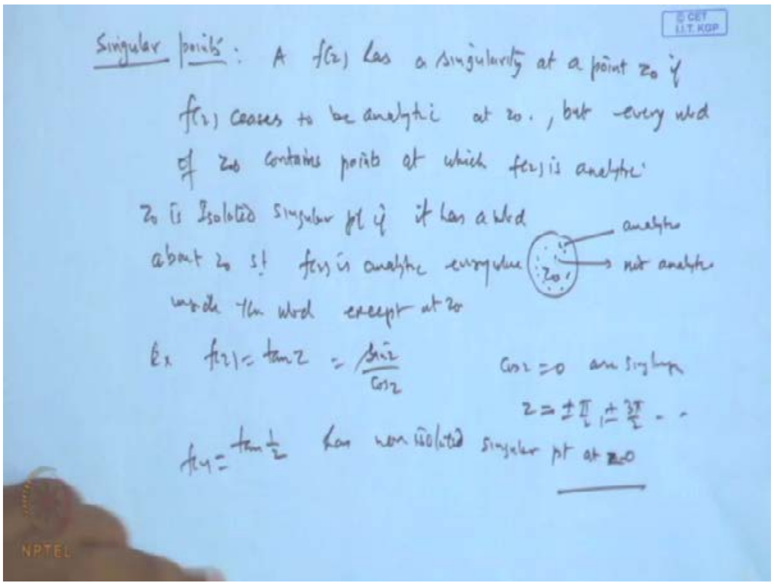

So, this is the way to compute Laurent series. Now let us come to the singularity, which is also singular points a function f z has a singularity at a point z naught a function f z has a singularity singularity at a point z naught if f z ceases to be analytic ceases to be analytic at the point z naught it means the function is not analytic is not differentiable of function is not defined then we say z naught is a singular point for this function. And so if a function z naught is a singular point that is if you cease to be analytic but, at every point but, every never would of z say contains points and which f is analytic.

So, z naught is z a point if I take any never would of the point z naught, then the function is not analytic at z naught. But it contains the point, where the function is analytic then we say z naught is a singular point, z naught is called as isolated singular point, if there exist never would, if has a never would isolated point, if it has never would about the point z not such that f z is analytic everywhere inside the never would except at z naught then we say z naught is an isolated singular point.

We will discuss the Laurent series expression about the singular point, which are isolated there are the points, which are non isolated has a for example, if we take function f z say equal to certain z then it is sin z cosine z. So, the point, where the cosine z become 0 are the singular point that is z becomes all the point plus minus pi by 2 plus minus 3 pi by 2 and so on, these are all singular point. But if I take tan of 1 by z, if I take this 1, then this sequence has a non isolated singular point, then it has non-isolated singular point at z equal to 0 at z equal to 0. So this we will discuss it next time, when you go for that thank you very much.