Advanced Engineering Mathematics

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Lecture No. # 17
Cauchy Integral Formula

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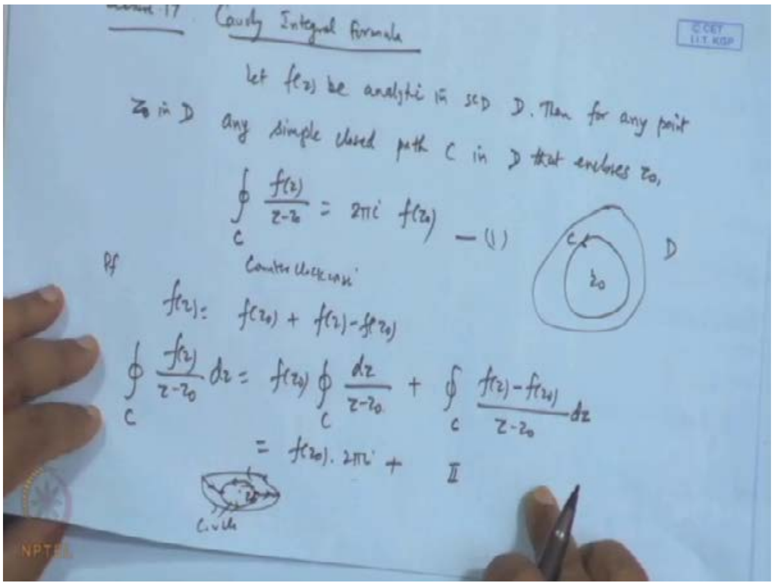

So, we were discussing the cauchy integral formula; and this cauchy integral formula said that if z be and I will taken a simply connected domain, let f z be analytic; in a simply connected domain D, then for any point z naught in D for any point z naught in D and any simple closed path C in D that encloses z naught. The value of this in t here f z over z minus z naught along the path C is equal to 2 pi i times the value of the function at a function that is, this is our domain d. z naught is this point and C is a curve which encloses the point z naught line in d. That direction is anti clock wise direction counter clock wise direction counter clock wise C is diverse in the counter clock wise the proof of this we already seen few examples just on this formula, let us peaceful proof of this.

So, we start with the z which can be written as f (z naught) plus f z minus f z naught, now substitute in one this is the one. So, integral of z, z, minus z naught along the path C d z is nothing but what f z naught is a fixed value. So, we can take it outside in theta C d z, over z minus z naught plus integral along the curve C, f z minus f z naught divide by z minus z naught d z. The first part of this is nothing, but the f of z naught into 2 pi i. Because this integral is a well known one where n is equal to minus 1 the value will come out to be 2 pi i, what is by the close path C may be this cycles. So, C is the cycle of close path there is a simple close path C is simple close path. In fact, it can be transfer to the circle easily look up by deformation of the path because if C is any close path suppose this is our C which is not a circle it improvises the point z naught.

Then the value of the integral along this path C will be the same as the value of this integral along this circle why the reason is like this that if you suppose picked up the two point here then the value of the line integral from this to this is the same as the value of the line integral along this path by deformation of the path, because the function is analytic. Similarly along this path the value will be the same as this path. So, value of the integral along any closed curve will be the same as the value of the integral along this circle centre it should not with that radius r. So, we can transfer and the value will come out to be 2 pi r. So, there nothing to value r then second part we wanted to show we this path will go to o this is our. So, second part we wanted to, what is our second part is f z minus f z naught over z minus z naught.

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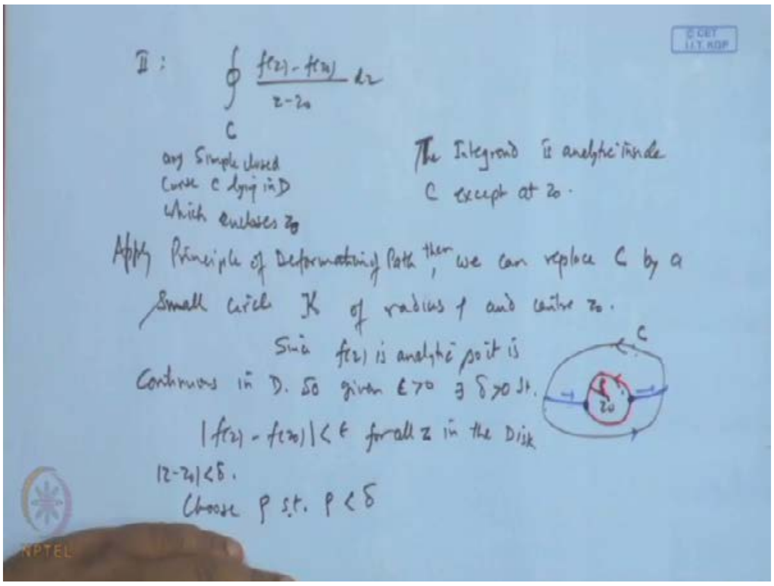

So, let us see now second part is integral f z, minus f z naught over z minus z naught. Along the simple closed, any simple closed curve C lying in D and which encloses the point z naught this was the we wanted the value of this integral will be 0. Now if we look this function f z minus f z naught over z minus z naught, now this function is analytic excepted z equal to z naught. That is the function the integrant is analytic insides C except at z naught. So, if we look function f z minus f z is analytic, but the point z naught is at this point the whole function is C is to be analytic. It is now apply the principle of deformation. So, apply principle of deformation of path.

So, we can replace C by the curve close curve C by a small circle k of radius say rho and centre z naught. Why this is our C is a known this was our C, because this curve C was this which imposes the point z naught, now a deformation of path we are replacing this C by means of a circle, center a z naught with the radius z rho this is a radius and direction remains the same. This is and this is possible because the deformation path because if I take two curve as I told earlier that if I take any point join this one then if I go along this direction and combine along this and going from here like this that this domain the function is totally analytic.

So, the integral of this function point; this two point along this path and the path in this x direction will remain the same. So, this is what a this one here and here if we take this integral from here to here we start from this whole like this and up to here this integral of the function of this path will be the same as the integral along this red line because throughout this the function f z minus f z naught over z minus z naught is analytic. So, this way by deformation path C can be reduce to this path, similarly this part of the C can be reduce to this portion. So, entire C can be reducing by a circle centered edge not by the shift radius. So, that is what by using the deformation of path we can replace C by a small circle of radius rho and centre same without altering the value of this.

Now further since the function f z is analytic. So, it is continuous throughout D, hence at particular it is continuous f z naught. So, for given epsilon greater than 0, there exist r delta there exist delta such that mod f z minus f z naught less than epsilon for all z in the disk not z minus z naught is less than. Now choose rho without rho is our own choice we can reduce rho we that is all.

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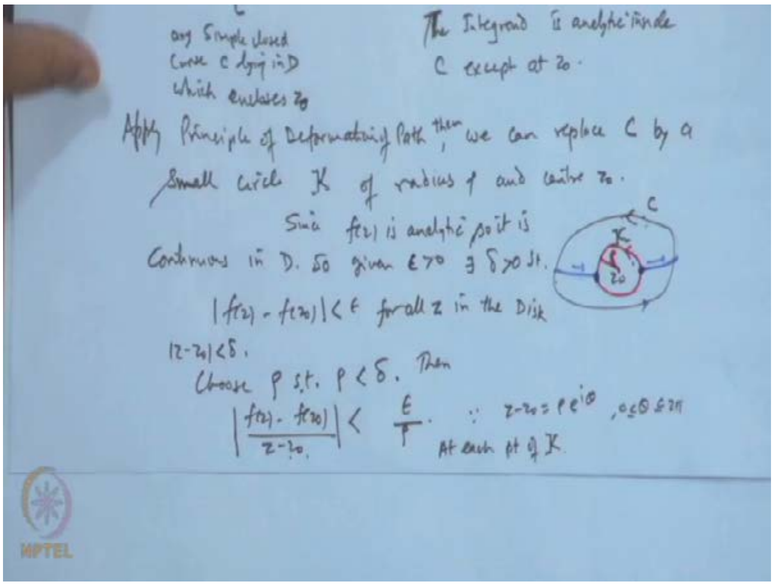

So, choose rho such that rho i is smaller than rho is less than delta is smaller than delta now once you choose then consider then mod f z minus f z naught divide by z minus z naught this will be a strictly less than because this part will less than epsilon z minus z naught is a circle centered z naught with radius rho. So, this is equal to rho. So, there because z minus z naught mod is rho e to the power i theta your theta e lying within this is a circle. So, z minus z naught mod becomes so we get this and this is to at each point of a circle k this is a circle k this now apply.

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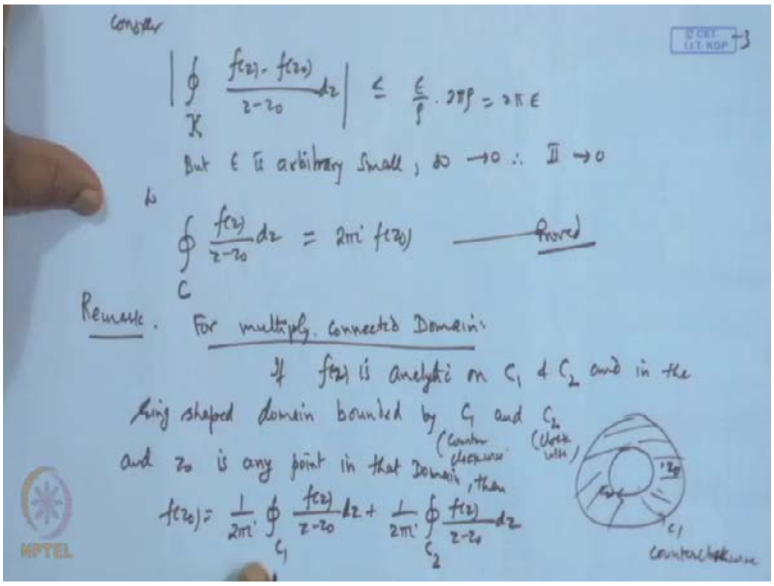

So, now consider mod integral f z, minus f z naught divide by z minus z naught d z modules of this along the path k; now this will be less than equal to now this part is less than epsilon this is less than… So, epsilon by rho and mod of in an integral mod d z over k is the circumference of the k. So, this is that 2 pi rho, so this becomes that 2 pi epsilon, but epsilon is arbitrary small. So, it goes to 0 as it can be this will go to 0 is in one a arbitrary small therefore, second part is tend to 0 this not. So, we get the integral of this path finally, we get integral of the function f z over z minus z naught along any simple close curve c which imposes the point z naught and f is generate a throughout will be equal to 2 pi i times that is proved.

So, this will be three now examples we have already seen. So, in all together now in this is done for a simple connected domain when D is a simply connected domain and C is a curve inside it. So, for a multiple connected domain they are cauchy integral formula we will take this shade. So, let us suppose if f z is analytic and c 1 and c 2 and in the ring shaped domain bounded by c 1 and c 2. That is this is our domain that this is c 1 here is c 2 we are taking opposite direction, here c 1 in clock wise c 2 in counter clock wise c 1 is c 1 is counter clock wise where c 2 is a clock wise.

So, this is counter clock wise and here is clock wise there is some this like this. So, suppose f is analytic on c 1 and c 2 and in the in sub domain case in sub domain case and z naught be and z naught be is any point in that domain. Then the value of the function at a point z naught is equal to 1 by 2 pi i integral along the curve c 1 of the function z over z minus z naught d z plus 1 by 2 pi i integral of the function f z over z minus z naught d z along the curves here the integer is outer is taking in a counter clock wise inner is taking in the clockwise and z naught is any point. So, this is this gives that cauchy integral formula for the domains now there is one derivative of the analytic functions.

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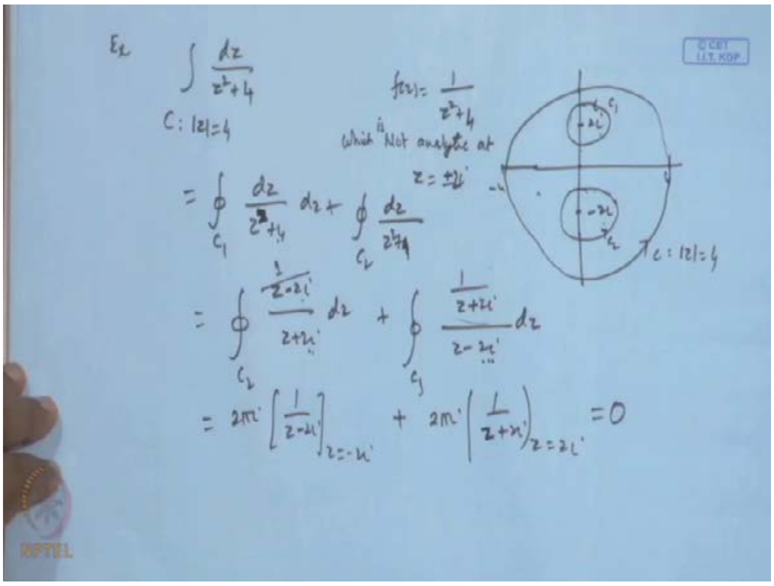

So, before that let we take one of the example is…

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That a eclipse let see this example one by integral d z over z square plus 1 over the cycle C, a C is mod z equal to 4 mod z equal to 4. Let us see this now the function f z is 1 by z square plus 1, which is not analytic which is not analytic at the point z equal to plus minus this at the point plus minus here z square plus 4 this is 4 z square plus 4. So, we get the mod z equal to 4 plus minus 2 i 4. So, align here z equal to 2 i z equal to minus 2 i the circle is center at 0 with the radius 4. So, minus 4 plus 4 and here is something.

So, this is our cycle this is our C. Now z equal to say 4, now the value of this integral we want this. So, it will look this function is two similarities. So, we can move these similarities going the two curves say c 1 and c 2. So, line integral along the pass is the same as the line integral in a boundary. So, this will be the same as integral along c 1 d z over z square plus 4 plus the line integral d z over z square plus 4. Now c 1 encloses the point 2 y. So, basically the function z minus 1 by z minus 2 i over z plus 2 i, so if we wake up z square plus 4 z plus 2 i and z minus 2 i then consider f z is 1 upon z minus 2 i and then z minus z naught is this. So, it is c 2 equals in minus 2 i. So, z equal to minus 2 i it is infinity and for c 1, you can write 1 over z plus 2 i over z minus 2 i along the pass c 1.

So, it be write it this way then what happen this becomes a function f z which is analytic throughout the domain is a not in because c 1 in project only z equal to 2 i. So, outside the function is analytic everywhere except this point. So, this is f z over z minus z 1. So, the value of this integral will be equal to 2 pi i times the value of the function 1 z plus 2 i at a point z equal to 2 i and similarly this also this will be equal to 2 pi i the value of the function at the point similar point z equal to minus 2 i. So, this total becomes 0. So, we can get this point; further results for this Cauchy integral that is the derivatives of the analytic functions.

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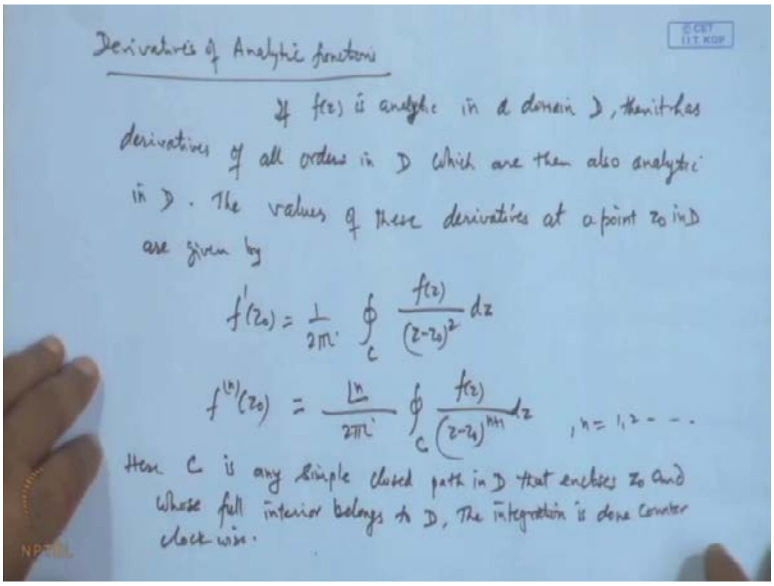

Now, we know if function f is continuous then it may or may not be differential, but if the function is continuous and differential then again we cannot talk about the second order derivative of the function of 0. Because it total depend what I would the function, but in case of the functions of complex variable if the function f is analytic then all of its derivative will achiest and that can be establish with the help of our Cauchy integral formula. So, the proof we are not give me just we are giving the formula for the derivatives of analytic functions that this results such if f z is analytic in a domain D then it has derivatives of all orders in D, which are then also analytic in D.

The values of these derivatives at…

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a point z naught in D are given by the formula given by f dash z naught is 1 by 2 pi i integral along the curve C f z over z minus z naught whole square d z and that derivative of f at the point z naught the formula is given by is factorial n over 2 pi i integral along the path C f z over z minus z naught to the power n plus 1 d z here n is 1 2 3 and so on. Here C is an simple close e f C is any simple close path in D any simple closed path in D that encloses that encloses the point z naught and the whole interior the full interior belongs to D and we integrate the integration is taking counter clock wise the integration is done counter clock wise integration.

So, what this result says is that f of function is analytic in a domain D, then it is all of it is derivative of higher orders all orders will exist and they will also the analytic. Further it we have the value of the function it not by the help of the Cauchy integral formula, then the corresponding values of the derivatives at the point z naught is given by this formula. The first derivative f line z naught the 1 upon 2 pi i integral C over C f z over z minus z naught whole square d z; and the another derivative at the point z naught is doing by this 1 factorial n over 2 pi i integral of the function f z over z minus z naught to the power n plus 1 over the curve C. Here C is any single closed path that integration is taking in the counter clock wise direction, and the curve must impose the function lies in the domain here the function is analytic and curve.

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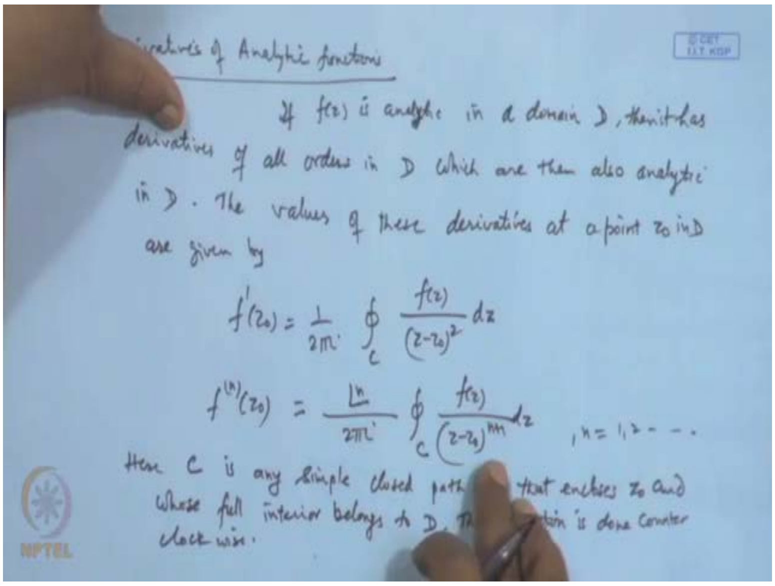

So, which curve this is the, but it will look this Cauchy integral formula then Cauchy integral formulas nothing, but the value of this is our Cauchy integral formula integral f z over z minus z naught is equal to 2 pi i f z conditions all exist is same. So, if you want the first derivative then it e g to by just assuming that we are differentiating the whole thing with respect to z naught keeping at thus variables. So, if I differentiate with respect to z naught the right hand side become 2 pi i f z naught and the left hand side f z is constant. So, it is one upon z minus z naught for this square into z that is what I checking getting. If you further differentiate factorial 2 sin come and by 2 pi i f z over z minus z q and like this. So, it is easy to remember that formula, if you know the Cauchy integral formula. So, we will go the proof of one only and this will follow.

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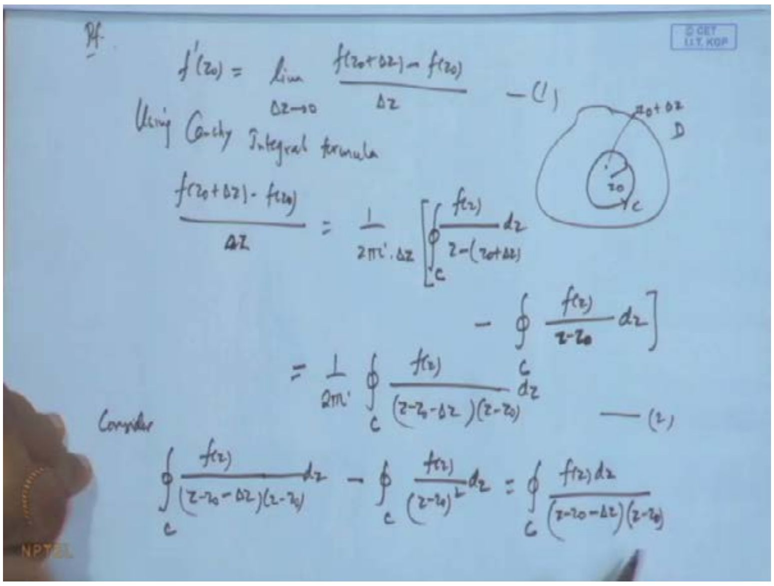

So, I will not the proof of first part. So, what we have to prove is the derivative explain z is nothing, but this what is the derivative of a function at a point z naught is nothing, but the limit of this f z naught plus delta z minus f z naught over delta z then the delta z tends to 0. So, this is enough. So, what we are doing is we are having this is a domain D here is the point z naught and this is a cell k. Now if I look, take a point another point z naught plus delta z which is very sufficiently close to this then the function f will also be analytic at this point and inside it encloses that curves C this is the curve C. It is include it the curve C will also enclose the point z naught plus delta z.

So, if you know the value of the function at a point z naught with the help of the Cauchy integral formula then we can also find the value of the function at the point z naught plus delta z with the help of Cauchy integral formula. So, apply the Cauchy integral formula or using Cauchy integral formula to get integral formula to get the value of the function f z naught and value of the function at a point z naught plus delta z we get f z naught plus delta z minus f z naught divide by delta z. Now this is nothing, but 1 by 2 pi i delta z will remain it is and then the value of this integral f z over z minus the point z naught plus delta z d z along the path C minus the integral of the function f z over z minus z naught d z along the pass because f of z naught is 1 upon 2 pi i integral f z over z minus z naught d z.

So, z naught by a z naught plus delta z get we are getting this simplifying just when you simplify it you will get the value will come out to be 1 by 2 pi i integral along the curve C f z over z minus z naught minus delta z into z minus z naught d z. Just we just simplify you will get the value.

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So, that let will be ones two this will one this two now what is what we want is we want it to prove this part that of this integer f z naught is this it means then you take the limit delta z goes to 0 the limiting value of this that is the limiting value of this must coincide with this or another words if I take this minus the difference of this then L delta goes to 0 must go to 0. Now, consider that difference of this two. So, if you take the difference of this integral because this integral integral minus the difference of this integral is in know.

So, consider this integral f z over z minus z naught minus delta z into z minus z naught d z along the curve C minus the integral of this f z over z minus z naught whole square d z s delta at goes to 0 delta 0 this must go to 0 then what is this value is nothing, but what f z over this thing is in know and then minus. So, we get this part equal to if I just take because this integral is the same along the same path. So, this integrant can be and we get the integral along the curve C which is comes out which comes out to be f z d z over z minus z naught minus delta z into z minus z naught. So, this is finally. So, we wanted to show that this goes to 0 again tends to infinity.

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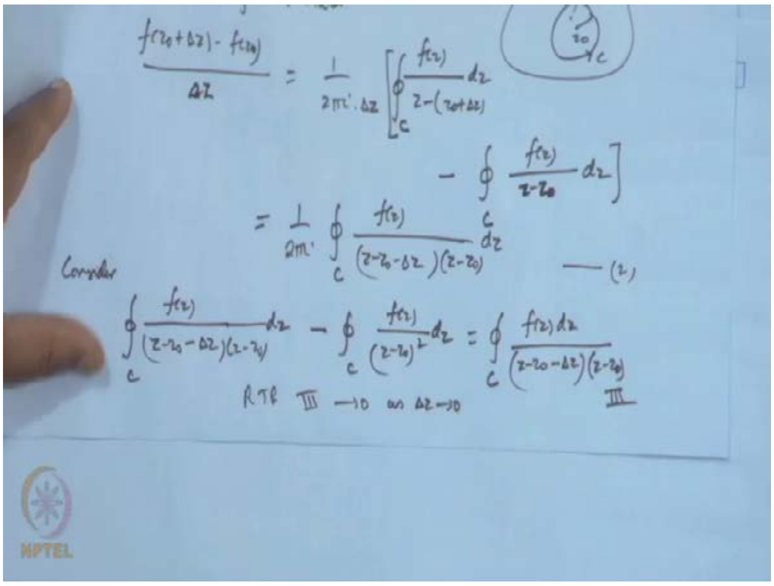

So, our aim is to show required to show is this part, let it be this part third tends to 0 as delta z goes to 0.

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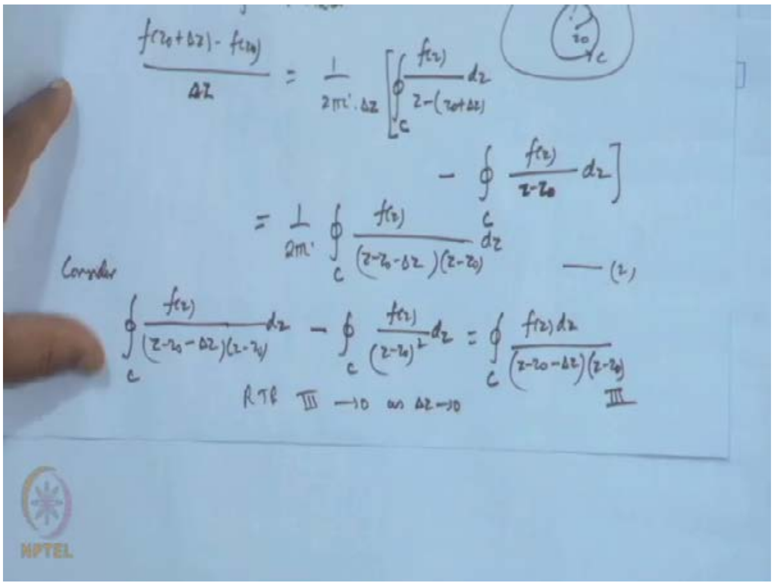

So, using them now to show this thing since f z is analytic. So, f z is also continuous f z is continuous one see must it is continuous then which is bound it every continuous function is bound it. So, it is bound it on C and once it is bound it in a should therefore, there exist a constant k such that mod f z is less than equal to k, for all z this is for all z belonging to C. Now let D be the smallest distance D be the smallest distance from z naught to the point of C that is this is our z naught here this is C. C is any close curve.

So, this a smallest distance it denoted by D. D is the smallest distance from this two these then for all z belongs to C we get we have mod z minus z naught which is less than equal to z minus z naught minus delta z plus this can be written as z delta z minus delta z. So, equality sign and which is less than equal to z minus z naught minus delta z and then plus mod delta z, but the z is an arbitrary point and D is the smallest distance. So, this is greater than equal to 2 D is less than equal to this number this. So, from we get D minus mod delta z is less than equal to z minus z naught minus delta z. Now, if I choose z delta z mod of delta z is less than equal to D by 2 that is possible because z naught I can choose the point here a z naught plus delta z in a such way that is f suit value is less than D by 2 because that any point z. Here any point z this is the point z.

So, mod z minus z naught is less than D is greater than equal to D because D is the smallest. If I choose any point z naught plus delta z distance can be made less than D by 2. So, if this we can say if it is there then minus D by 2 greater than this. So, this part is less than half D is less than equal to this portion D minus delta z because this is greater than equal to D minus D by 2 and which is less than equal to z minus this and this. Therefore from here, what we get 1 over z minus z naught delta z is greater than a, this is greater than this. So, 1 by this is less than equal to 2 by that is.

Now, use this part because we wanted this third tends to 0. So, what is the mod of third means mod of integral along C f z over z minus z naught delta z into z minus z naught d z this is mod not if you take mod apply the M L in equality. So, this will be less than equal to mod f z is bounded by k along the pass C z minus z naught mod of this less than equal to 1 upon this less than equal to 2. So, 2 by d z minus z naught is nothing, but z minus z naught this is just D a small. So, it is greater than equal to D this is 1 by this is less than equal to D, so again D.

So, 2 by D and then this is yes this will be D a square will be there I think that there is a left and this part will be equal to is square. So, if square is left I think we have made a mistake. So, here it will be is square. So, this is D square and D square now this approaches to 0 as delta z approaches to 0; why because when delta z approaches to 0 what happens to this delta z approaches to 0 then this mod of delta z is also there yes this mod delta z this is mod delta z. So, d z is mod delta z is it not? So, this mod delta z will go to 0 because rests are the constants. So, this will go to 0 at mod delta this and delta z goes to zero therefore, this integral will go to 0 this integral goes to 0 it means this limit will go to this and once this limit goes to 0 means this limit will be equal to this.

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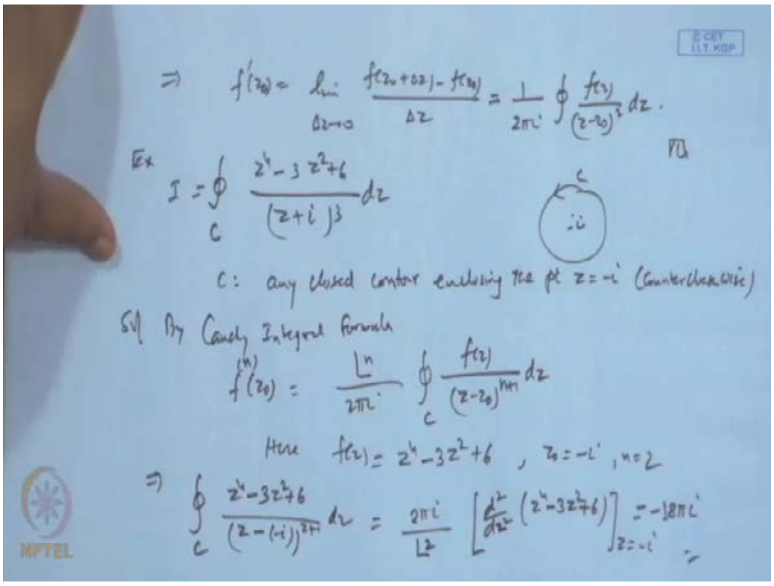

So, this implies that derivative a function z naught is equal to limit z naught plus delta z minus f z naught by delta z tends to 0 and which comes out to be same as 1 by 2 pi i integral f z over z minus z naught whole square d z is in know. So, that would be the Cauchy integral formula a let us see few examples and then we go for the suppose we had this integral say integrate the z to the power 4 minus 3 z square plus 6 divided by z plus i whole cube d z along the pass C we are C any closed counter enclosing the point z equal to minus i counter clock wise. So, if we look this function C is any close counter. So, z equal to minus i this in minus i and C is this which encloses the point minus i.

Obviously, the function z 4 minus 3 z square plus 6 is an analytic function, but because of that z plus i in the denominator total function C is to be analytic in minus i is in know. So, this is equivalent to f z over z minus z naught whole cube. So, if you look the formula for this the formula for C is by Cauchy integral formula that derive a for z naught the any derivative of function at a point z naught is nothing, but factorial n over 2 pi i integral along the close simple close path C in D which encloses the point z naught of f z over z minus z naught to the power n plus 1 d z. So, it be compare this integral in with this f z is this here f of z is z 4 minus 3 z square plus 6 and what is z naught is minus i and n becomes 2.

So, basically this is the same as the f 2 z naught the second derivative of function at the point z naught means i. So, if you take this curve then the value of this equal to this is i and the integral f z means z 4 minus 3 z square plus 6 divide z minus i minus i power 2 plus 1 n plus 1 d z along the simple close curve C which encloses the point minus i will be equal to 2 pi i times factorial 2 is a not and then second derivative of this. So, second derivative of this function z 4 minus 3 z square plus 6 and then value of this a z is equal to 1 minus i a z equal to minus i. So, second derivative of this becomes 4 12 z a square minus 6 f z equal to minus i and when you go for this the value will come out to be minus a 2 pi i.

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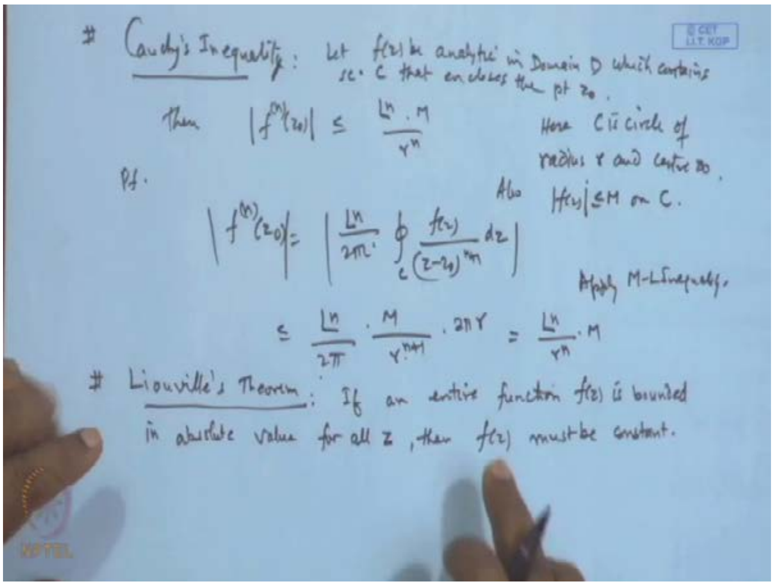

So, this were source the value now as a consequence of this Cauchy integral formula we have few more concepts which are very useful one is the Cauchy inequality Cauchy's inequality what this Cauchy inequality says is the f modules of f n at a point z naught is less than equal to factorial n into n divide by add to the power n we are we choose C in a circle centre we are let f be analytic in a domain the which contains are simple closed curve simple closed curve C that encloses the point that encloses the point z naught inside it z naught inside it.

So, this and let us C is a circles suppose C is a circle which centre which contains suppose here C is a circle circles of radius r and center z naught in centre z naught then we have this the modules of f n z naught is less than this now one more thing is also the function f z is bound it y on C with them boundary. In fact, is analytic it will let us see the proof is follows from the what is our M n f n z naught the any derivative of a at the point z naught is equal to what the formula says it is factorial n 2 pi i integral along the pass C f z z minus z naught n plus 1 d z. So, take the modules apply the M L in equality M L inequality.

So, this will less than equal to factorial n 2 pi then now mod of f z is dominated by M on C. So, it is less than equal to n then z minus z naught because is a circle we have to use in with radius r. So, r to the power n plus 1 d z mod of d z integral C with the circumference 2 pi r. So, we get this part and which is nothing, but what all are get cancel 2 pi get cancel. So, this is nothing, but what factorial n by b r to the power n into n. So, this curve this point that f n is this now from here we also conclude another results which is known as the Liouville's theorem what this theorem says is if an entire function if an entire function f z is bounded in absolute value in absolute value for all z then f z must be constant.

So, we can say in solve their boundary entire function is always a constant function. So, this result also gives one very important results that is if we take the sin z when z becomes real it is a boundary function. So, sin x is boundaries cosine x is boundaries. So, for x is real, but when x is replace by a complex that is the function is defined over the complex field then in that case this sin z becomes unbounded, because if it is bounded according to this result it must be constant. So, it that functions is analytic everywhere throughout them.

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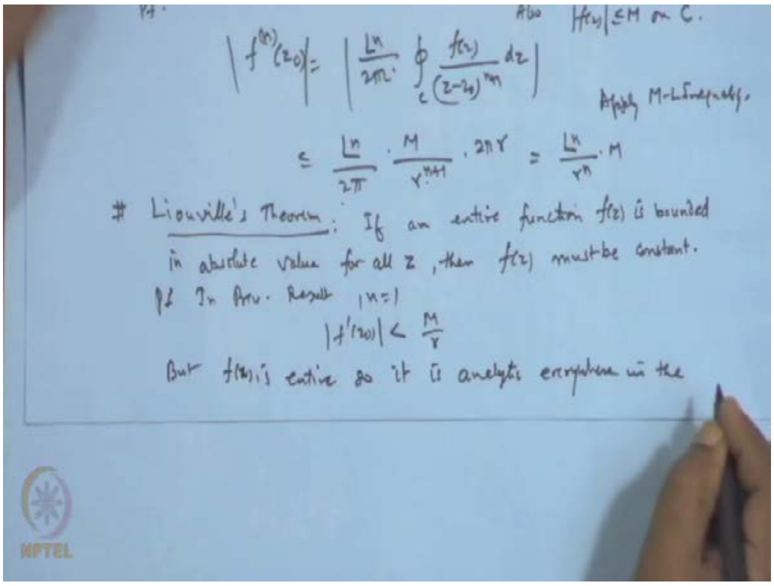

So, let us proof of this is very simple. Now in the previous case, in previous these are if I take n is equal to 1, then what type it mod of f line z naught is strictly less than what M over r, but function f is analytic a entire function. So, f is analytic everywhere in the complex explain, but f is f z is entire.

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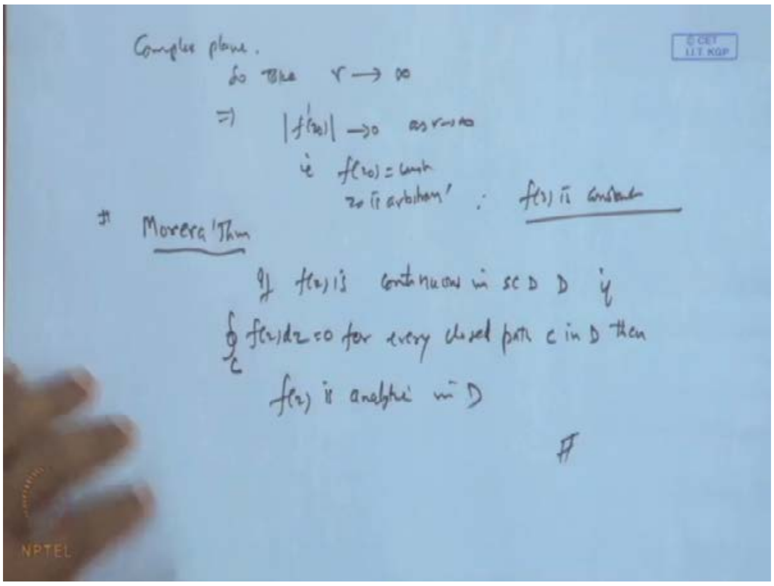

So, it is analytic everywhere in the complex plane therefore, take r passing with infinity therefore, it implies that mod of f line z naught tends to 0 as r 10 that is f of z naught is constant, but z naught is arbitrary. So, we can say the f of, but z naught is arbitrary we can replace z naught. So, the function f is constant good and the converse part of this is the Morera theorem just I will give the result which is the converse. If the function f z is continuous in a simply connected domain D and if integral of the f z along any close path C is 0 for every closed path C in D, then function f z will be is analytic in D. This is the converse of the Cauchy integral theorem. If the function is continuous in simply connected domain D and in integral along any close path 0, then function must be analytic in D thank you very much that is all.