Advanced Engineering Mathematics

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Lecture No. # 02

Vector Spaces, Subspaces, linearly Dependent/Independent of Vectors

Now we are ready to define a vector space or a linear space or which is the basic object of a linear algebra.

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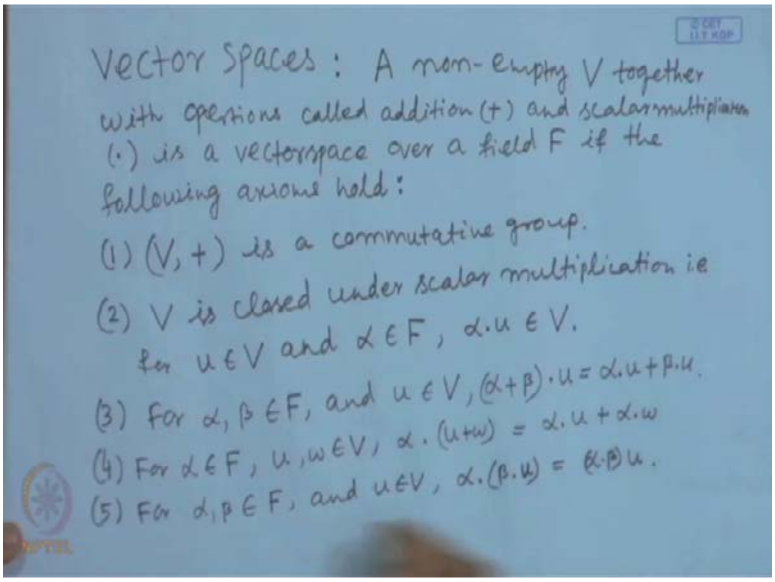

So, we will say these vector spaces. So, this is also starts with a non-empty set with some operations. A non-empty set given together with operations called addition and scalar multiplication scalar multiplication is a vector space over a field F, if the following axioms hold: first one is this; V with respect to this addition operation is a commutative group. Second property is that, V is closed under scalar multiplication; that is, for any element u in V and alpha belongs to F, this alpha dot u belongs to V. Third property is this, for alpha, beta in F and u belongs to V, alpha plus beta dot u is equal to alpha dot u plus beta dot u. Fourth property is for alpha belongs to F, and elements u and w in V, alpha dot u plus w is equal to alpha dot u plus alpha dot w. Fifth property is that, for alpha beta in F, and u in V, alpha dot beta dot v sorry beta dot u this equal to alpha dot beta u. And this sixth axiom is this, 1 dot u is equal to u, for every element u in V, and 1 is the multiplicative identity of F, identity of this field. So, if a non empty set together with two operations called addition and scalar multiplication, that is called a vector space over a field F if this axioms hold.

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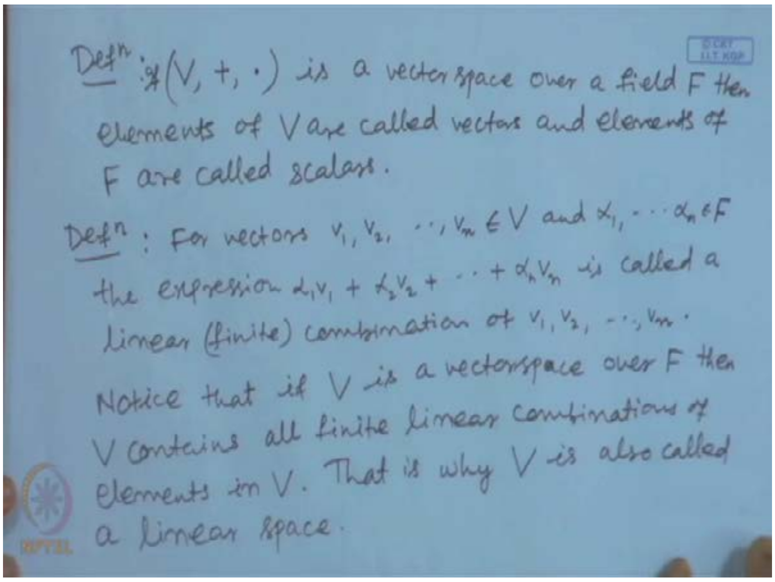

Here, whenever this V together with addition and scalar multiplication is a vector space. So, if this is a vector space over a field F then, elements of v are called elements of V are called vectors, and elements of F are called elements of F are called scalars.

So, we will have also linear combination of vectors that, for vectors v 1, v 2, v n in V and alpha 1, alpha 2 to alpha n in F. The expression alpha 1 v 1, plus alpha 2 v 2 plus alpha n v n is called a linear. Of course, this is also made this set as finite linear combination of v 1, v 2 to v n. Notice that, if V is a vector space over F then, V contains all finite linear combinations all finite linear combinations of elements in V. That is why V is called a linear space. that is why v is also called a linear space

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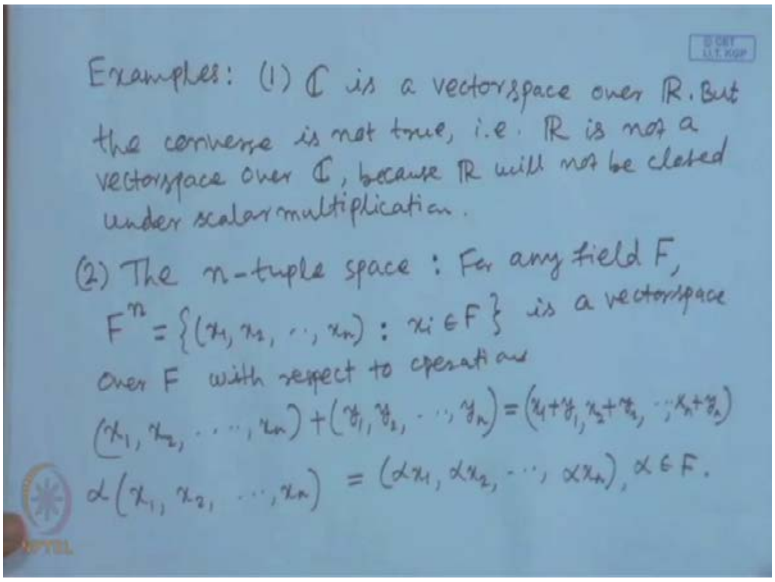

Next, we will have some example of vector spaces. So, the first example that is very natural one is that, set of complex numbers C is a vector space over the real field R, but the converse is not true, but the converse is not true that is R is not a vector space over C, because R will not be closed under scalar multiplication.

So, the second example is basically, that is called the n-tuple space the n tulle space that is, for any field F, for any field F to the power n that is Cartesian product of F with itself, n number of times that, F to the power n consist of all n tuple that x 1, x 2 up to x n such that, each x i is an element of F. This is the F to the power n is a vector space over F with respect to the operations. That for any two elements that addition is defined like this: x 1, x 2 to x n plus y 1, y 2 to y n is equal to x 1 plus y 1, x 2 plus y 2, like this x n plus y n coordinate wise addition. And scalar multiplication is alpha times this x 1, x 2, x n that is equal to alpha x 1, alpha x 2, alpha x n for any alpha in F.

So, here we can observe that, this R to the power n is also vector space because. So, as a particular case as a particular case as a particular case R to the power n is a vector space over R, this R to the power n is also called that n dimensionally Euclidean space.

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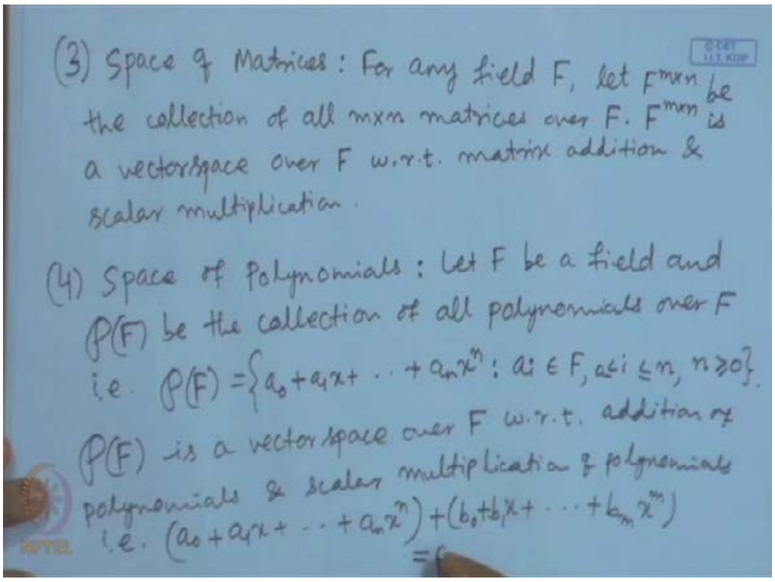

Next, we will have this example, third example that is, space of matrices: space of matrices, that collections of some matrices also form vector space. So, here for any field F, let us denote that F to the power m into n be the collection of all m into n matrices over F. Then this F to the power m into n this is a vector space over this field F with respect to matrix addition and scalar multiplication. with respect to matrix addition and scalar multiplication

So, next we will have this fourth example, it is space of polynomials: space of polynomials. So, let F be a field and P(F) be the collection of all polynomials over F that is, P(F) is this set of all polynomials a naught, plus a 1 x, plus a n x to the power n, such that each a i that comes from this field F, i from greater than or equal to naught, less than or equal to n, and this n is an integer that is greater than or equal to naught. So this, P(F) will be the collection of all this polynomial. So, over this field F is a vector space, over this field F with respect to addition of polynomial and scalar multiplication of polynomials with respect to addition of polynomials and scalar multiplication of polynomials that is, you were having two polynomials that a naught, plus a 1 x, plus a n x to the power n plus another polynomial say b naught, b 1 x, plus b m x to the power m then, their sum is the polynomial c naught, plus c 1 x up to this c k x to the power k.

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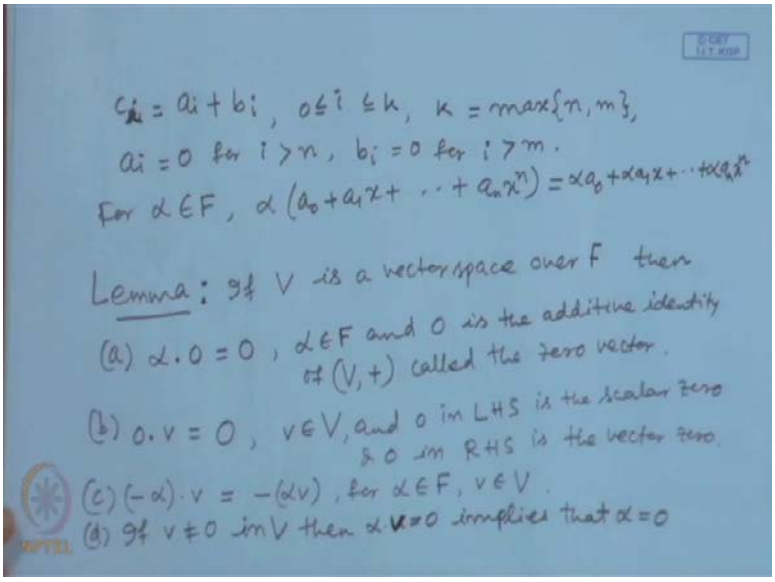

Where this c k is equal to for this c i is your c i is equal to a i plus b i, for i greater than or equal to naught, less than or equal to k and k is equal to maximum of {n,m} and of course, this a i is equal to naught for i greater than n, and b i is equal to naught for i greater than m. And for any scalar alpha alpha, in to any polynomial that a naught, plus a 1 x, plus a n x to the power n is equal to alpha a naught, plus alpha a 1 x like this alpha a n x to the power n. So then, we can have some natural properties of a vector space that we will write in a lemma and then of course, the proof of this lemma is not difficult to check and one can take this is an exercise.

So, these properties are very trivial, but useful we use them again and again. So, if V is a vector space over a field F then, we will have the following results. A is this: alpha dot naught is equal to naught, for alpha belongs to F and naught is the additive identity of additive identity of (v,plus) called the naught vector. Second property we can have is this, this naught dot any vector v that is equal to naught, for any vector v in the vector space, and naught in L H S is the scalar naught and naught in R H S is the vector naught. Third property we can having this, minus alpha into v is equal to minus alpha v, for any alpha in the field F, and that v in the vector space. Fourth property you can having this, if v is not equal to non naught vector not equal to naught or a non naught vector in v, then alpha dot u sorry alpha dot v is equal to naught implies that alpha equal to naught; that means, for any non naught vector v if this alpha dot v, v equal to naught then alpha equal to naught for any alpha in F.

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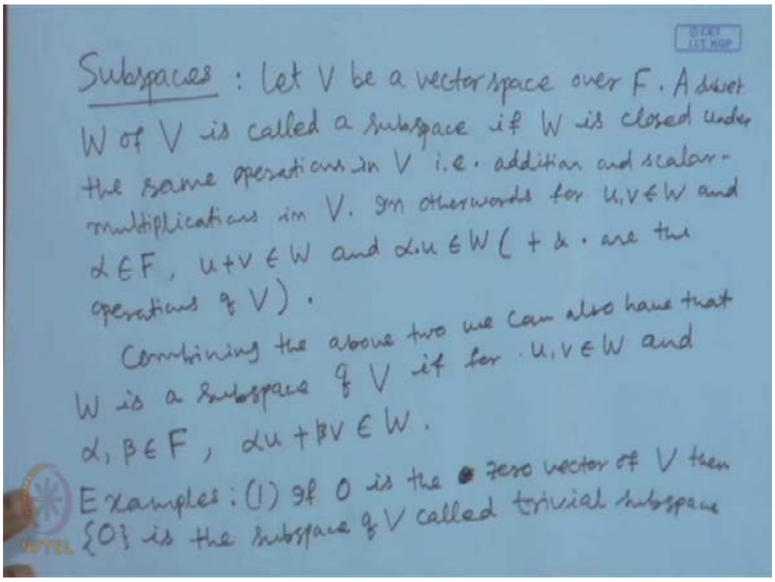

So, the next we will write this subspaces concept, whenever we are having some algebraic structure we have it is sub structure. So, we will have these subspaces. So, let V be a vector space over a field F. Then a subset a subset W of V is called a subspace. If W is closed under the same operations same operations in V that is, addition and scalar multiplications scalar multiplications in V. In other words in other words for u, v belongs to W and alpha belongs to F, u plus v belongs to W and this alpha u belongs to W (plus, and this dot plus, and, dot are the operations of V). Operations of V; One can also combine this two and write in a single condition that combining the above two we can also have that W is a sub space of V if for u, v belongs to W and alpha, beta belongs to F, alpha u plus beta V belong to W.

So, let us see some example of subspaces, let us set some example of sub spaces sub. So, that trivial subspaces are if this naught is the naught vector of is the naught vector of V then, this set consist of the naught vector only is the this is a sub space this is the sub space of V called trivial called trivial subspace.

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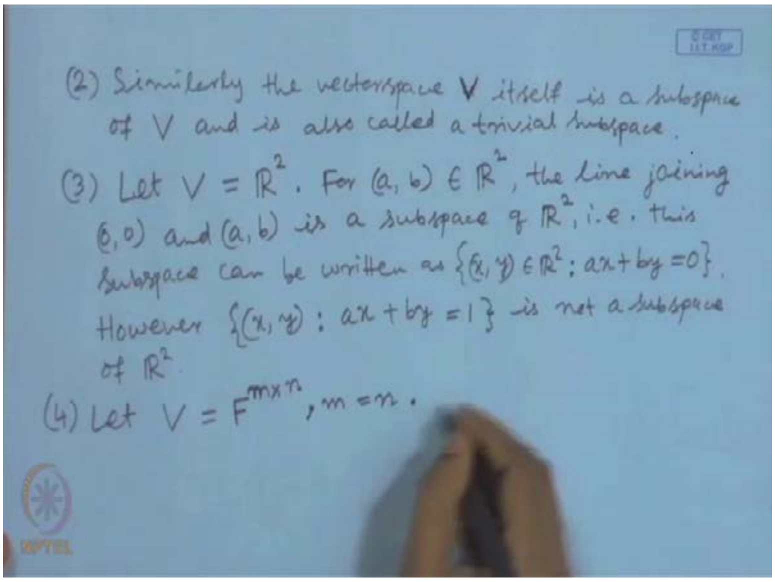

Similarly similarly the vector space V the vector space V itself is a subspace subspace of V and is also called a trivial subspace. So, next will have another example, let us consider this R to the power 2, we will take this Euclidean plane R to the power 2 then, this for any point (a, b) in R to the power 2, the line joining the origin (0, 0) and this point (a, b) is a subspace of R to the power 2, that is; this subspace can be written in set of all (x, y) in R to the power 2 such that a x plus b y is equal to naught; however, if we consider any other line that is not not passing through the origin; however, this line that collection of all (x, y) such that a x plus b y is equal to one, this is not passing through origin is not in subspace of R to the power 2.

So, next we will have different kind of vector space and sub spaces, that is they are matrices. So, now you consider this set of all m by n. Let V be the space of this n-tuple no we will sorry. We will consider this V with this set of all m into n matrices over F, then the collection of all with this m into n matrices with m is equal to n then, this set of all m into n symmetric matrices over F form a subspace of V. then the collection of all n by n symmetric matrices over F forms a sub space of this V.

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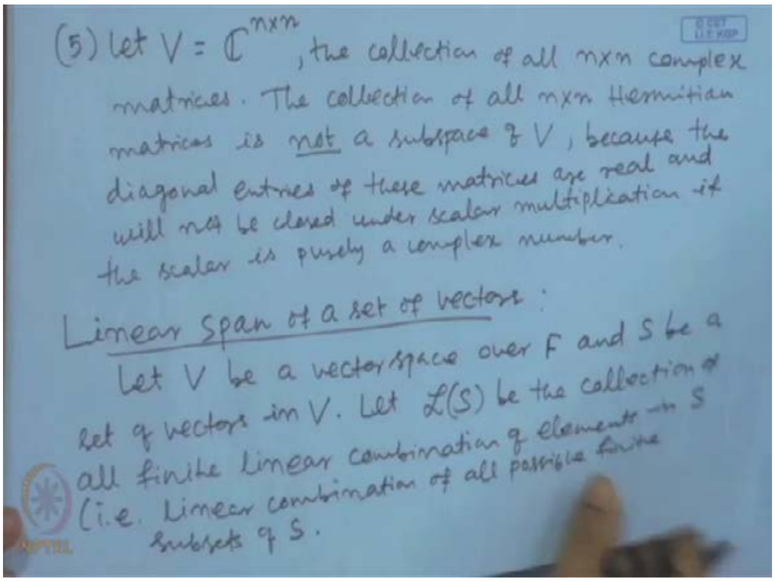

Similarly we will write another example that, fifth example, that the set of all hermitical matrices. Let, again we will consider this vector space V be C to the power m into n or in other words, the collection of all n into n complex matrices. This is the collection of all n into n complex matrices. Then here the collection of all n into n hermitian matrices hermitian matrices is not a subspace not a subspace of V, because the diagonal entries of this matrices are real and we will not be closed under scalar multiplication scalar multiplication, if this scalar is purely a complex number a complex number. So, these are some example of subspaces then we will we will define another important concept that is a linear span of a set. Linear span of a set of vectors. So, let us have a vector space let V be a vector space over field F and s be a set of vectors in V. So, let us denote, let L(s) be the collection of collection of all finite linear combination of finite linear combination of elements in S. Finite linear combination means; linear combination of finite number of elements in S that is, linear combination of linear combination of all possible finite subsets of S then, this L(s) satisfy some properties that we will write that as a lemma of course.

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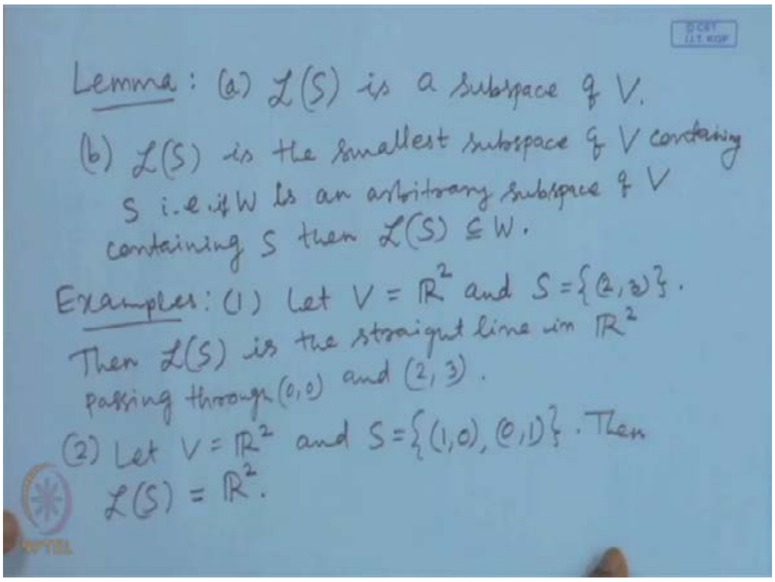

This proof is easy and that is, then let us and exercise it is not difficult to check, that first first property is this L(S) is a sub space L(S) is a sub space of V. And second, that one can see is that. In fact, not only this is subspace, this L(S) is the smallest subspace smallest sub space of V containing containing S. L(S) is the smallest sub space of V containing S, in other wards that is another the W is, that is if W is an arbitrary subspace of V containing S then, this L(S) will be contain in W. This you mean that, this L(S) is the smallest subspace. It will be contained in a subspace if that subspace contains this set S. So, let us have some examples.

So, first example is that, we can take this vector space is equal to R to the power 2 and this S is consist of only one element 1 element or 1 vector of this vector space, that is (2, 3) then L(S) is actually the straight line passing through origin and (2, 3), then L(S) is the straight line in R to the power 2 passing through the origin (0, 0) and (2, 3). This point, second second example is that, again if you take this V is equal to R to the power 2 and S is consist of the points (1, 0) (0, 1) then L(S) is equal to the whole space R to the power 2.

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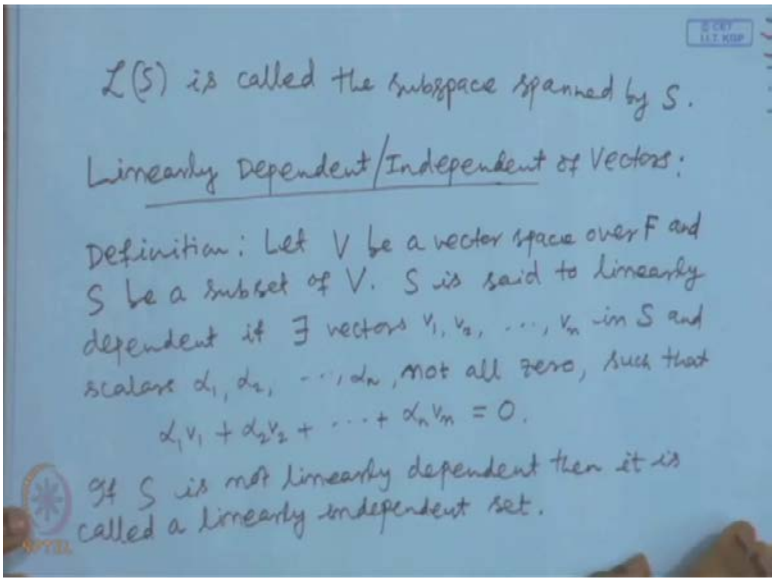

So, this linear span of sets will be useful and this L(S) is also called this subspace spanned by S. L(S) is called the sub space spanned by the set of vectors S. Then we will discuss about linearly dependent and independent of vectors. So, that is very important concept in a linear algebra, linearly dependent or independent of vectors. So, this is the very important concept in linear algebra. So, give this definition. So, let V be a vector space over a field F and S be a subset of V. subset of V. S is said to be a linearly dependent said to be linearly dependent if there exist there exist vectors v 1, v 2 to v n in S and scalars alpha 1, alpha 2 to alpha n not all naught of course, alpha n not all naught, this is very important such that, this linear combination alpha 1 v 1, plus alpha 2 v 2, plus alpha n v n is equal to naught. This naught is the naught vector. If S is not linearly dependent not linearly dependent then it is called a linearly independent set. linearly independent set.

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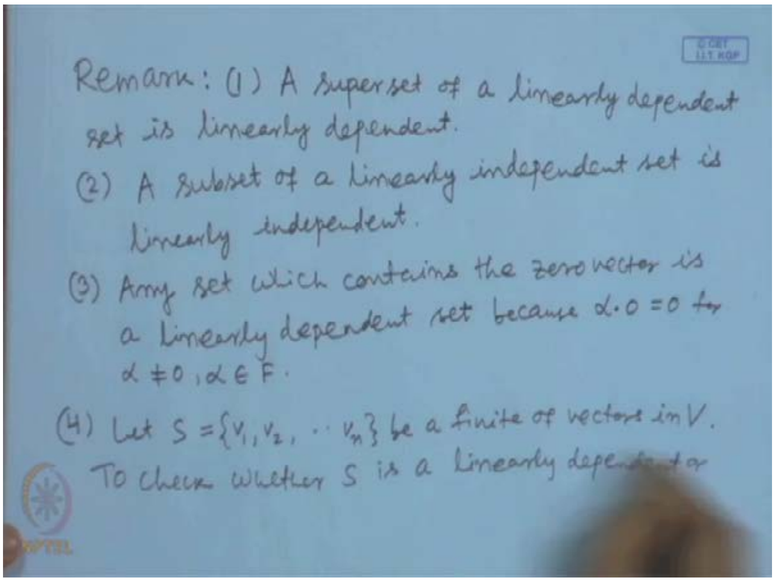

Here we will have some results in this linearly dependent and independent sets they are trivial. So, we can write them as a remark set of thing of course, these are trivial, and trivially one can get using the definition of linearly dependent and independent of vectors. So, first one is like this; a super set a super set of a linearly dependent set of a linearly dependent set is linearly dependent. Similarly you can have, that a subset every subset of a linearly independent linearly independent set is linearly independent. And third observation one can have is this, any set which contains the zero vector which contains the 0 vector is a linearly dependent set linearly dependent set because alpha dot naught equal to naught for alpha not equal to naught be a scalar.

Then fourth remark we can have that, here one can check this and a linearly dependency of a finite number of vectors suppose, a finite number of vectors are given and we want to check whether they are linearly dependent or independent. So, how should we do? What is the working formula for that? So, this is all about the definition that we have given, but how to check? And what is the method for checking either a given of course, finite set of vectors are linearly dependent or independent? So, using this definition also one can check directly and we will also give another method for verification of linearly dependent and independent by using some special type of matrices. So, here we consider that, I mean let that S be subset of vectors that v 1, v 2 to v n be a finite set of vectors in V. So, to check whether S is whether s is a linearly dependent or independent set independent set we do the following that is, we consider a linear combination of vectors in S and equal to zero.

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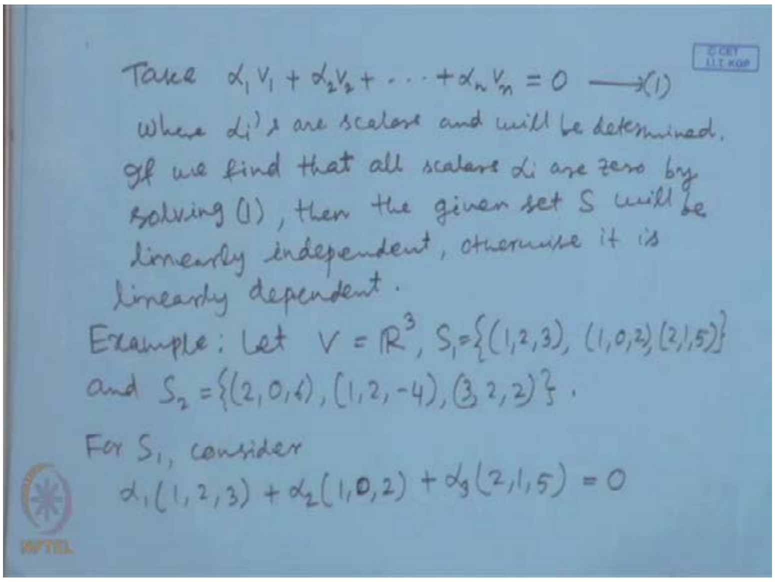

Take this alpha 1 V 1, plus alpha 2 V 2 plus alpha n v n equal to naught. Where alphas Li's are scalars and will be determined. So, if we can find that if we find that all scalars Li's are 0, scalars alpha are 0, say this equation 1 by solving equation one, then then the given set S will be linearly independent, otherwise it is linearly dependent. So, let us take one example and explain this concept. That you take a V be R to the power 3 and S 1 be the set of vectors (1, 2, 3) (1, 0, 2) (2, 1, 5) and S 2 be the set of vectors (2, 0, 6) (1, 2, minus 4) (3, 2, 2) subset subsets of V. So, we will check linearly dependency or independency of S. So, for S 1 consider this alpha 1 (1, 2, 2) plus alpha 2 (1, 0, 2) plus alpha 3 (2, 1, 5) take this linear combination of vectors in S 1 and equal to naught.

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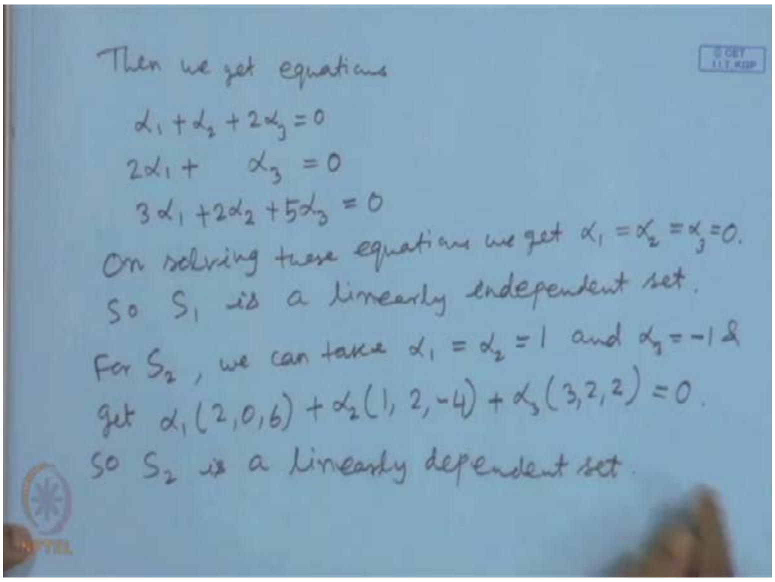

Then we will get equations like this, then we get equations alpha 1 plus alpha 2 plus 2 alphas 3 is equal to 0. Two alpha 1 plus alpha 3 is equal to 0. 3 alphas 1 plus 2 alpha 2 plus 5 alpha 3 is equal to 0.

On solving these system on solving this equations these equations we get alpha 1, alpha 2 and alpha 3 are all equal to 0. Therefore, this S 1 is a linearly independent set; however, for S 2. So, S 2 we can take alpha 1 is equal to alpha 2 equal to one and alpha 3 equal to minus 1 and get that, this alpha 1 times (2, 0, 6) plus alpha 2 times (1, 2, minus 4) plus alpha 3 times (3, 2, 2) is equal to 0. So, S 2 is a linearly dependent set. this is a linearly dependent set. So, there is another way also to verify that whether the given set of vectors are linearly dependent or independent and for that we need a type of matrix it is called echelon matrix.

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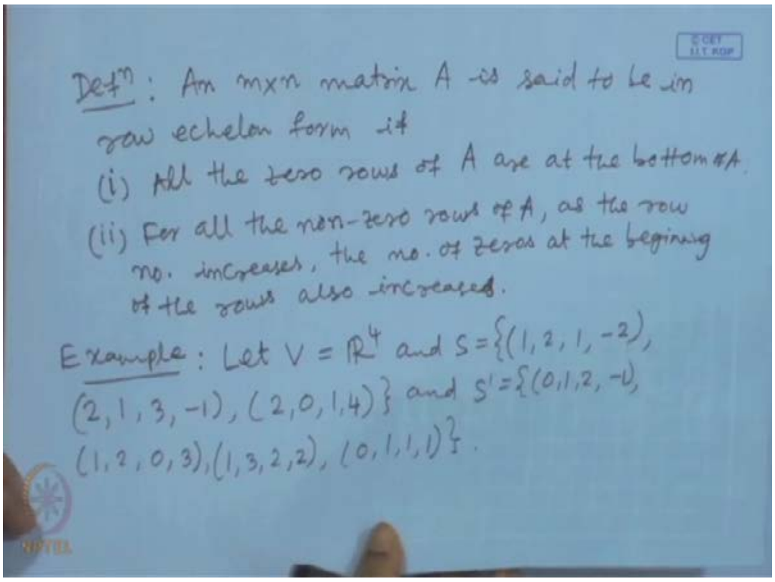

So, here we give this definition of an echelon matrix. So, an m into n matrix A is said to be in echelon for said to be an row echelon form. Echelon form, if it satisfy the following. That first condition is this all the 0 rows all the 0 rows of A are at the bottom of A. And second is this, for the entire non zero rows all the non-zero rows of A, as the row number increases, the number of 0's at the beginning at the beginning of at the beginning of the rows also increases also increases. So, let us see one example here, how to apply this echelon matrices and verify whether it is a set of vectors are linearly dependent or independent. So, let us take V be the vector space R to the power 4 and S be the set of vectors that (1, 2, 1, minus 2), and this (2, 1, 3, minus 1), (2, 0, 1, 4) and S prime be the set of vectors (0, 1, 2, minus 1), (1, 2, 0, 3), (1, 3, 2, 2), (0, 1, 1, 1). So, let us verify whether they are linearly dependent or independent sets. So, for this set S, this subset S of vectors we form a matrix by taking the vectors in S, S rows.

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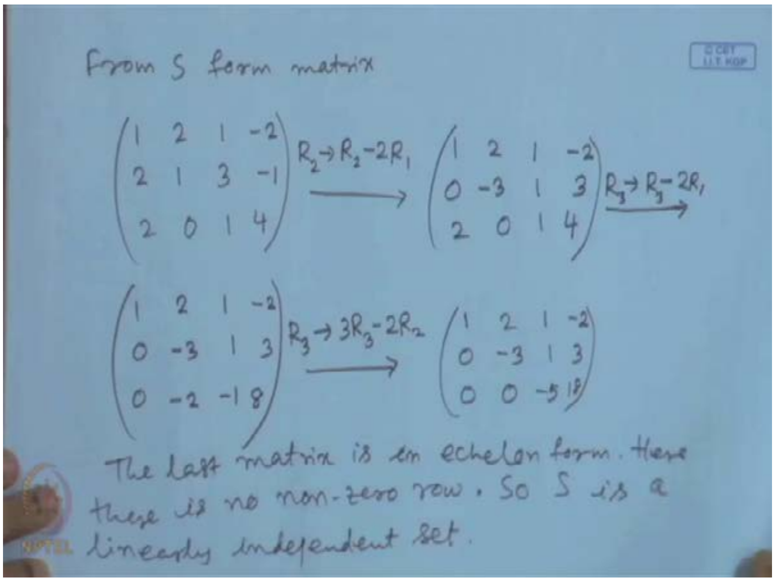

So, from S form matrix, that (1, 2, 1, minus 2) taking the vector says rows of this matrix, and the (2, 0, 1, 4). So, applying this elementary row operations or replacing R 2 by R 2 minus twice R 1, we get this matrix that (1, 2, 1, minus 2) this is unchanged and (0, minus 3, 1, 3) (2, 0, 1, 4). So, again from this matrix we can have this elementary row operation R 3, that we replace R 3 by R 3 minus twice R 1 and get this matrix (1, 2, 1, minus 2) (0, minus 3, 1, 3) (0, minus 2, minus 1, 8) and again by elementary row operations, that R 3 we replace by this 3 R 3 minus twice R 2 and we get this matrix that (1, 2, 1, minus 2) (0, minus 3, 1, 3) (0, 0, minus 5, 18) the last matrix matrix is in echelon form. Here there is no non-zero zero there is no non zero row. So, S is a linearly independent set linearly independent set.

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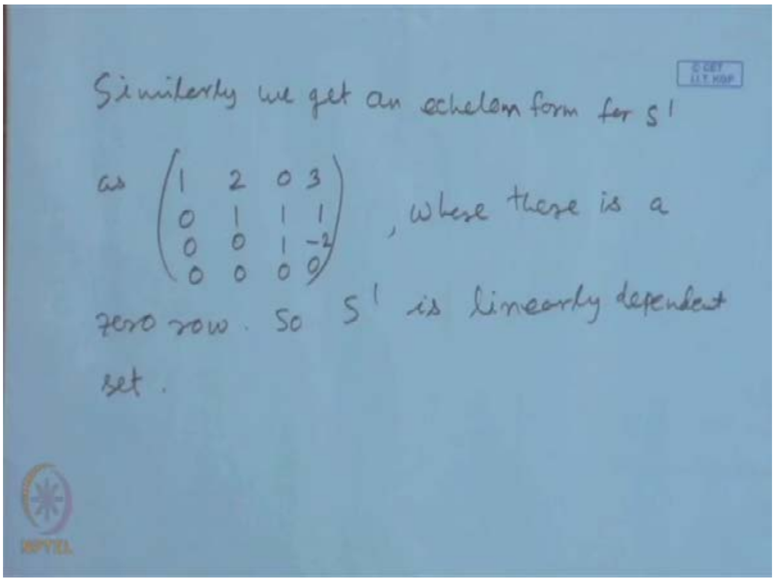

Similarly, one can get this echelon form of S prime like this. Similarly, we get an echelon echelon form for S prime as this (1, 2, 0, 3) (0, 1, 1, 1) (0, 0, 1, minus 2) (0, 0, 0, 0) where there is a zero row. So, S prime is linearly dependent linearly dependent set. This is how we verify that whether set of vectors are linearly dependent or independent by using this echelon form of matrices, that given any set of finite set of vectors we form a matrix by taking the rows of the matrix is the vectors in that set and we convert top echelon form. If in the echelon form all the rows are non zero then, this set will be linearly independent. And if there is a zero row then these set of vectors will be linearly dependent set of course, we will repeat this echelon form again in our next lecture while finding basis, and dimension of vector spaces that is all for this lecture. thank you.