Advanced Engineering Mathematics

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Module No. 01

Lecture No. # 18

Power & Taylor's Series of Complex Numbers

So today we will discuss the power series of complex numbers and in particular case a taylor series of this. So, before going for the power series say what is that series and the some test few of this test will be used to identify whether the given series is convergent or divergent one.

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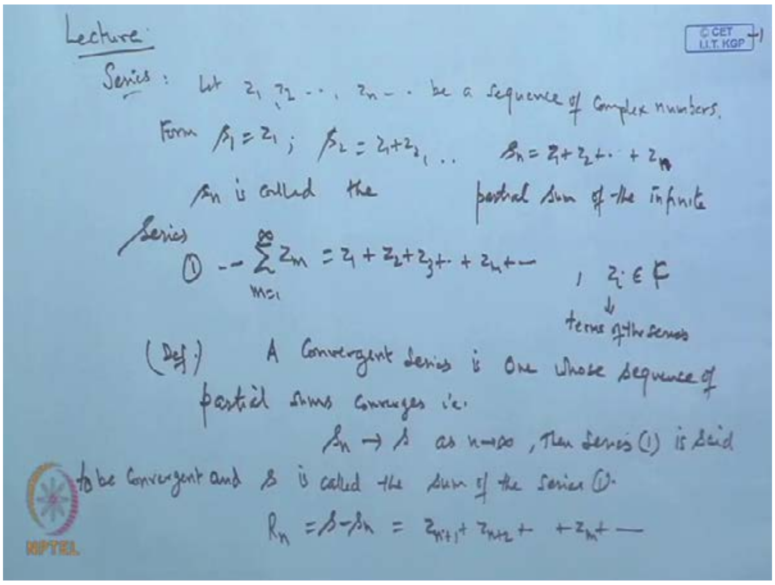

So, we defined the series as given a sequence let z 1 z 2 z n and so on be a sequence of complex numbers. With the help of this series if we form the sequence of its partial sums; say S 1 which is the first term z 1, s 2 sum of the first two terms and s n with the sum of the first and terms of this sequence z n. Then, s n is called the partial sum of the s n is called the partial sum of the infinite series sigma z m m is 1 to infinity that is S series, having the term z 1 plus z 2 plus z 3 plus z m and so on, where all these z i's are complex numbers, they are all complex numbers.

Then, z 1 z 2 z i these are called the terms of the series. Which are called the terms of the series, say 1 this is the series one. Now this series one is set to be a convergence series, if the sequence of a partial sum converges. So, as a convergent series a convergent series is one whose sequence of partial sums sequence of partial sum converges. That is this equation one is said to be convergent, if the sequence of partial sum S n will converge.

That is the sequence S n which we are getting as a sequence of partial sum; this is s n. So, this sequence S n if it goes to S as n tends to infinity; then,, we say the series one is convergent. And S is called, and then S the series one is said to be convergent is said to be convergent and s is called the sum of the series one. This is what we have.

Now, when we take the S minus S n, then basically you are getting the remaining terms of the series of z n plus 1 z n plus 2 and so on; up to say infinity. Then, this we denote it as R n and is called the remainder of the series. This is known as, the remainder of the series of the series one. Now in case of the series is converges, then remainder goes to 0. So obviously, clearly if S n converges to S; if series 1 converges then the remainder term R n will go to 0 R n will go to 0 (( )). And that will give the criteria for this.

Now, as we know these are all complex numbers, and in case of the real we know the all sort of results, for the test for the converges of the series; the similar case similar type of the result continue to hold good for a complex series of complex numbers.

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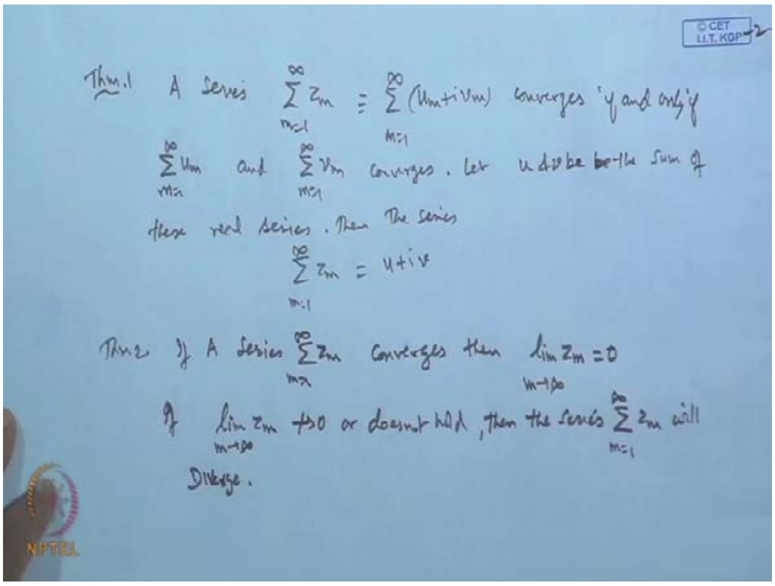

And we have a basic result says, a series sigma z m m is 1 to infinity; where the z m is U m plus i times of V m say, m is 1 to infinity here u m is the real part of this complex number z. V m is the complex imaginary part of this complex number z. So, a series of this converges, if and only if the series sigma of U m m is 1 to infinity and the series sigma of V m m is 1 to infinity converges. And the sum is it convergent and let u and v be the sum of be the sum of these real series. Then, the series sigma of z m m is 1 to infinity will converge and converse to the sum i plus u v. So, that is the first result which we; another result which we have for the test, that if a series converges it if a series sigma z m m is 1 to infinity converges, then the n-th term of this series that is z m, n-th term of this, when m tends to infinity must go to 0.

That is the necessary condition for a series to be a convergent one that is now if limit of z m, as m tense to infinity; if this limit does not tends to 0 or does not hold, then the series will diverge; sigma z m 1 to infinity will diverge. So, this is basically a test for a divergence of the series diverge for this. Again the proof is same similar as the proof of in case of real variables. So, we are not going to touching this; now here is we have seen in this theorem, that this one way is true; that is if the series converges then the n-th term will go to 0. However, the converse is not true; if the n-th term of series goes to 0, then the corresponding series may or may not be convergent one that we can see.

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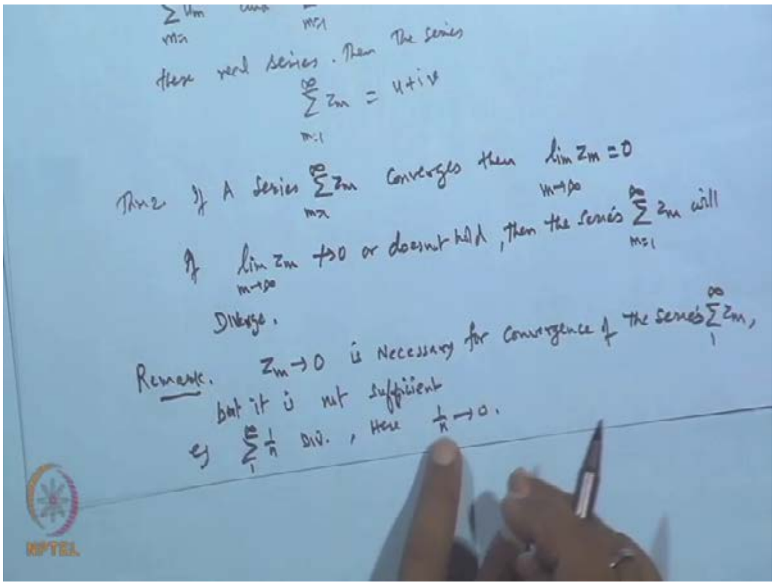

So remark is, the remark says that the z m the n-th term goes to 0, is the necessary condition only. Is necessary for convergence of series of the series sigma z m 1 to infinity, but it is not sufficient. And the reason is very simple; suppose I take a example say series sigma 1 by n 1 to infinity, now this is an harmonic series which diverges. But here the n-th term 1 by n tends to 0 so though the n-th term goes to 0, but the series is not convergent. So, it is only a necessary condition, but not sufficient condition.

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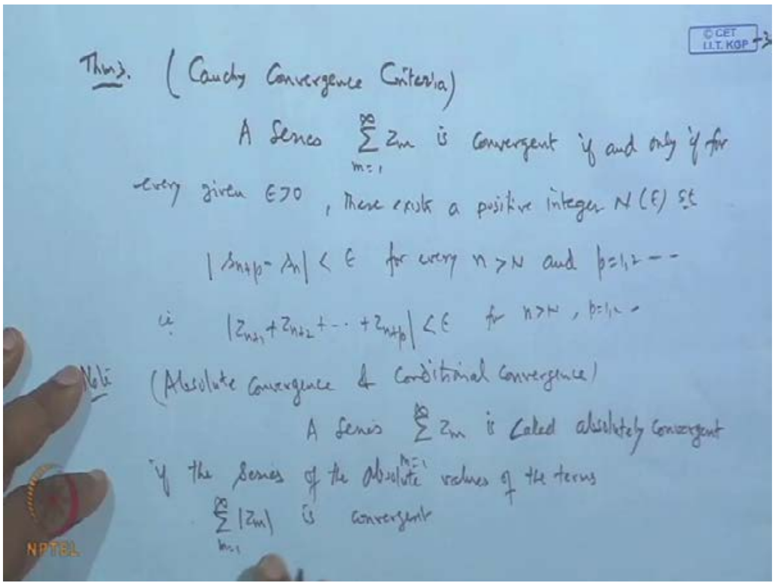

Another results which we also useful is the cauchy convergence criteria. Which is valid for a function of a real variables or series of a real variable, the similar things also holds good in case of a series of complex variables or complex numbers. So, what this said a series sigma z m m is 1 to infinity this series is convergent, if and only if for a given f channel for every, for given f channel for every given f channel greater than 0. There exists a positive integer capital N, such which depends on f channel such that modulus of S n plus p minus S n. This modulus is less than f channel for every n greater than N and p is 1 2 3 and so on. That is that is mod of z n plus 1 plus z n plus 2 and so on; up to z n plus p this is less than f channel for all n greater than N p is 1 2 3 and so on.

So what does this shows is, that a series will be a convergent series if the sequence of its partial sum satisfy this cauchy convergence criteria; s n plus p minus s n remains less than f channel. Note a series said to be absolutely convergence, absolute convergence and conditional convergent; a series sigma z m of complex number z m is called absolutely convergent. If the series of the absolute values of the terms that is mod of z m, that is sigma of mod z m m is 1 to infinity is convergent. So if a series, with its absolute terms absolute value of terms converges then we say the original series converges absolutely.

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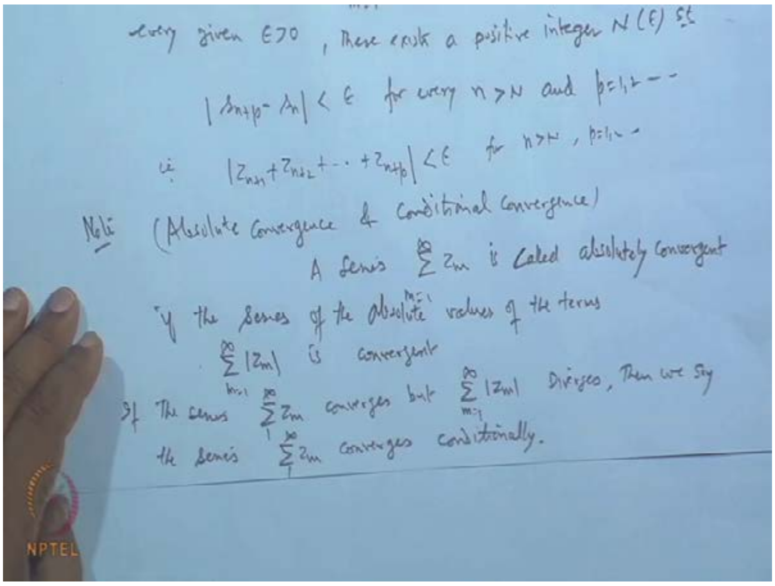

Now if the series, if the series sigma of z m 1 to infinity converges, but the corresponding series of its absolute terms diverges. Then we say the series converges; then we say the series sigma of z m 1 to infinity converges conditionally.

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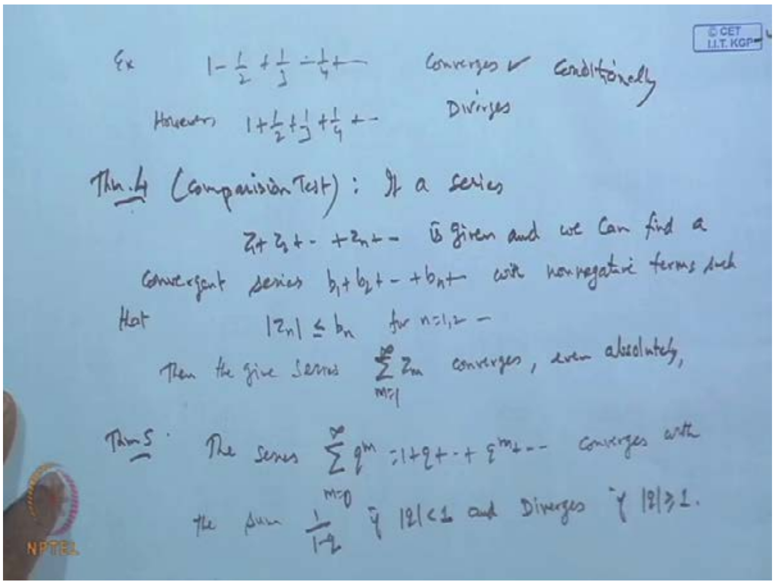

So that is the conditionally convergence series. For example, we have seen that alternating series if it look the series 1 minus half plus 1 by 3 minus 1 by 4 and so on. Now this series converges, by leibniz test if the terms are alternative positive negative, they are of decreasing nature in absolute value and limiting value tends to 0; so leibniz test say this series converges. However, if we consider a series with of all of its absolute terms then this becomes a harmonic series which diverges.

So we say, the original series converges conditionally. That is what we got. Obviously, if your series is convergent absolutely it is convergent also. Now next result which will give you some test, that is a comparison test. What this comparison test says, if a series is given and we can find a convergent if a series z 1 plus z 2 plus z n and so on. We given is given and corresponding to this series, if we are having another series convergence series of the scalars and we can find a convergent series.

Say b 1 plus b 2 plus b n and so on. With non negative terms or terms of non negative with non negative terms, such that the corresponding terms of the series; that is z n is dominated by b n. For each n and this is true for each n. Then the comparison test says then the given series sigma z m 1 to infinity will converge, and in fact it converges absolutely and in fact even absolutely; this is the result.

So, what problem what is the main idea is, that if we want to test given series sigma of z m to be convergent; then with the help of this terms of the series identify a sequence b n’s of positive terms such that this inequity is settled. That is b n always dominates to mode of z n over z n mode of z n. Then if the right hand side series converges, then the left hand side will also converge. And, since absolute terms are less than so it will converge absolutely also, so that is one.

Then geometric series of course is very standard one, the geometric series sigma of course, q to the power m m is 1 to infinity. Here this is 1 plus q plus q square plus q to the power m and so on this is terms. So, m is 0 to infinity say m is 0 to infinity converges, with the sum 1 over 1 minus q if mode q is less than 1, and diverges if mod q is greater than equal to 1. Again we are not going for proof (( )) just on the parallel lines as we did in case of real variables.

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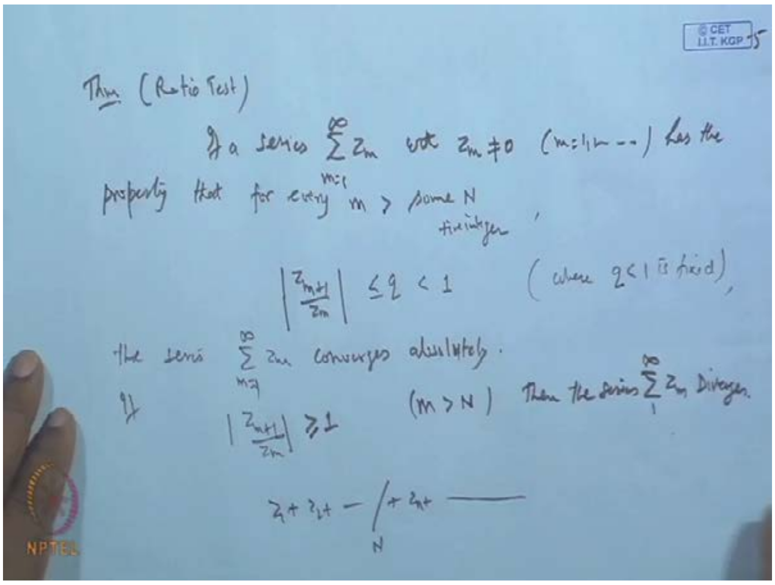

Then another test is which known as ratio test; for the series. If a series sigma z m m is 1 to infinity if the series with all terms to be non zero with all terms are non zero. With z m is non zero and m is 1 2 3 and so on, has the property that for every n greater than some number some N some positive integer N. That is greater than that is mode of z m plus 1 over mod of z m greater than and this for some n and mod of z m is less than equal to q which is less than 1.

Where q is fixed then this the series sigma z m m is 1 to infinity, this series converges absolutely. Now if this ratio is strictly greater than one, strictly greater than or equal to 1; for all n greater than say n and then the series then m sorry m then the series, sigma z m 1 to infinity will diverge. So, this is what ratio test is. So what is the ratio test is say a series is given and we want to test convergence or the divergence of this series then what this ratio test says, that first you look all the terms of series are non negative or non 0.

If so, then find out the ratio of the term two is preceding terms in absolute values z m plus 1 over z m take this absolute value; because these are all complex numbers. Now, if this absolute value for all m after certain stage remains less than equal to some fixed number q and if this q is less than 1, the corresponding series will converge. Now, in case if this ratio is always be greater than or equal to 1; after certain stage then the series will diverge. So, that is the ratio test. Then another ratio test which is also called ratio test, this is basically a comparison type. Is it not?, you are comparing the next term to its preceding terms, and then finding ratio with that. Another test which is also ratio test where we take the n-th root of this.

So, n-th root test also we say or (( )). Now, here what we have seen is that, we are taking these terms z 1 plus z 2 z n and so on. This is the series we are fixing N, and once you fix up N; then after this N the ratio of term true is preceding term is always be less than equal to q. So, this inequity must be satisfied after a certain stage on (( )). Which is not so easy to find out the q; so, in that case if as this ratio has a limit when m is sufficiently large. Then the same ratio test can be put up in a very simple way and (( )) you need not to find q. So, that is the another version of this slightly better result. So, we can say this is the ratio test.

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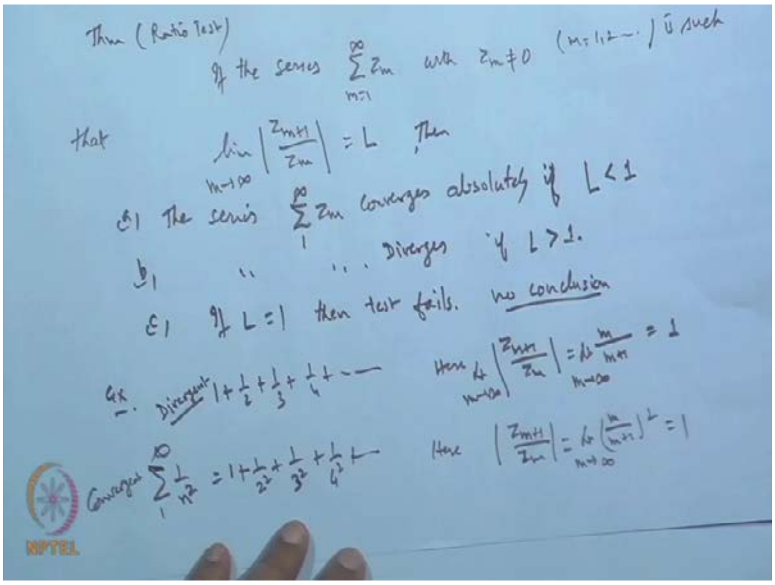

In another form, if the series sigma z m m is 1 to infinity with z m is not equal to 0 m is 1 2 3 and so on, is such that the limit of this ratio z m plus 1, over z m as m tends to infinity. If this limit is suppose say l, then the series sigma z m 1 to infinity converges absolutely; If L is less than 1, the series sigma (( )) diverges, if L is strictly greater than 1. And for L is equal to 1, if L is 1 then test fails. It means nothing can be predicted, then nothing can be predicted no conclusion. In this case, a series may be convergent may be divergent. Means we have an example we have this ratio test is 1, series diverges ratio test is 1 series converges also.

For example, so first taken proof we have this dropping. In support of see, we give an example; suppose I take a series say series 1 plus 1 by 2, 1 by 3, 1 by 4 and. so on. This series here z m plus 1 by z m mod of this, this is nothing, but what, z m plus 1 by z m means m over m plus 1. So, when you take the limit m tends to infinity limit m goes to infinity, this is divided by m, so finally you are getting one. But this series we have seen is a diverging one, divergence series. Then this series 1 plus 1 by 2 square, 1 by say 3 square, 1 by 4 square and so on. Now this series is of the form sigma 1 by n to the power p, where p is 2. 1 by n to the power p, where p is 2. It means this series is a convergent one, because it is a standard result 1 to infinity.

So, this is convergence series. But, here if I take the ratio mod z m plus 1, over z m. What you are getting is that m, over m plus 1 whole square, and limit of this as m tends to infinity again it is 1. So, the l is 1 you cannot give any guarantee, about the nature of the series. It may be convergent, may be divergent. That is what this is.

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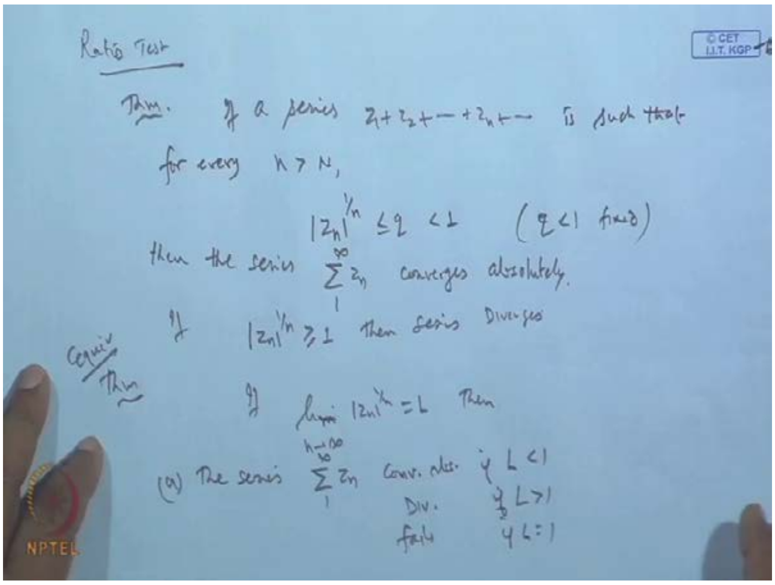

Then another form of ratio test, if the series is such. If a series z 1, plus z 2 plus z n and so on, is such that for every n greater than some integer capital N. The mod of z n to the power 1 by n is less than, equal to q which is less than 1; where q is fixed, then the series converges.

Series sigma of z n, 1 to infinity converges again absolutely. And, if mod of z n power 1 by n is greater than equal to 1, then the series diverges. Equivalent to this, if this limit exists; in case if the limit exists then we can say if the limit of this limit of mod z m to the power 1 by n as n tends to infinity, is suppose l. Then the series sigma of z n 1 to infinity converges absolutely. If L is less than 1, diverges if L is greater than 1 and test fails if L is equal to 1. Again, for in this support for 3.

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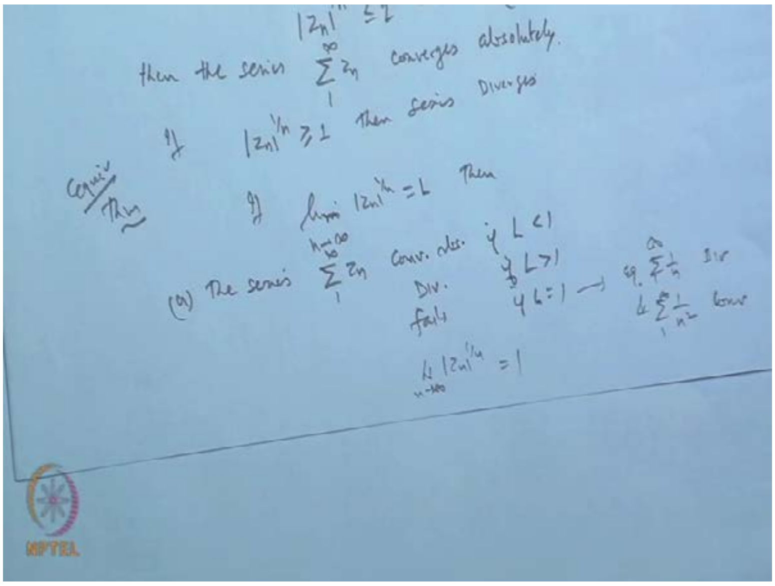

If I take this support of three, again choose the example sigma 1 by n and sigma 1 by n square 1 to infinity. This is diverging, this is convergent and in the both cases mod z n to the power 1 by n, as n tends to infinity comes out to be 1. So, this is what we get it. So, this gives a rough idea about the various tests which will be used, for getting the nature of this series given series whether it is convergent or divergent part. And, it is very effective tools these are particularly when we judge for the radius or reason of convergence of the power series.

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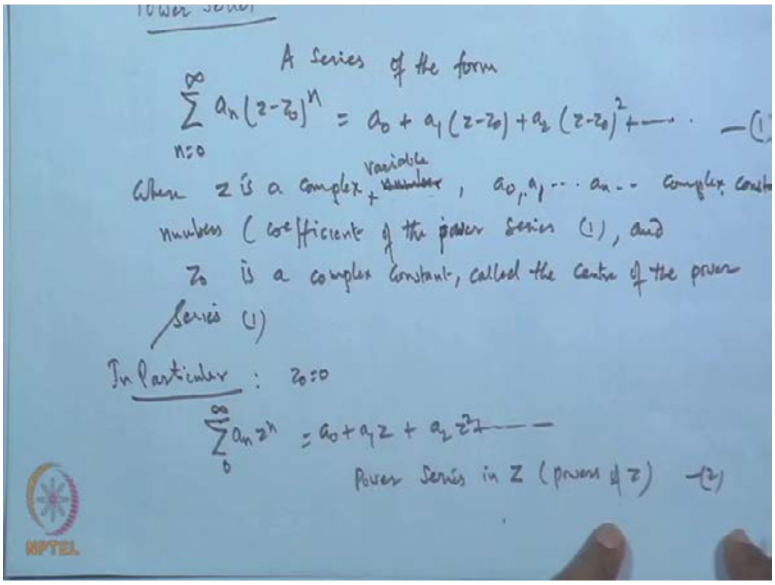

Now, let us come to the main thing is the power series. The power series a series of the form sigma a n z minus z naught, to the power say n when n is 0 to infinity. That is if I expand it it gets a naught plus a 1, z minus z naught plus a 2 z minus z naught whole square and so on. Where z is a complex number or complex variable a naught, a 1, a 2 these are all complex numbers and these are known as the coefficients of the series of the power series one, and z naught is a complex constant.

Now, here z is a complex variable you can say variable complex number these are the complex constant numbers. These are the constants. Complex constants, these are the complex variables. Because, you keep on giving the value of z, you get it different points some of this power series. So, z naught is a fixed complex constant, it may be real also. Then, n is called the centre of the power series called centre of the power series one.

So, what is the power series? A series of this form a naught plus a 1 z minus z naught plus a 2 z minus z naught whole square and so on. Where z naught is a fixed point which is called the centre of the series; a naught, a 1, a 2, a n these are the complex constants or real constants, and z is a variable one. Then this form of the series is known as the power series. Now, in particular case when we take z naught to be 0, that is the series of the form a naught plus a 1 z a 2 z square and so on. That is the sigma a n z to the power n 0 to infinity. Then this is a power series in z disinterred 0. This is the power series, in z in the complex variable z, in the power series, in the powers of z, in the z means in the powers of z disinterred 0. Now again when we look this power series either one or may be two then since z is a variable one.

So, when we give a particular value to z then it becomes it gives one series one power series or one series of complex numbers that is all. So, that series may converge or may diverge also. So, it depends on the z point to point the series will converge diverge like that. But, in the case of the power series of complex numbers, we have a very pick real things; that if a series converges, then it will converge at every point inside the disc which centered z naught and diverges outside. So, we have to identify the reason of this convergence where the series is convergent or where does it diverges. So, let us see a few behavior of this convergence behavior of power series.

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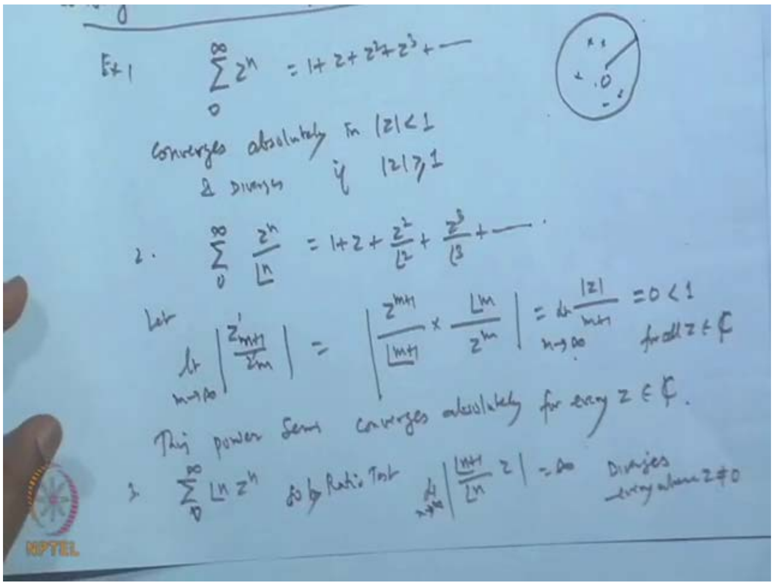

Convergence behavior of power series. So, what is say there are the power series which will converge everywhere in the complex plane. There are the power series which converges only at singleton point there are the power series which converges only in the region in the form of the disk.

So, let us see the various examples where the series converges. First is let us consider the series sigma z to the power n n is 0 to infinity, that is 1 plus z plus z square z cube and so on. Now, this is a geometric series and obviously, this series when mod z is less than 1 converges, when mod z greater than equal to 1 diverges.

So, this series converges absolutely; in the region mod z less than 1. It means this is a power series centered at 0, with a radius 1 if I draw then at all points inside this the series converges absolutely and diverges outside, if mod z is greater than 1. But, what about this point when you take the point on this circumference, we cannot say right now what will be the behavior. Say for example, z equal to 1, 1, 1 then it will diverge. If we take the z equal to minus 1 then what happens some point is plus minus alternate series. So, again it will diverge.

So, we cannot say anything about this except when sorry for z is equal to 1 what happens is is 1 plus 1 plus 1, so it will diverge; so we can get greater than equal to 1. Now, another example let us take, if we look the series sigma 0 to infinity, z to the power n factor 0 is 1; so z to the power n factorial n.

If we look the series is 1 plus z plus z square by factorial 2, z cube over factorial 3 and. so on. Now, this series for some point of z, whether it is convergent or it converges to all point of z or it never converges let us see. So, it is the series of the complex variables Apply the comparison test or ratio test. What is our earlier test which we have seen? Apply this test, then we get ratio test. So, what is our z m plus 1 over z m, let us apply this test.

So, let us consider this. This is the first u m plus 1. So, I will say z dash is z m plus 1 over factorial m plus 1, and z (( )) m is z m factorial m modulus of this, take the limit as m tends to infinity. Now, what happens to this. This is the mod z over m plus 1, limit m tends to infinity. Now, whatever the z may be once you fix up value it will have a finite value mod z.

When m is sufficient large the values will be 0. So, it is always less than 1 for all z lying in a complex plane. It means the series is convergent for every point in the complex plane. So, we say the power series converges so this power series converges absolutely for every z; in the complex plane c.

Now, let us look the other series suppose I take third example; look the series sigma factorial n z to the power n 1 to infinity or 0 to infinity even 0 is first term is 0. First term is 1 because this is 1 plus z. Now, if we look the series what is this? All the terms of series is positive mod z. So, if we apply this ratio test, by ratio test what we get z n plus 1 over means factorial n plus 1. Factorial n mod z limit of this as n tends to infinity goes to infinity. Because, whatever the z may be this n plus 1 will come and it will tends to infinity. It means this series diverges everywhere. Except what? Everywhere where z is different from 0. Because when z is 0 it reduced to a single value only 0. Or the terms of sorry one only one. So, it is convergent. And in fact, every power series converges at the center, that is the result.

So, what we have seen through these examples that, behavior of the power series depends on the coefficients basically. Because this z to the power n this is the n-th term with the coefficient a n. So, if a n's are changing the correspondingly the behavior of the series changes. So, we wanted to have a result, where the convergence or the divergence of the series can be easily identified with the help of the coefficients n's only. That is if we apply that result on the coefficients you will find out you will say whether the series is convergent divergent or neither means only one point or all points like that.

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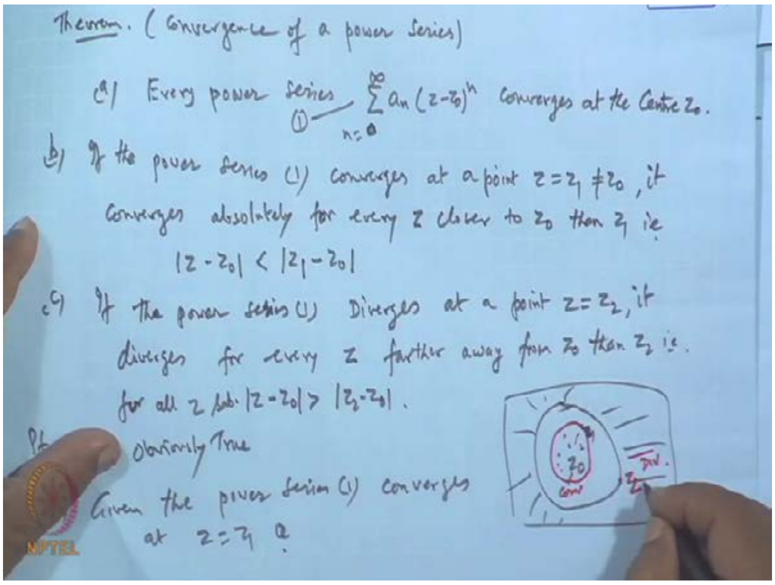

So, before that let us see few results which are valid for a general power series the result is theorem convergence of a power series. So, first the result says every power series, that is sigma n n is 0 to infinity a n z minus z naught to the power n n is 0, to every power series converges at the centre, at the centre z naught. Second result says, if the power series one let it be one. If the power series one, converges at a point say z 1 which is different from z naught. Then it converges absolutely for every z closure to z naught closer to z naught then z one.

That is the set of those points z will satisfy these condition. That mod of z minus z naught is strictly less than mod of z 1 minus z naught. Then all such point then the series will converge absolutely. And third result says, if the power series one diverges at this point z say z equal to z 2; then it diverges for every z. Further away from z naught then z 2. That is the point z minus z naught is greater than the point z 2 minus z naught that is for all z which are satisfying this condition.

So, what is the meaning of this is; suppose I have a power series, whose centre z naught and a point suppose this is the point. This is the region of the convergence here this is the region of divergence. What it says is suppose we have a point z 1 here, and if the series converges at a point z 1 then it will converge at every point inside this z.

If z 1 is this point then it will converge at every point inside this. And if suppose z 2 is somewhere here, this is our z 2. And if the series diverges then it will diverge here. Here it will converge, and yet the point z naught always it is convergent. So, the first part is follows quickly because when z is z naught, this reduced to a single point a naught. So, series is convergence. So, the second part be a obviously true; b part if the power series converges at some point z 1, this is given. So, suppose given the power series one converges at the point z equal to z 1. So, it means that is the when we substitute z equal to z 1.

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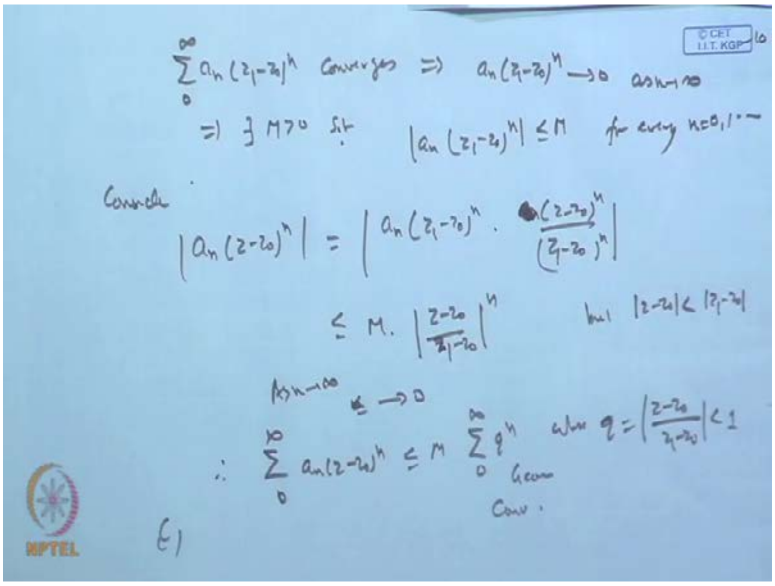

Then you are getting this series sigma a n z 1 minus z naught to the power n 0 to 1 converges. And the necessary condition for convergence is the n-th term must go to 0; as n tends to infinity. Because this is the necessary condition for a convergence of this series. So, if the an n-th term is tending to 0, it means it means bounded form. After certain n stage remain less than equal to m.

So, we can say from here there exists m positive such that mod of a n z 1 minus z naught to the power n less than equal to m, for every n 1 0 1 2 3 and so on. Now, consider this term a n z minus z naught to the power mod n. Now, this will be equal to mod a n z 1 minus z naught to the power n and then mod a n z minus z naught a n is common. Divided by, z 1 minus z naught to the power n. Now this term is less than equal to m.

So, it is less than equal to m and this term is z minus z naught this n z 1 minus z naught to the power n. But, z minus z naught is strictly less than z naught minus z naught. So, this is less than 1 basically. So, which is this less than as n tends to infinity, this will go to 0. So, it is tending to 0. So, this will go to 0 basically. Therefore, this will (( )). So, the series a n z minus z naught to the power n 0 to infinity is dominated by this series sigma q to the power n 0 to infinity.

Where q stands for z minus z naught, z 1 minus z naught, which is restrictedly less than 1. So, this is geometric series convergent. Therefore, this series will converge. So, what is in the series converge? Third part c follows in same suppose it diverges, here we want it to say diverge. Suppose it is convergent then by the previous result it should converge at every point here is it not. If at any point will converges z 2 also which contradiction. So, that third part is show.

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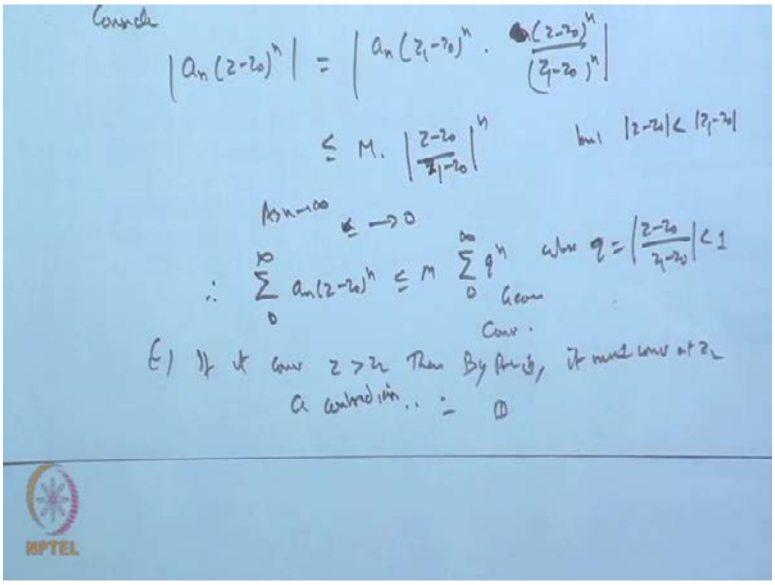

So, third part is show is assume if it converges all z greater than z 2, then by previous be it must converge at z 2 a contradiction. And contradiction leads contradiction shows the proof is okay. So, thank you that is all.