Advanced Engineering Mathematics
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Lecture No. # 19
Power & Taylor's Series of Complex Numbers (Contd.)

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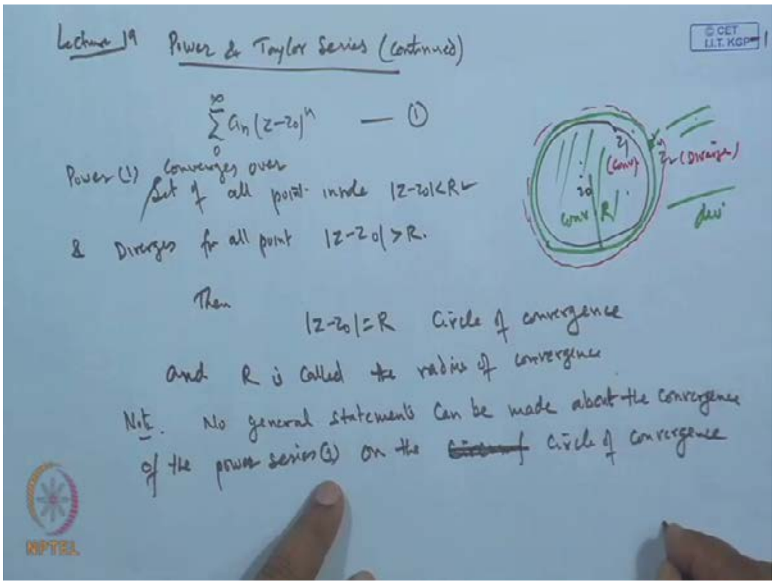

So, we were discussing the power series sigma a n z minus z naught to the power n - n is 0 to infinity. That was the power series and we have seen that this power series, convergence behavior of the power series depends on the coefficients. So, for variable z, z is a variable once you fix up z, then it becomes a power series. Where all the results and ratio test, comparison test can be applied over this.

So, if we use (( )) then we will we have seen, that the power series there are power series which converges only at single term point, that is z naught itself or it will converge throughout the entire domain or there are some series; which converges inside a disc and diverges outside. So, basically the convergence behavior of the power series; that is interstate and particularly those power series which behaves like some inside some disc it converges and outside it diverges.

So, we can identify a circle of the convergence. So, that all the point inside the circle we can say the series converges and the point which is outside of this diverges. And this follows from the previous result which we have already discussed; that if a series if this series which center z naught and converges at a point z 1, then it will converge at every point inside this z 1. And if it diverges at the point z 2, then it will converge it will diverge outside of this.

So, this is the here it will diverge. This is the point for diverging, where this is the point for convergent; where the series converges. Then what we have seen is that, all the points at inside this series will converge. While the point outside of this it will diverge. Now, slowly if we keep on increasing the point this one point here here and there and reducing here, then you will get one circle and that circle I can identify that circle.

So, that when you take this circle mod z minus z naught say radius R if I take, then we can say that; the set of all points inside the circle, mod z minus z naught less than R the series is convergent. Series converges the power series 1 converges over the set of all points inside the circle. And suppose and diverges for all point over the set, all points where mod z minus z naught is greater than R. Then, this mod z minus z naught equal to R, this is known as the circle of convergence. And R is called the radius of convergence. That is what we are interested.

Now, that we have seen that if R is the radius of the convergence. Then, all the points inside the region mod z minus z naught less than R the series will be convergent; and the point outside of this region z minus z naught is greater than R those point will lie in this region will diverge. What about the point on the boundary, on the circumference. So, nothing can be said, note it we can say no general statement. Statements can be made about the convergence of a power series 1, convergence of power series on the circumference, of the on the circle of the convergence. In this (( )) circumference, let us write circle of convergence.

It means the behavior of the power series over the circle of convergence varies. There are the power series which may not converge at any point inside, over this circle of convergence. There are the series where it converges at every point on the circumference convergence or may be a series, which converges only at few point and nowhere else.

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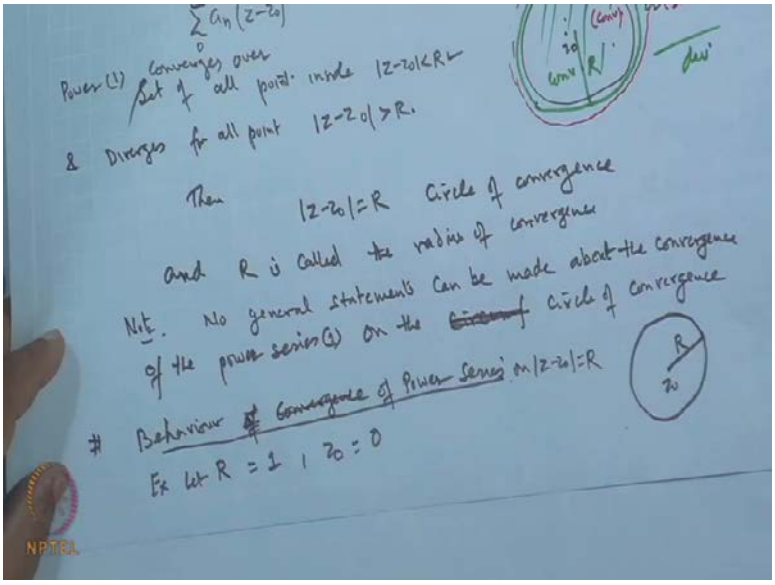

So let us, see the behavior of, the convergence of the power series. Or, behavior of power series on the circle of convergence. Convergence power series on the circle of convergence mod z minus.

So, behavior of the power series on circle of convergence. So, let us take the few examples. Suppose, I take the example here. Let R becomes 1. Let us choose R equal to 1 and z naught equal to say 0.

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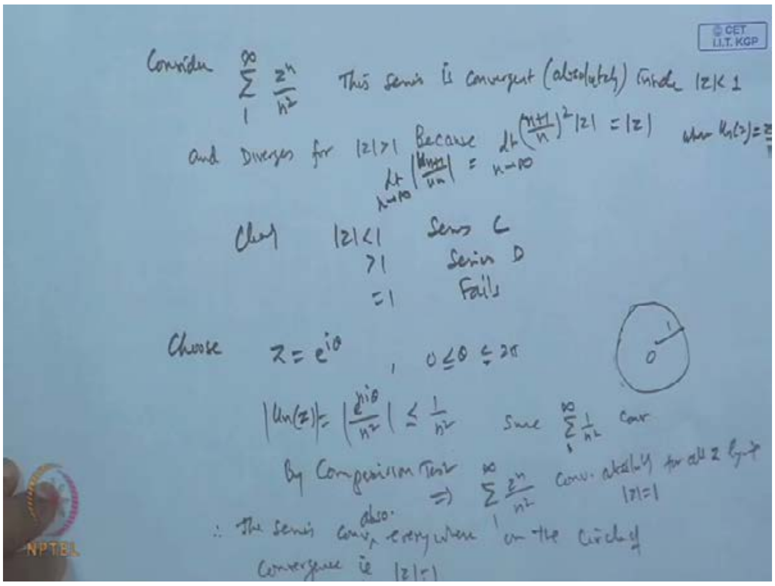

So, with center 0 and radius 1. So, let us draw the consider the power series sigma z to the power n over n square, n is from of course, 1 to infinity because other way it is not defined 1 to infinity. Now obviously, this series is convergent. In fact, absolutely convergent, inside the circle mod z less than 1. Because, and diverges for mod z greater than 1. The reason is; because if we use the ratio test then what is the ratio test says that, u n plus 1 over u n, that is n plus 1 over y n whole square into z mod z. This is u n plus 1 over u n. This limit as n tends to infinity is mod z.

This is the ratio test. Say u n plus 1 over u n; u n stands for u n z stands for this function. Where, the u n z is z to the power n over n square. This is our u n z z square. So, if we apply the ratio test, then you are getting limiting value to be this. Now, if this is less than 1, clearly if mod z less than 1 series converge it; greater than 1 series diverges and equal to 1 test fails.

So, let us see that series where it converge. So, suppose I take choose z equal to e to the power i theta. Now, e to the power i theta where theta lying 0 and 2 pi 1 side is 2 pi. So, it is a point lying on the circumference of the circle centered a 0 with radius 1 these are all points we will cover. So, what will be our u n? The u n becomes u n z, this becomes z means e to the power n i theta by n square.

Now, this mod of this thing is dominated by 1 by n square and 1 by n square sigma is finite; since sigma 1 by n square 1 to infinity is a convergence series. So, this is each term of this series is dominated by a non negative real numbers, where the series is converging. So by comparison test, we can say that the series this z to the power n over n square 1 to infinity converges absolutely for all z lying on the circumference mod z equal to 1; or satisfying the condition this. It means this series converges everywhere on the circle of convergence. So, those are series converges everywhere absolutely. Everywhere on the circle of convergence. That is mod z equal to 1.

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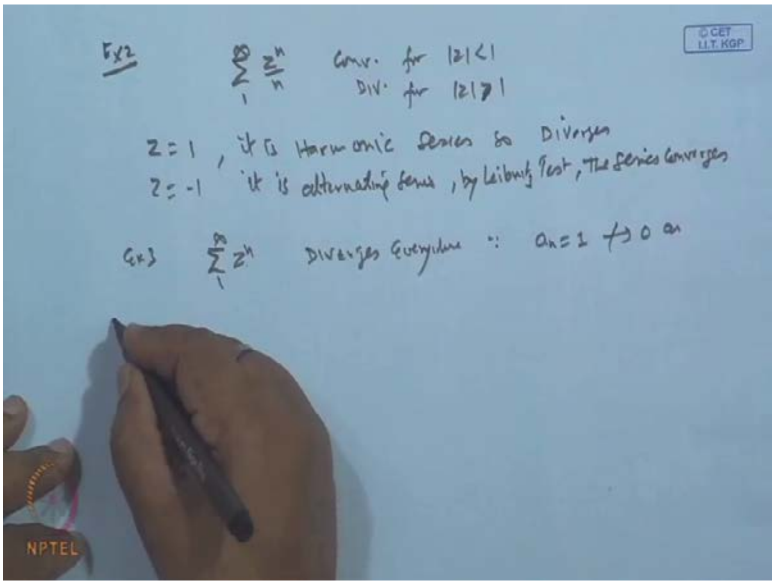

Now, let us take another example; this is the example one. Let us take the other examples two. Choose the z to the power n by n 1 to infinity. Now, if we look the series then obviously, this series converges for all z modulus less than 1, diverges for all z which satisfying this condition mod z greater than 1; and for one. When z is equal to 1, it is harmonic series so diverges. Where z equal to minus 1, it is alternating series alternating alternating series. So by leibniz test, the series converges. So, what we can say here that, it converges at minus 1; but diverges at one. It means the points where it converges and diverges.

Then third case is 1 and minus 1 it will converge and diverge. Similarly, i and minus i also you can establish the similar things for it. Now, if we look the sigma z to the power n 1 to infinity or 0 to infinity, then it diverges everywhere. Why? Because what is the coefficient a n, a n is 1 and that does not go to 0 as n tends to infinity. And in fact, what will happen is, non 0 term and there is one term if the sigma u n z to the power n is there.

And if all the terms are non negative and this does not go to 0 neth term must go to 0. So, here the z n, n is one of course. So, when you take the mod n or you can choose like this. Say z equal to 1, it will diverge z equal to minus 1. Whatever may be mod z it will always be 1.

So, it will diverge everywhere in the complex plane. Because this follows from the (( )). A series who neth term must go to 0, but neth term will not go to 0 because (( )), so it diverges. So, what we see here there is a series, whose convergence on the circumference is granted everywhere. There are series which converge only at few point, but nowhere else. And the series which converges, which nowhere convergent. So, the behavior of the power series on the circle of convergence cannot be definitely set; it depends on the series. So, this is what we cannot say anything (( )). Now, as we have seen that this convergence and the divergence of the series totally depends on the coefficients.

So, why do not we have a results or formulae; which depends only on coefficients and but capable of identifying the given series to be convergent or divergence. So, that will also give the formula for the radius of convergence. So, that you can find the reason of convergence where the series converges and the other reason where it diverges.

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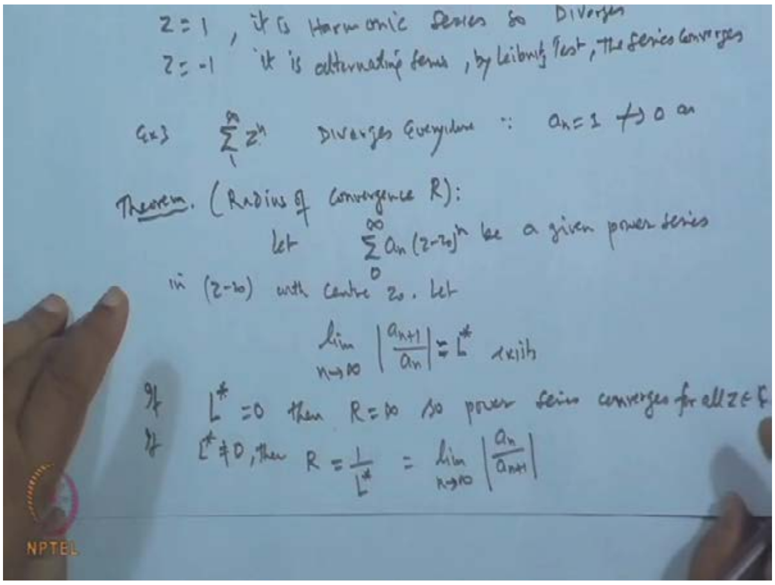

So let us see, the result which is in the form of the theorem. The result is about the radius of convergence and we denote this by R. Suppose, that the sequence n converge with the limit. Let sigma a n say z minus z naught to the power n, n is 0 to infinity, we are given power series; in z minus z naught in the power of z minus z naught, which centered z naught. And suppose let us, limit of this limit of mod a n plus 1 over a n as n tends to infinity, is suppose this limit exist and suppose L star exist and say it is L star.

Now, if the result is; so what it says is that, if L star is 0, then the radius of convergence R will be infinity and the power series converges, at every point for all z in a complex plane C. If L star is not 0, then we can define radius of then the radius of convergence R is nothing but 1 by L star. And that is equal to, limit of this as n tends to infinity mod of a n over a n plus 1.

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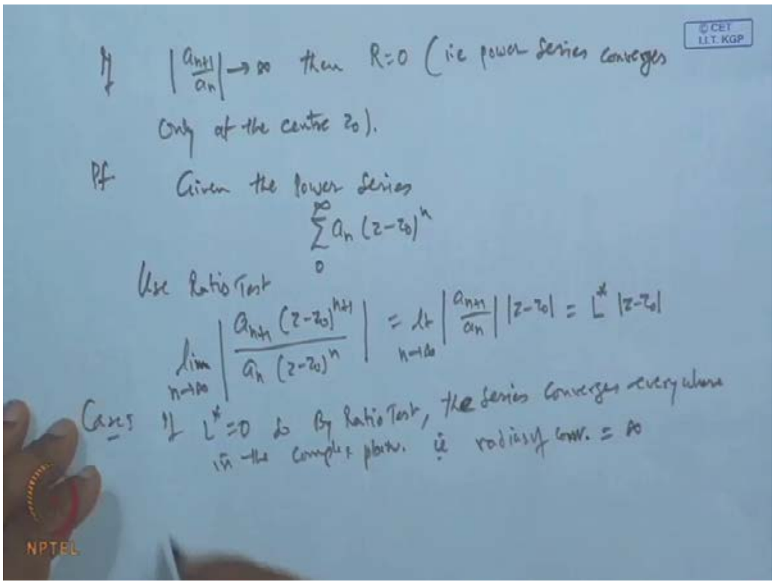

Now, third case if our mod of a n plus 1 over a n tends to infinity, then the radius of convergence R will be 0. That is converges only at the center. That is power series converges only at the center z naught. So, this is the result and this result is given by cauchy hadamard. So, it is also known as the cauchy hadamard theorem, this is also known as cauchy hadamard formula. Let us, see the proof of this.

Proof is simple. What is given is, given the power series sigma a n z minus z naught to the power n 0 to infinity. Which centered z naught, in the power of z minus n. Now, use the ratio test. So, if we take the ratio test u n plus 1 over u n. That is a n plus 1, z minus z naught to the power n plus 1 over a n, z minus z naught to the power n this u n this is equal to what mod of a n plus 1 over a n into mod z minus z naught. By it is given this limit of this exist. So, if you take the limit of this as n tends to infinity, this is equivalent to limit of this as n tends to infinity mod z minus z naught is free from n.

So, basically it is equal to L star mod z minus z naught. Now, if L star is 0 case one. If L star equal to 0, it means this ratio test is strictly less than 1. So, by ratio test the series the power series converges everywhere.

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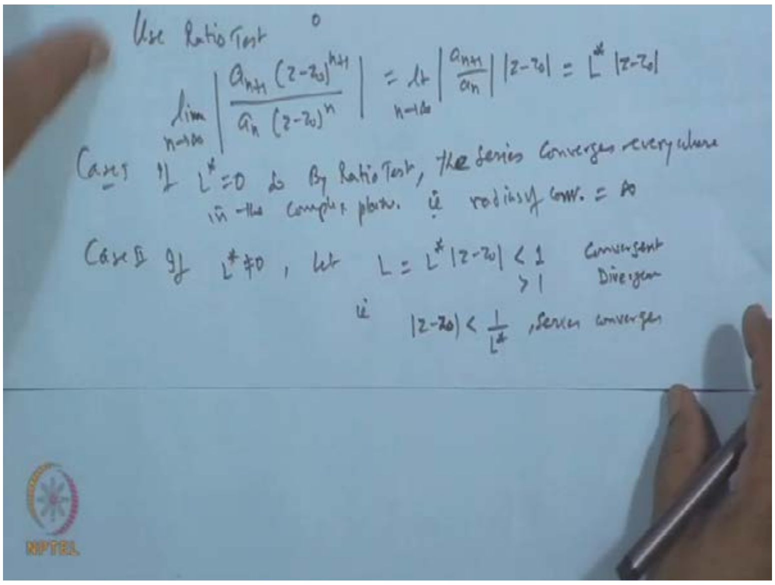

So, this series converges everywhere in the complex plane. That is the radius of convergence R will be equal to infinity. R becomes infinity, because it converges everywhere in the complex plane. And if L star is not 0; then let us write it this is suppose L. Then let L is equal to L star mod z minus z naught. Now, if this is two cases either this will be less than 1 or greater than 1.

So, if it is less than 1, then convergent. If it is greater than 1, then diverging. So, then it implies greater than one divergent. That is mod z minus z naught, this is strictly less than 1 by L star then the power series converges. And if it is greater than 1 by L star, then the series diverges.

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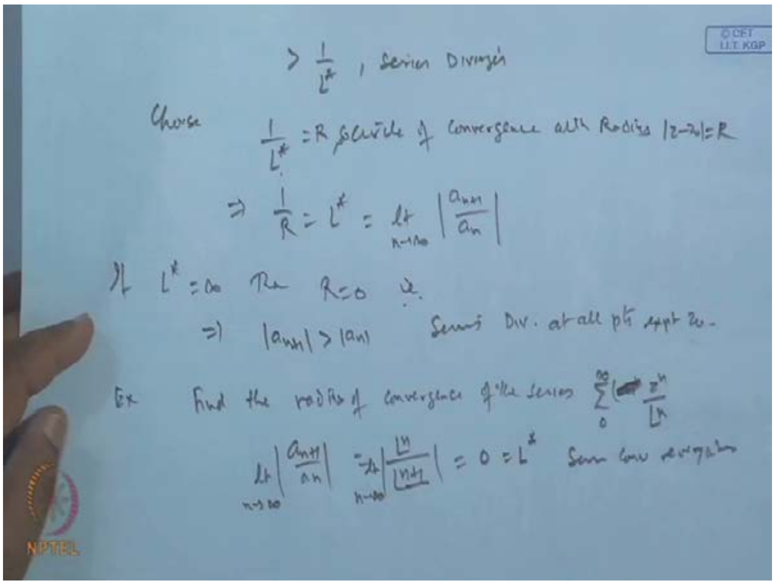

So, it means if I take it 1 by L R choose 1 by L star is R; radius of convergence. So, this is the circle of convergence. That is with radius R. So, circle of convergence radius R means, mod z minus z not equal to R. And then, what we are getting is that if R becomes 1 by L star; so R L star 1 by R that is formula is this. L star is this. So, from here we get, 1 by R is equal to L star, that is equal to limit as n tends to infinity, mod a n plus 1 over mod n, where R is the radius of convergence.

So, this proves that convergent. So series converges inside. Now, what happen if the L star does not exist it goes to infinity. If L star goes to infinity, then R becomes 0. So, in that case the series is at converges only at center that is all. So, that is the this greater than or sufficiently law. So, implied series diverges for all L star is infinity is it not.

So, what do you mean by this will be L star is infinity. So, we are getting what? mod of z minus z naught this is basically the terms are greater than 1 is it not? mod of a n plus 1, this shows that mod of a n plus 1 is greater than, mod of a n is it not? These terms will keep on increasing.

So, this series diverges at every point. So, all is 0 means, the series diverges at all points; except 0 except z naught. There's no point ratio test, imply divergent. Because, this entire thing is greater than 1. If it is greater than 1, then we get this divergent L star. Now, take some examples. Suppose, I take this series, find the radius of convergence of the series say sigma minus 1 to the power n, z n z minus z cube. So, 0 to infinity 1 minus z square.

So, this will be the equal to factorial n or minus 1. Suppose, I take this series, then what happen is mod a n plus 1 over a n. This is equal to what? mod a n plus 1 is factorial n plus 1 mod of this. And limit of this as n tends to infinity is the same as limit of this as n tends to infinity, which is nothing but 0. So, L star is 0. L star is 0 means R becomes 1. So, this is our L star.

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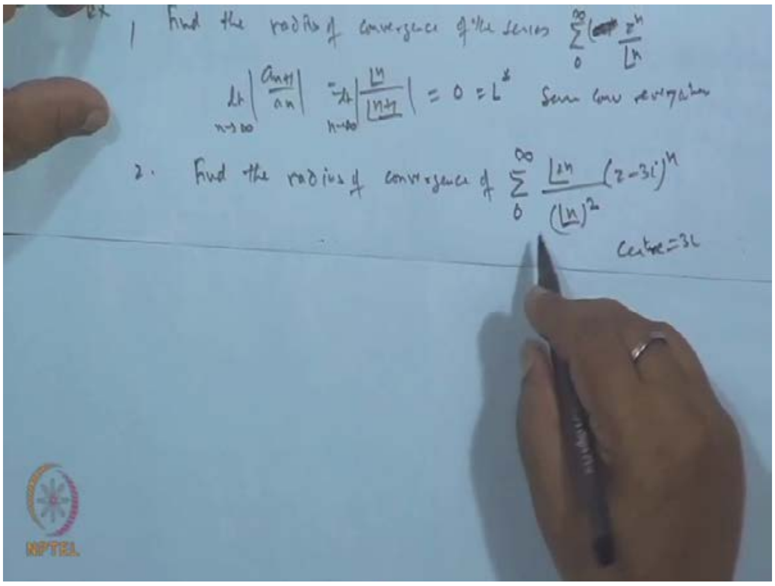

So, series converges everywhere like this. Similarly we can go for others. Now, if you take some point. Suppose, another example if I choose say find the radius of convergence of the power series, sigma factorial 2 n over factorial n square z minus 3 i to the power n, n is 0 to infinity. Now, here this is a power series centered 3 i.

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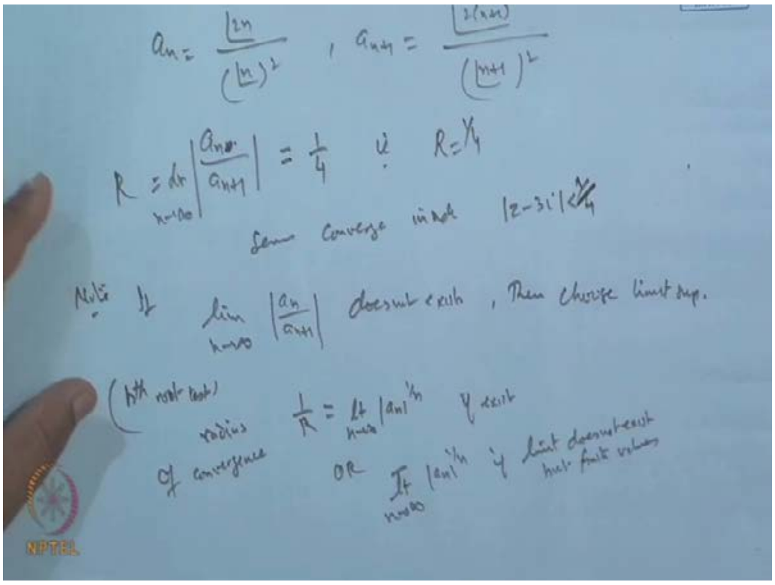

So, let us find out the first radius of convergence. So, what is our a n? a n becomes factorial 2 n over factorial n square. So, a n plus 1 is factorial 2 n plus 1 over factorial n plus 1 whole square. So, a n over a n plus 1 is the radius of convergence limit of this as n tends to infinity. So, what will be the radius of convergence is a n over n if I complete it, then this comes out to be what? and basically just you find a 1 by 4. Limit will come out to be 1 by 4. Using the factorial values another it means, that is radius of convergence R is 4.So, the series converges inside the mod z minus 3 i less than 4. So, radius converges is 1 by 4. Now, in case if the limit does not exist, then we will choose the max limit superior of this.

Note: If the limit of this mod a n over a n plus 1, as n tends to infinity, if this limit does not exist then then choose limit superior of this.

So, if that is n plus 1 over a n. If limit does not exist, then choose the upper part of it like this. Then, choose the limit superior. Similarly, for other case same results is another result is for ratio test neth root test. The radius of convergence R having the formula 1 by R. (( )) limit mod a n to the power 1 by n as n tends to infinity or if limit exists or limit superior as n tends to infinity, mod a n to the power 1 by four, if limit does not exist, but finite values. This is not the infinity.

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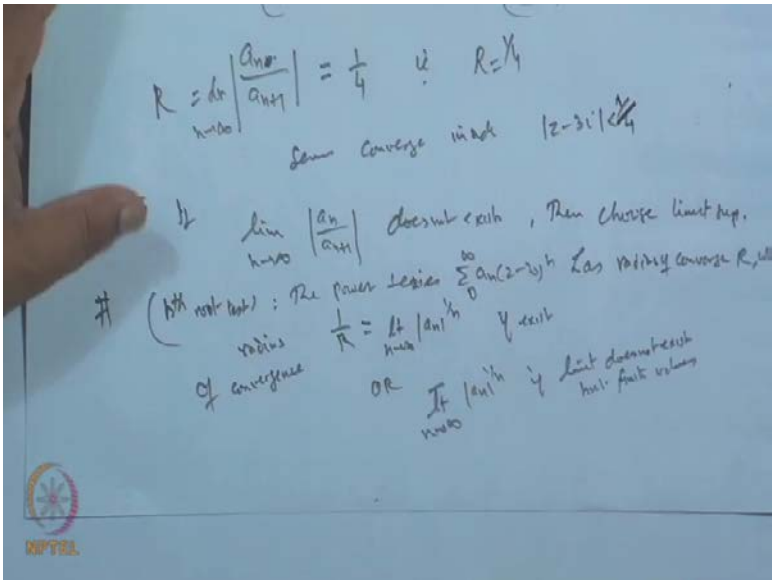

So, here this is the power series sigma a n z minus z naught to the power n 0 to infinity, has radius of convergence R. Where 1 by R is given by this formula, has radius of convergence R where 1 by R is given this form.

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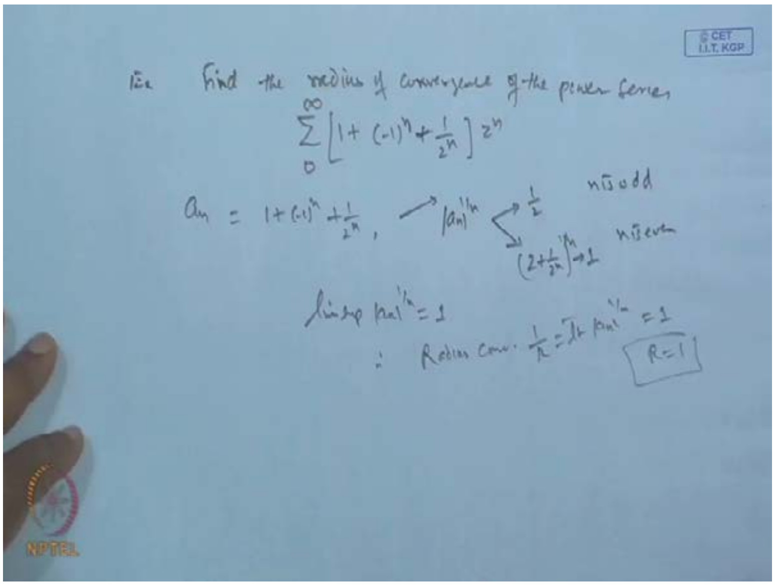

So, for example if we take this function. If a series find the radius of convergence of the power series, sigma 1 plus minus 1 to the power n plus 1 by 2 to the power n then z to the power n this 0 to infinity. Now, if we look this series what are the a n n as 1 plus minus 1 to the power n plus 1 by 2 n. If we look this say then mod n to the power 1 by n it does not go to because, this limit will tends to always half. So, clearly for odd when n is odd, then mod of a n to the power 1 by n this will go to half when n is odd because it is becomes minus. So, this will be 0 and power half. But when n is even, the limit will come out to be what? 2 plus 1 by 2 n power 1 by n as n tends to infinity. So, it will go to 1. So, limit will be half n 2.

So, what is the limit superior? Limit superior of mod a n to the power 1 by n is 1 therefore, radius of convergence will be 1 by R is the limit superior mod a n to the power 1 by n that is 1. So, R becomes 1. That will be the answer. So, we can find out the corresponding.

Now, next about we have a different difference between the series of the real variables and series of the complex variable. The major difference is here. In case of the series of real variable, when the series converges we have a function objects and the power series convergence we have a function object which is continuous. We can find out derivative of it. Suppose, after that we cannot say the further derivative is possible. But here in case of function of complex variable, the power series if it is convergent it will represent a function object.

Which is not only analytic, in the region of convergence. What is all ours derivative also be analytic. So, this will be justified in the following discussion. So, let us see the functions given by power series.

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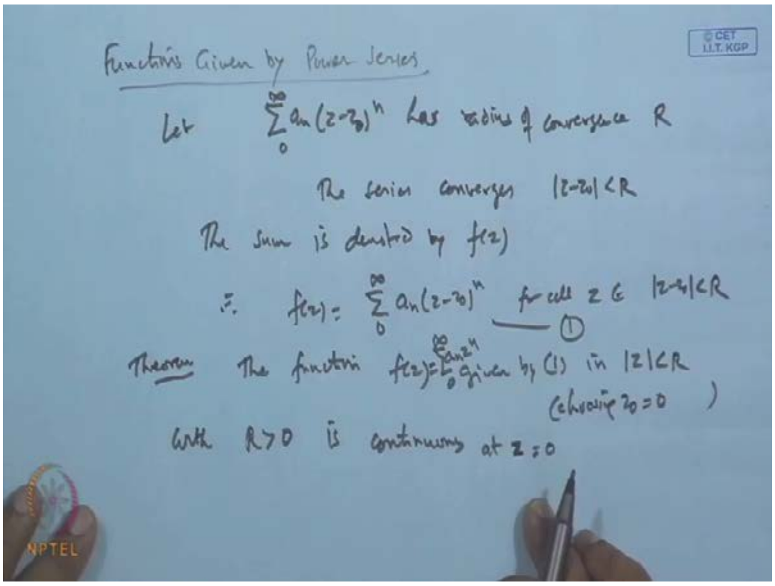

Now, when we have a power series sigma. z minus z naught to the power n 0 to infinity. Let this power series have radius of convergence say R. It means inside the circle z minus z naught less than R, the series converges. Once it converges then for each z it will represent some function. So, the sum of this series we denote it by f z. So, sum is denoted by f z, which is a function of z. It means when we say f z equal to sigma n z to the power n. So therefore, f z will be 0 to infinity z minus z naught to the power n and for all z which belongs to the region z minus z naught less than R. And, this will convergent represent a function f z.

Now, what the result says is the function f z given by 1 in the circle mod z. I am just saying mod z less than R choosing z not to be 0, particular case. Choosing let for simplicity. This is for simply we can choose also given by one is less than R or even z minus z naught also we can choose. So, equivalently z naught is zero, is less than R with R positive is a continuous function at z equal to zero. It means, the function f z sigma a n z to the power n zero to infinity given by one z in the mod z less than bar is a continuous function in this.

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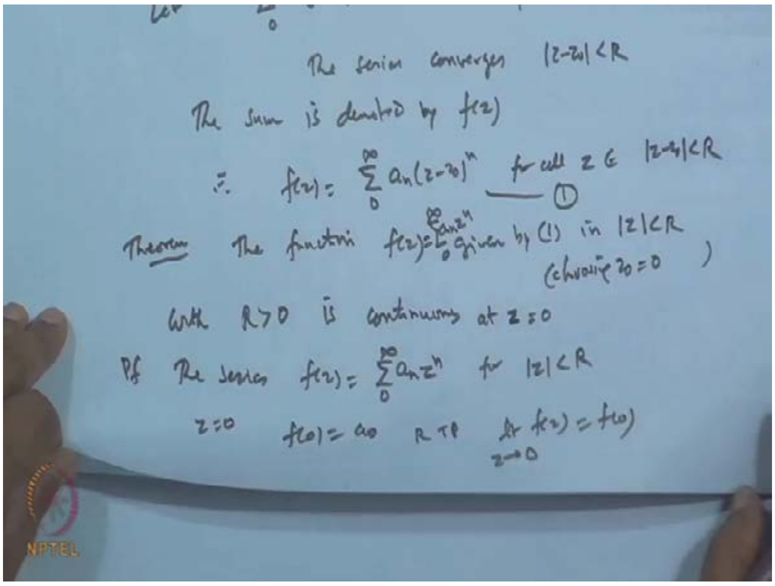

So let us, see how it is proved. Once we prove the continuity then all the things will come easily. Now, if we take z equal to f z is sigma a n z to the power n 0 to infinity for z lying between mod z less than R. So, if we take z equal to 0, it means f of 0 is a 0. So, if I prove required to prove is, that limit of the function f z. When z tends to 0 is f of 0, this is to prove. It means for any f channel greater than 0 if you find the delta such that when mod z less than delta implies mod f z minus a 0 is less than f channel, then our result is okay. Then it is continuity follows for this.

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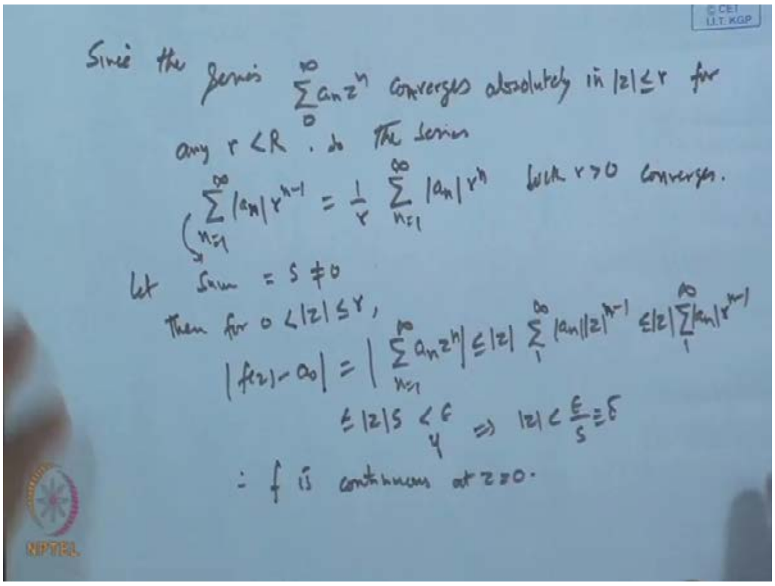

Now, since the series sigma a n z to the power n 0 to infinity, will converge absolutely inside the circle in mod z less than equal to R. Where any R is strictly get less than R. Because, this we have already shown the series is not only convergent it converges absolutely. So, converges absolute everywhere inside circle.

So, we take the circle or disc with a radius model which is strictly less than R. This is true. Then, the series sigma mod a n R to the power n minus 1 this n is 1 to infinity this is nothing but 1 by R sigma n is 1 to infinity mod a n R to the power n with R greater than 0. Now, this series is convergent.

So, this is convergent. Let the sum be s, which is different from 0 because it non negative terms. Now, for then for 0 less than mod z less than equal to R the difference of this mod of f z minus s which is equal to sigma a n z to the power n and n is 1 to infinity and that will be equal to less than equal to mod z sigma mod a n mod z to the power n minus 1 1 to infinity and that is again less than equal to mod z because mod z is less than equal to R sigma mod a n R to the power n minus 1 1 to infinity, but this is our what? s because this difference is total is s sum is we are taking to be s. So, this is what is equal to mod of z or less than equal to this part is less than equal to mod z into s. But, what? This mod of z is less than f channel.

If I choose this to be less than f channel then we get delta to be this. So, if this is less than f channel, then this implies a mod z is less than f channel by s which is nothing but delta.

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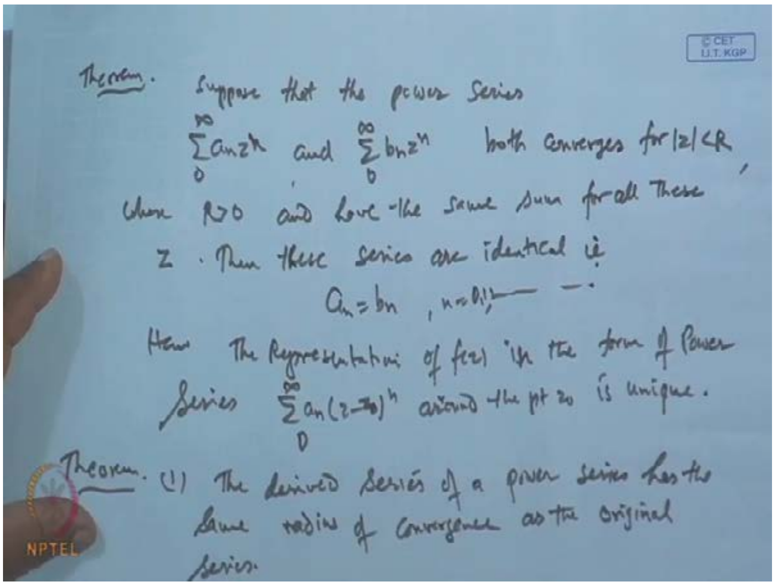

So, the function f is continuous at 0. This proves that and immediately this result will give another result which is also the result is: the two power series. Suppose, that the power series sigma a n z to the power n and sigma b n z to the power n 0 to infinity. If both converges for mod z less than R, where R is positive. Then and have the same sum for all these z then these series are then these series are identical. That is a n is equal to b n for n 1 2 3 and so on. In fact, 0 also 0 1 2 3 and so on. If the function f z has power series represented with the center. Then this representation of the function f in the form of power series sigma a n z minus z naught to the power n around the point z naught is unique.

So, that goes well with the help of this previous result just you can find out. It means, if your function f is given, then and z naught is fixed. Then we can have a unique representation of the power function in terms of the power series. So, that is interesting result and proof follows so, I will just skip that proof. Then, there are the few results and again I am saying in the form of theorem that differentiation of the power series. The first result say that derived series the derived series of a power series has the same radius of convergence as the original series.

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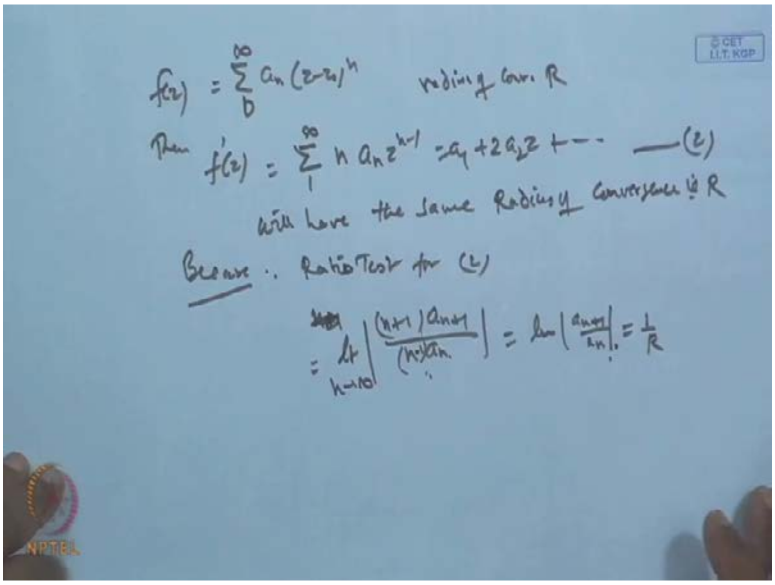

It means, if the series suppose a series is given sigma a n z minus z naught to the power this n is 0 to infinity. This is a series represented by a function f z and this series has a radius of convergence R. Then the derived series of this means f prime z when you take the derived series will be equal to n is 1 to infinity n a n z n minus 1, that is equal to a 1 plus 2 a 2 z plus and so on. This is also the power series will have the same radius of convergence that is R. The reason is because if we apply this ratio test for the power series two then what happen?

The n plus 1 over n plus 1 so n plus 1 or what is the ratio do not go for the ratio test, n plus 1 a n plus 1 over n a n this is our term is it not?. So, if you get for this limit of this. Now, this is our n minus 1. So, neth term will be equal to n plus 1 a n plus 1 and this is the n minus 1eth term. So, we can say this is equal to a n is reverse way is it not?. So, we can get the derived series will be n plus. So, u n plus 1 limit of this n plus 1 n a n.

So, limit of this mod of this as n tends to infinity is the same as limit of this is 1. So, it is the limit of mod a n plus 1 why a n? Why it is? The neth term will be a n plus 1. So, it is n plus 1 and this is below. So, you will get the n minus 1 term n minus 1. It is now there is no problem n plus 1 and then n plus 1 n a n. So, this is the same as this. So, it will be the same a n plus 1 because this limit is 1.

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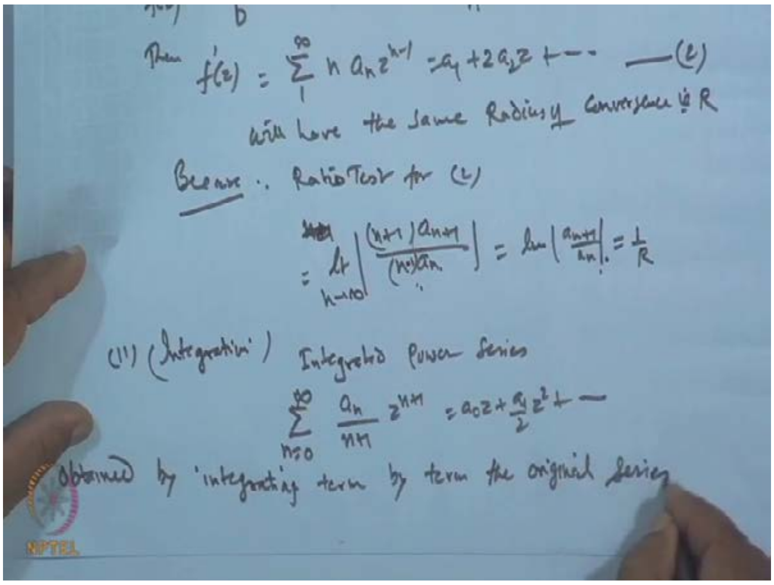

So, we get this it means, this is nothing, but the 1 by R. So, it is the 1 by R, it means it will have the same divergence. Similarly for the second part the integration the power series integration. If we integrate the whole power series the integrated power series, that is equal to sigma n is 0 to infinity a n over n plus 1 z n plus 1. This is the integrated power series that is a naught z a 1 by 2 z square and so on. Obtained by integrating term by term the original series.

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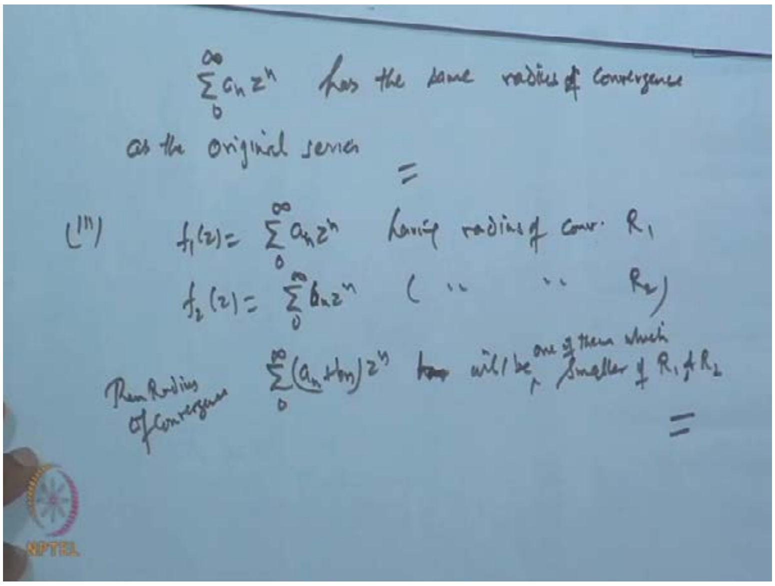

That is sigma a n z to the power n 0 to infinity. Original series has the same radius of convergence as the original series. So, we get this one original series clear? Now, the reason is again simple because if you go for the neth term by divide by neth term again you will get the similar things so, it will follow from here.

Then third is if suppose we have the two power series say, f 1 z is 1 power series, a n z n 0 to infinity having radius of convergence R 1 another power series having radius of convergence R 2 and if i add these two series b n. If I take a n plus b n z to the power n 0 to infinity. Then the radius of convergence of the sum of the two power series that will be equal to power series radius of convergence R 1 it is power series radius say is model of R 1 and R 2 has the radius of convergence of this power series will be smaller of R 1 and R 2 will be one of them. That which is smaller than is smaller of R 1 and R 2. So, that is the result which we have.

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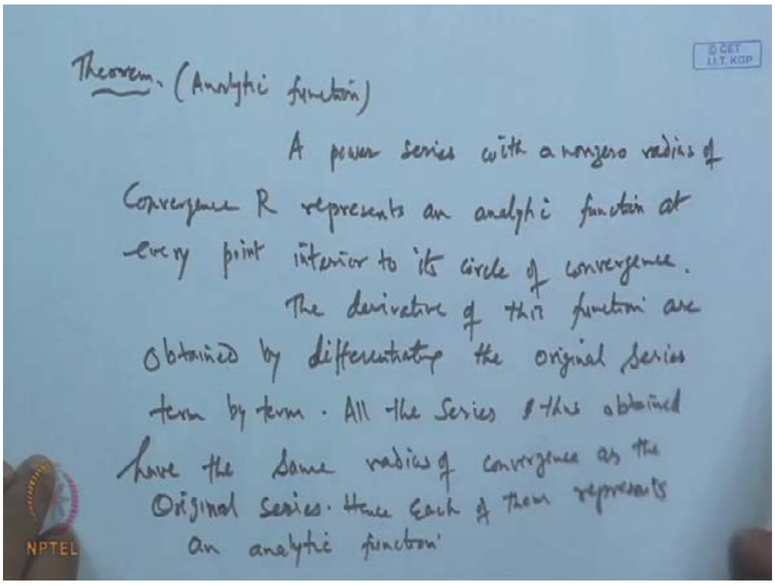

Then lastly we have one theorem without proof of course. So, that result is shows that every power series represents the (( )) a power series that is analytic functions and they are derivative a power series with a non zero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivative of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, each of them represents an analytic function.

So, this is the ways results which we are interesting that the power series with a radius of convergence R represents analytic functions at every point. And since, the derived series will also be a represent a power series with the same radius of convergence. So, derived series will also represent a function which is analytic and or such derivatives of the function which is represented by a power series are basically analytic and all its derivative are also there and that's what this zero. Thank you very much.