Advanced Engineering Mathematics

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Lecture No. # 16

Cauchy Integral Theorem (Contd.)

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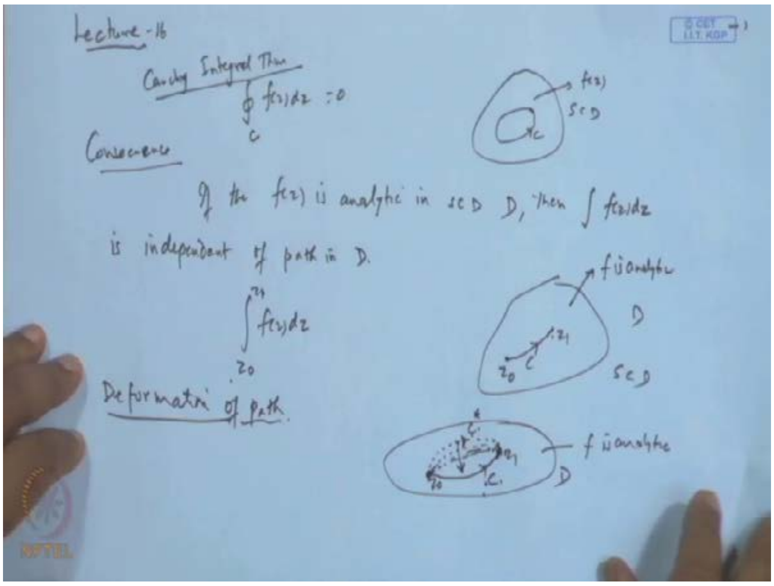

In the last lecture, we have discussed the Cauchy integral theorem; and the Cauchy integral function if the function is analytic in a simply connected domain D. Then for every simple closed curve C line inside the integral of the function f(z) dz along this Cauchy will be 0, this is the Cauchy integral theorem. And as a consequence of this, we have already proved as a consequence to this that the line integral is integral path. If the function f(z) is analytic in a simply connected domain D. In a simply connected domain D then the integral of f(z) dz is independent of path in D; that is if D is the simply the connected domain, and function f is analytic that is finite.

Then if we independent of one means if we pick up the two point z 1 z naught and z 1 suppose, then integral of this function f(z) dz, from z naught to z 1 will depends only at the point z 1 in z naught. But does not depend on the path of integration, what about the path you choose? Joining these two point the value of the integral will remain the same.

So, this comes out as a consequence of the Cauchy integral theorem, application to Cauchy integral theorem. So, this must be. Then solve them a deformation of path is also another use from Cauchy integral theorem one can write deformation of path the deformation of path.

Suppose we have this curve say z naught, and z naught we have the two points, and this is the path of integration C. Since, function f is analytic inside this domain D which is simply connected domain then the value of the integral everyday will depends only that point z naught and z 1 independent of the path. It means if we take any other path joining the z naught to z 1 then the value of the line integral will not change. So, we can think that this C path, this is suppose C is star, then if I consider infinitely many path joining z naught to z 1 is such a way that it keeps shown will do same and going up to the C. In this process the point z naught and z 1 is fix means the path will always join the z naught and z 1.But it approaches to over C and that point through which this curve process we have the function should be analytic.

Say if the function is analytic throughout this domain means whatever the path we choose it will dependent of the integral of the path. So, the C star when it touches to C the value of the C star along c star, and value of along the integral of the function along C be remain the same, and this is called the deformation of the path. This is it means if the path is given in a very uncomfortable way, then one can transfers it is C star to a comfortable path provide it, the initial an terminal points are induct and the function is analytic throughout this process.

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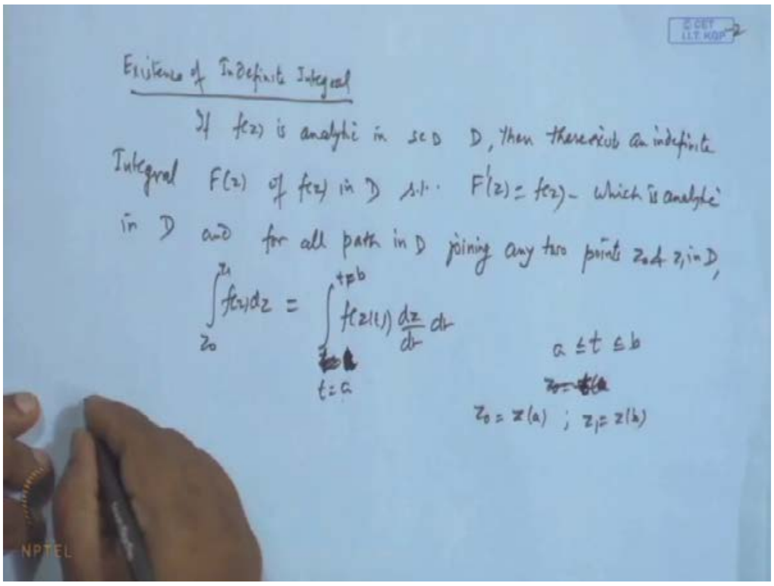

So, this will give the relation. And for a let us see the example for this, suppose we have this curve. Let us we will see the example after word let us see first, now one more result which we have mentioned earlier the existence of indefinite integral this we will just a state without the proof. But this if the function f(z) is analytic in a simply connected domain D then there exist there exists; an indefinite integral capital f(z) of a small f(z) in D, such that f prime z equal to f(z) which is analytic f is analytic. So, f prime z is also analytic in D. And for all path in D joining any two points z naught and z 1 in D, the integral f(z) dz z naught to z 1 can be evaluated with the help of this t f of(z) t dz by dt dt. When t varies from t naught to t 1, t varies from should a to b t varies from a to b. And t lying between a to b, z naught is corresponding to point t of a z naught the which z of t dz and z 1 correspond to the point b.

So, this is. So, we can get the value of this line integral in terms of the definite integral from a to b f of z(t) dt dt. We have already use this path. But the existence of is because of the if function is analytic then definitely the derivative of this function this integral will exists and equal to the proof. We are not now we have seen the Cauchy integral theorem in case of a simply connected domain, but if the domain is not simply connected. Then also we have a Cauchy theorem for multiple connected domains. What we do is here? We convert the multiple connected domain into a simply connected domain first, and then apply the Cauchy theorem for this simply connected.

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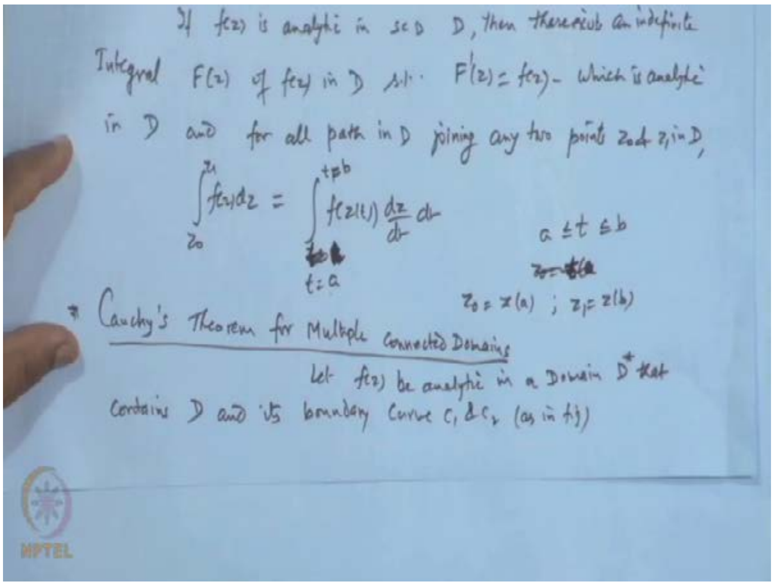

So, this is what Cauchy theorem is? Cauchy’s theorem for multiple connected domain multiple connected domains; Cauchy theorem suppose we have a function let f(z) be f(z) is analytic let f(z) be analytic in a domain D star that contains determine D, and its boundary curves c 1 and c 2 as in figure.

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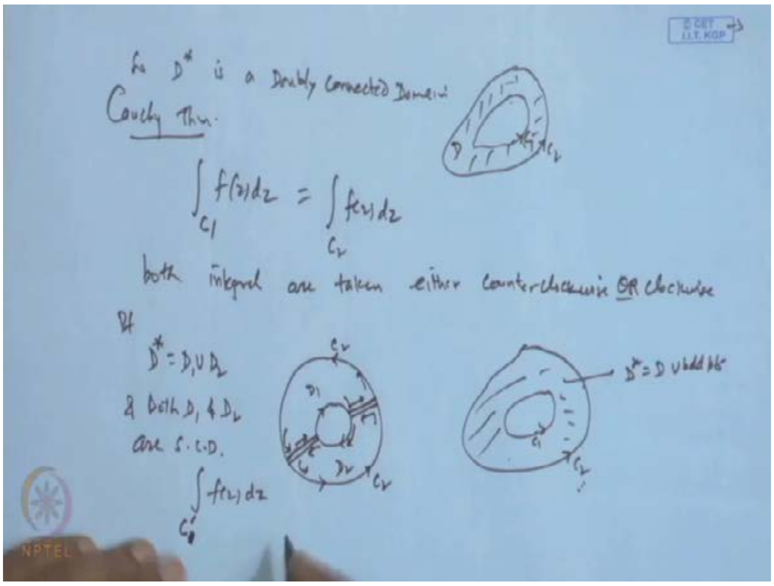

Figure is that suppose this is our D star, this term this is our D contains between that two curves c 1 and c 2 in between this is the domain D. And D star is the domain which contains the points of the as well as the point of the boundaries c 1 and c 2. So, basically

D star is a multiple connected domain is not a simply connected domain multiple connected domain. So, D star is a doubly connected or multiple connected doubly connected domains is not, because this as the two boundaries. And this portion is not there simply connect, then this terms says then the Cauchy theorem for multiply such then. Cauchy theorem say give that integral along the curve c 1 of the function f(z) dz will be the same as integral of the function f(z) dz along the path c 2. We are both integrals are taking either counter clockwise or clockwise clock counter clockwise or clockwise we send both are taken.

So, this is the Cauchy theorem for a multiple connected that is the value of the line integral along the outer boundary is the same as the value of the in line integral along the inner boundary when the direction is that is all. So, let us see c 1 and c 2 then what we say is proof suppose we have a multiple connected domain, we say this two boundary say c 1 and c 2. So, this is our c 1 and here is c 2 and these domain now this is the domain D. D star which is D union the boundary points c 1 and c 2. So, this is our domain D. Now let us convert this thing into a simply connected domain by putting the two cuts along this. So, this is our one cut. So, we start through this way come back up to here and then go from here along this direction; and then come back this way again this direction then you are getting one domain say D 1.

Then similarly when you are going along this direction come back through this an again this direction up to here, then come back and get like this. So, we get D 2 another, it means the D star. Now has been break up into the two domains D 1 and D 2 and both are simply connected domains. So, this is our now here when the path c 1 and c 2 are same is it not then direction is the same, but here the direction reverses. So, that we will take carefully. So, what we have the D star is nothing, but the D 1 union D 2 and both D 1 and D 2 are simply connected domains. So, we can apply the Cauchy theorem for this therefore, the line integral of the function f(z) dz along the path along the path this that is this value c 2 this is c 2. So, if I go along this path c 2 come back this way, and then come back.

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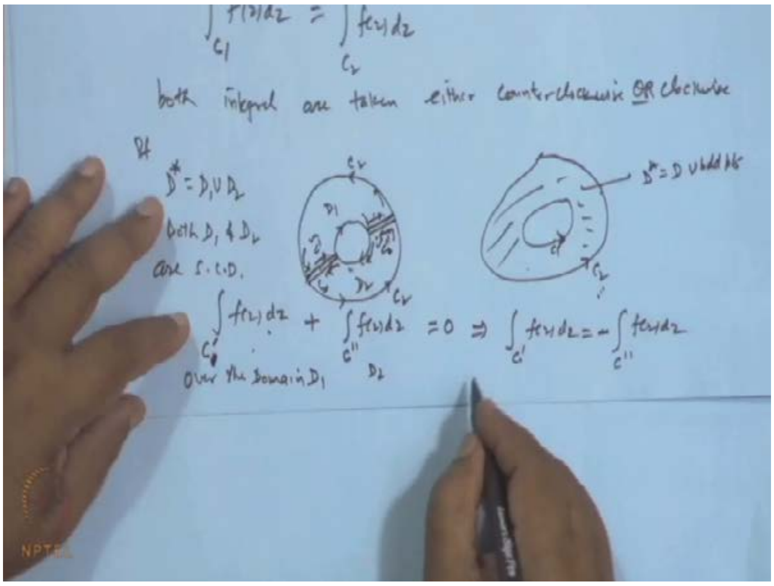

So, let us see the path this is c 1. Let us c 1 dash or c dash cos c dash this along over the curve over the domain D 1, and then the integral of the function f(z) dz over the domain D 2. D 2 we have the boundary is this c 2 plus this and going like this. So, when we consider this integral along this close boundary and function is generating the value of this will be 0 value of this will be 0 and total is 0. So, we get from here is the integral along c dash f(z) dz is the integral minus times of integral along f(z) dz c double dash. The value of the integral along these curves that is this side and this side get cancels similarly this side and this side are opposite. So, the value of the integral along these two cuts will not be continued it will be 0.

So, we do not have at the value either added or subtracted over the arc this c 1, which we call it as c 1 dash delta, and here is c 2 dash c double dash delta this 1. So, we do not have now c dash is this curve; we are this direction is taking in this way that is the direction is opposite to the direction, which we are taking earlier c 1 is this direction and it is opposite this is going opposite of the, so this become the minus of c 1.

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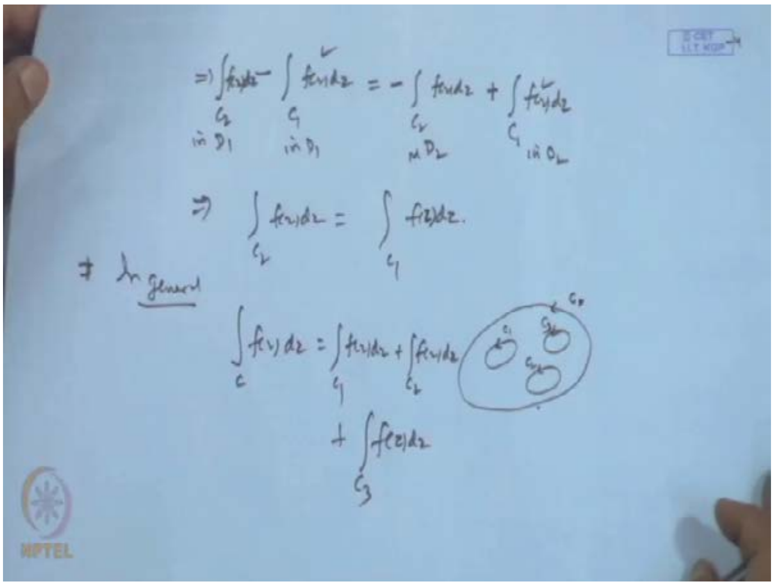

So, this is equal to this becomes equal to minus of c 1 f(z) dz equal to minus. Now c 2 dash c double dash is the curve along this integral along this curve. So, the direction is the same as c 2 basically it is part of the c 2 plus this c double dash, but these are all 0. So, except c 2 it will come. So, c 2 plus this curve c 2 plus this. So, basically this we are getting this c 2 and so this 1; c dash is completely this zero as c dash is what we are getting this curve here c 2 plus this c 1. So, here c 2 will also come c 2 f(z) dz part of it in D 2 in D 1, and this will come the c 2 in D 2 in D 2. In the D 2 plus this portion which is the opposite of this.

So, opposite of this means plus of this c 1 f(z) dz, because this is opposite to this. So, it is c 1. So, part in D 2 this is also in D 1. So, this is the portion lying in D 1 this portion is lying in D 2. So, it be combined these two transfer, then this is the total c 1. So, this implies that integral along c 2 f(z) dz is integral along c 1 f(z) dz is it again repeat what we did, we wanted to prove that integral of the function along the outer boundary is the same as the integral of the inner boundary, then the direction is taking same.

So, what we did we are met the two cuts? So, by making these two cuts the entire domain D divided into two parts D 1 and D 2. Now D 1 consists of a part of the c 2 arc and the arcs c 1 then D 2. Consist the boundary consist of part of c 2 and c double dash arc. So, the value of this integral is 0 means the value along this plus value along this must be 0. Similarly, the value of this it means value along c 2 and these 2 are 0, but these two all are these two are and the same path function is same value will not. Therefore, the value of the integral along outer boundary is the value belong the inner boundary and that is proves the.

So, that is now this can be extended in general suppose there are this is our domain where there are many holes are there, say this was. So, it is a multiple connected domains, and the boundaries are c 1 here c this is the boundary say c 1, this is the boundary say c 2 this is the boundary is c 3. Same direction then the line integral of the function f(z) dz along c will be the same as the line integral of the function f(z) dz along c 1, plus line integral of the function f(z) dz c 2, plus the line integral of the function f(z) along c . Now c 1 is this direction of c and direction of c 1 c 2 c 3 must be the same either both clockwise or may be the anti clock wise, this also now before going for this some more examples here.

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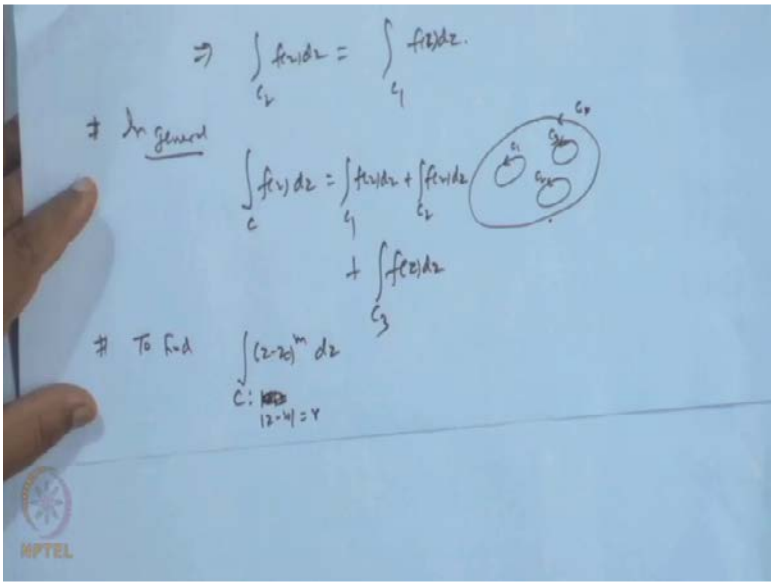

Let us see the first which is interesting to find the value of this integral, because it will be used dz z minus z naught to the power m dz over the c. We have c is the circle any close curve circles mod z circle z minus z naught is equal to say r this we wanted to show.

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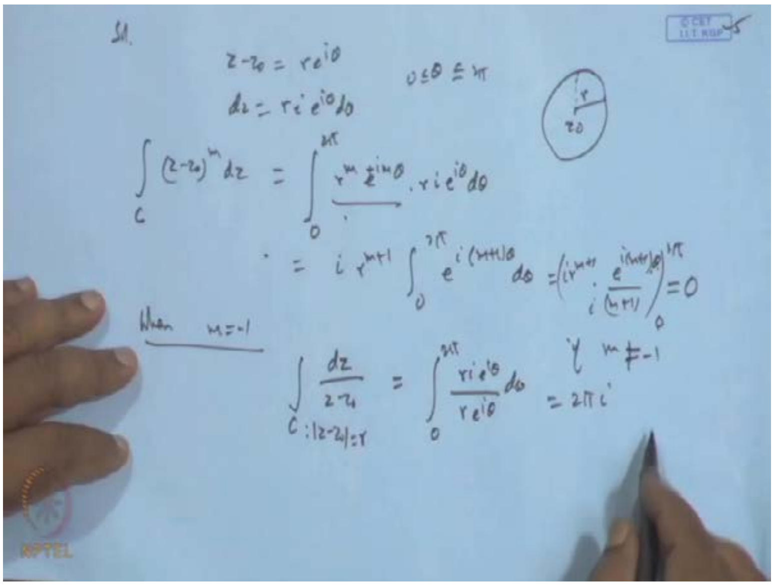

So, it is give in z minus z naught is r e to the power i theta where theta lying between 0 and 2 equal to the circle of r centred edge z naught with the radius r. So, any arbitrary point z if I take it then z can be written as z naught plus r. So, dz will be equal to r i e to the power i theta d theta. And therefore, the integral z minus z naught to the power m dz along this path c will be, z minus z naught is r to the power m e to the power i n theta divide by then dz is r i r i e to the power of r i e to the power i theta d theta now when the n values 0 to 2 pi.

So, basically it becomes i r m plus 1 integral 0 to 2 pi e to the power i m plus 1 theta d theta is it not. And this value will come out to be i r n plus one e to the power i n plus 1 divided by i m plus 1 and limit 0 to 2 phi theta. So, the value of e to the power i some constant theta to 2 pi will be one and at the point 0 is also 1. So, the value will come out to be 0 if m is different from minus 1, but when m is minus 1, then we have to evaluate this integral separately and we get z minus z naught along the path c. We are the c is this circle z minus z naught is r. So, if I evaluate this the value will come out to be z minus z naught is r e to the power i theta dz is equal to r i e to the power i theta d theta and 0 to 2 pi the value will come to the 2 pi i. So, this result says the value of this integral will be 0 when I may different from minus 1 and equal to 2 pi i, when m is equal to minus 1.

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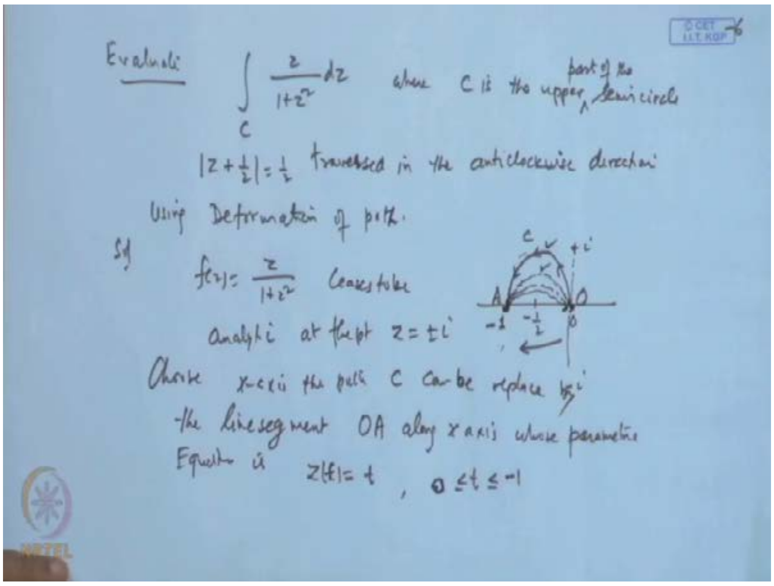

This will be now as we have seen that few examples we will go this, suppose we have evaluate the integral z over 1 plus z square dz along the path C. Where C is the upper semicircle, upper semicircle mod z plus half equal to half upper semicircle is the upper part of the semicircle like traversed in the anticlockwise direction. Using deformation of path deformation of the path, now if we look this circle the circle is basically centre at minus half with the radius half. So, it will be the path and traversed in the anticlockwise direction. So, we are moving in this direction; let now this is our zero the function f(z) which is z over one plus z square cease to be analytic ceases to be analytic that is it is not defined or derivative this cease to be analytic at the point we have to denominator zero z equal to plus minus i and plus minus i will be somewhere line here this is plus i this is minus i.

So, it is does not fall at any point on this circuit on the path of integration this is path of integration therefore, if I join these two point, I kept this is zero, this is one if I kept these two point in and draw the curves joining these two points then the integral along the this curve will be the same as the integral along this curve because the function is analytic all the point and the join initial and terminal point are intake. So, if I keep on changing, then I should I will take only the simplest path and simplest path if this from 0 to 1 along x axis. So, here choose x axis and x axis the path C. This is the path C can be replace by the line segment OA, this is a OA along x axis whose parametric equation is z t equal to x t plus by t, so x t that is t by 0.

So, x t varies from minus 1 to 0, 0 to minus 1, why 0 to 1? Because is taking in a 0 to minus 1. Say we take z equal to t then t varies from 0 to say minus 1 this is minus 1. This is minus 1 then 0 to minus 1, then integral along this path C can be replace by integral of the function along the x axis, along the line segment OA, which is on the x axis joining O to A.

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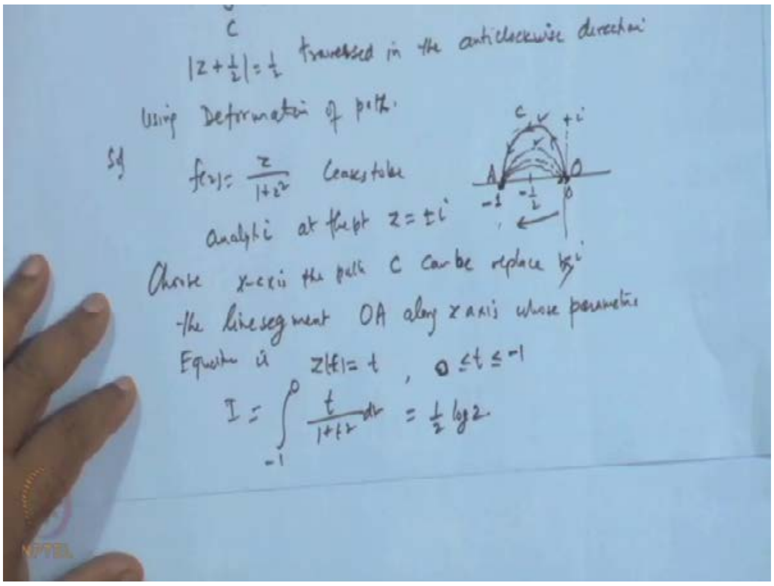

So, this integral I reduces to the integral z equal to t. So, looked as t over 1 plus t square dt and minus 1 to 0, and this value can be easily integrated because 1 plus t square put it to be t, then we are getting this value is coming to be half log 2.

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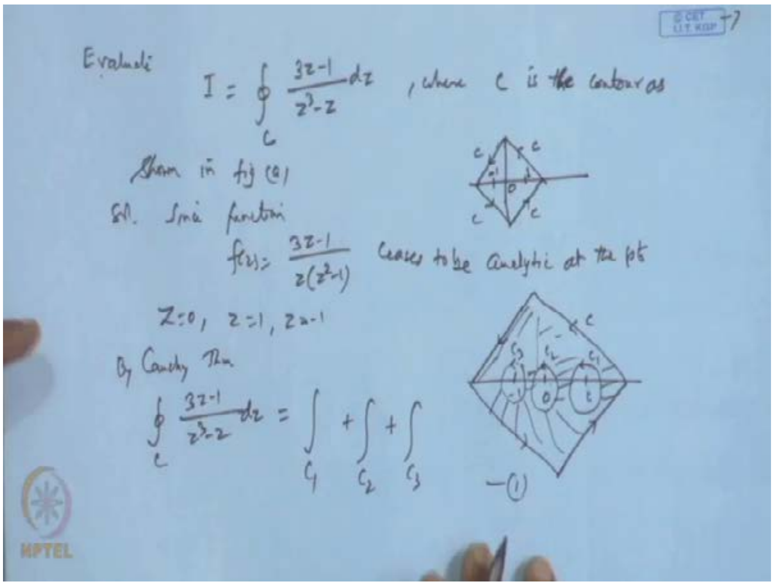

So, by using the deformation of the path one can easily get the integral, one can change this path of integration suitable and get now. Let us take few other evaluate using the Cauchy from for multiple connected domains. Evaluate this integral I along the path C of the function 3 z minus 1 over z cube minus z dz, where C is the contour close curve as shown in figure a. The figure is suppose we have this curve C is the path of integration like this is our C, this C which the point minus 1 0 and 1. These are the points we are interested in finding the value of the line integral of this function along this path now this close curve C is the close curve.

Now if I look the domain which is inside this close curve C, then you see the function f is not analytic at this point. Since the function f(z) which is 3 z minus 1 z, z square minus 1 ceases to be analytic at the point z is equal to 0, z equal to 1 and z equal to minus 1. So, these are the 3 points we are ceases to be analytic means the function is not defined at this point because the denominator. So, we have to denote this point because the function is not analytic. So, if I consider the domain, which is free from the these points that is 0 minus 1 1 draw this circles 0 and 1 minus 1. If I draw this curve which are say c 1 c 2 c 3 these are the close curve which enclose the point 1 0 and minus 1 respect to be.

So, if I look this domain then this domain and the boundary is this C this is the boundary of this value then this entire domain becomes basically simply connected, if I this vertical connected because gets. So, this entire domain is a multiple connected domain.

So, by Cauchy theorem the value of the function f(z), this function not 3 z minus 1 over z cube minus z this function if we take the function is not analytic at this point. So, if I note this point then of the function is analytic. So, by Cauchy theorem the value of the integral of this curve dz along the path C is the same as the value of this function along c 1 plus value of this function along c 2 plus value of the function c 3 is in not.

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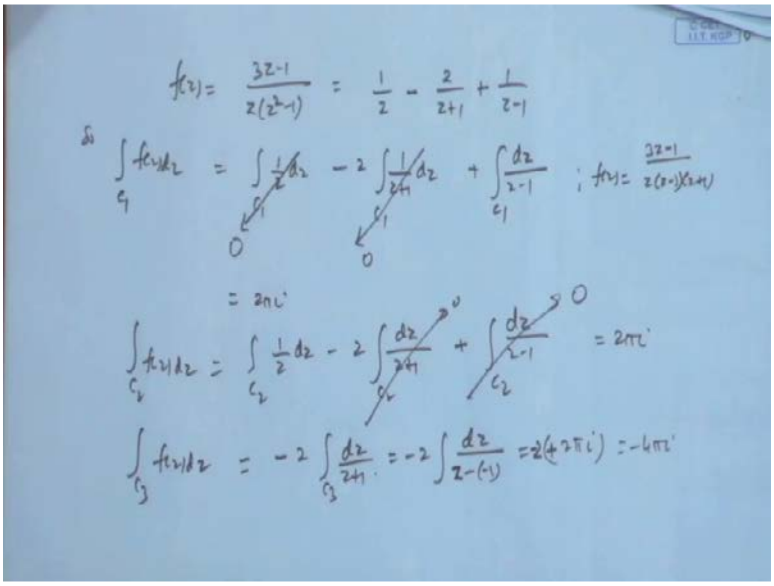

So, we get this, now c 1, c 2, c 3 let us compute separately. So, let it one now, what is our c1? c 1 is that first is the function f(z) this is 3 z minus 1 z z square minus 1. So, we can if we partial friction for this the partial friction will come out to be 1 by z minus 2 by z plus 1 over z minus 1. This is just simple z minus 1 and we write on the partial fraction and get this. So, integral along c 1 f(z) dz this is equal to now integral along c c 1 here, what is the c 1? Is integral along c 1 this is integral along c 1 on1e by z, dz minus 2 times integral over this along c 1 plus integral of this along c 1. But what is c 1? c 1 is this curve which enclosed the point z equal to 2 1 and 0 and minus 1 outside. So, if we look the function f(z), which is 3 z minus 1 over the z z square minus 1.

So, in this case the function f(z), which can be written as 3 z minus 1 over z z minus 1 z plus 1. So, if we look this function z plus 1 this point z plus 1 this point is no longer this point z equal to 0, 0 is not in put it side c 1. So, the value of this integral will be 0 becomes it becomes analytic one by z dz z equal to 0 is the point we are similar point, but inside the c 1 0 is not 0 lying outside. So, function f(z) one by z becomes analytic.

So, integral Cauchy integral theorem says the value of this integral must be 0 then z plus 1 z equal to minus 1 again z equal to minus 1 lying outside.

So, again the value will be 0, but for this z equal to 1 lying inside. So, once inside then with the value of this integral is nothing, but the value of dz over z minus 1. But we have already seen one example the value is coming to be this yeah z minus z naught to the power m in which minus one the value is 2 pi i. So, the value of this integral will be equal to 2 pi i, then integral along c 2 f(z) dz in similar way we can say c 2 1 by z dz 2 dz over z plus 1 then plus dz over z minus 1, but where is the c 2 is the point which enclosed 0.

So, minus one and plus one is out it means z plus one or z minus 1 1 by z plus 1 z minus 1 becomes analytic becomes analytic. So, these are 0, and this z equal to 0 will give only the value which is just if we compute z equal to etcetera the value will come out to be 2 pi. So, here the value also will come out to be now third case when integral c 3 f(z) dz c 3 is now this is our z c 3 z plus 1. So, z plus 1 this will give the value. So, this will give the value star 0 f(z) 0, and this is nothing but minus 1 plus 1. So, the value will come out to be minus dz over z minus 1, that is the same as z minus z naught. So, again the value will be 2 pi. So, it is minus 2 pi i and multiply 2. So, this is minus this. So, it is equal to minus 4 pi i. So, what we get it here the integral along the close boundary. So, we get according to the first.

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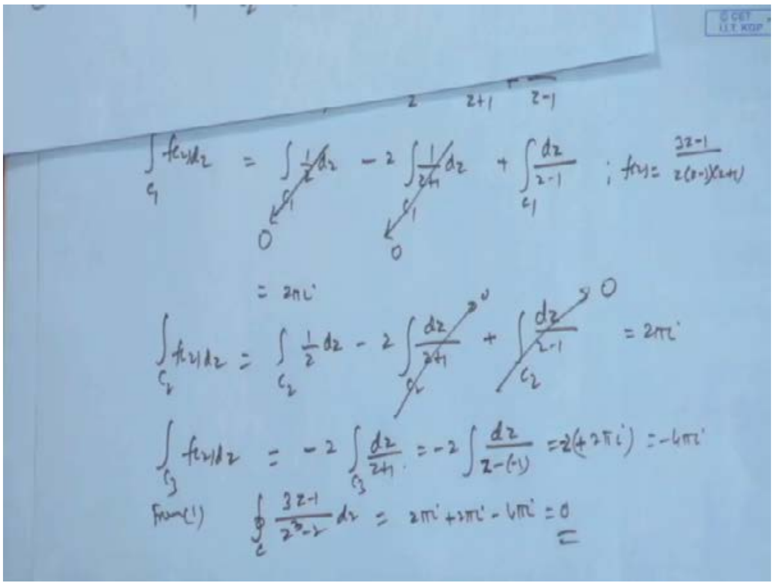

So, from first integral of this function along the outer boundary c, if the some of the integral c 1 c 2 c 3. Therefore, the some value will be 2 pi i plus 2 pi i minus 4 pi I that comes out to be 0.

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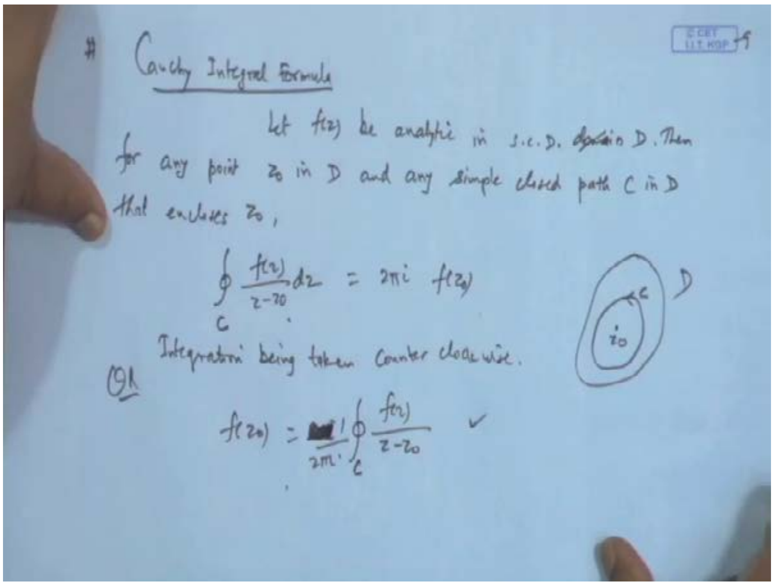

So, that will be the. So, this will be now let us take one more problems that is Cauchy, now we have another application of the Cauchy integral theorem is the Cauchy integral formula. The Cauchy integral formula this is the consequence of the Cauchy integral theorem which says the value of the function at a point z naught inside the domain D we have to function is analytic can be computed. So, what we say Cauchy integral, let f(z) be analytic in a simply connected domain in a simply connected domain D then for nay point z naught in D and any simple closed path C in D. That encloses dependent z naught the value of the integral f(z) over z minus z naught dz along this path C equal to 2 pi i times the value will be function f(z) the integration being taking counter clockwise counter clockwise.

So, it means if we are interested finding the value this is suppose if f(z) dz is analytic in a simply connected domain D, which has a point z naught inside this take any simple closed curve path C in D, which encloses the point z naught. Then the value of this integral f(z) over z minus z naught along the path C will be 2 pi i times the value of the function f(z) naught. So, another words we can say that interested finding the value of f function f at an arbitrary function z naught in a domain D, where the function is analytic then we simply considered calculate by integral of f(z) minus f(z), f(z) over z minus z naught along any simple closed curve C which encloses the point z naught multiply by this by 2 pi divide by this 2 pi i divide this by 2 pi i, then this will give the value of the function at a point z naught.

The prove of this we will see the prove, but let us see advantage of this the advantage is that suppose we want to calculate the say integral of this function say z cube minus 6 over 2 z minus i suppose, we want to conclude this integral.

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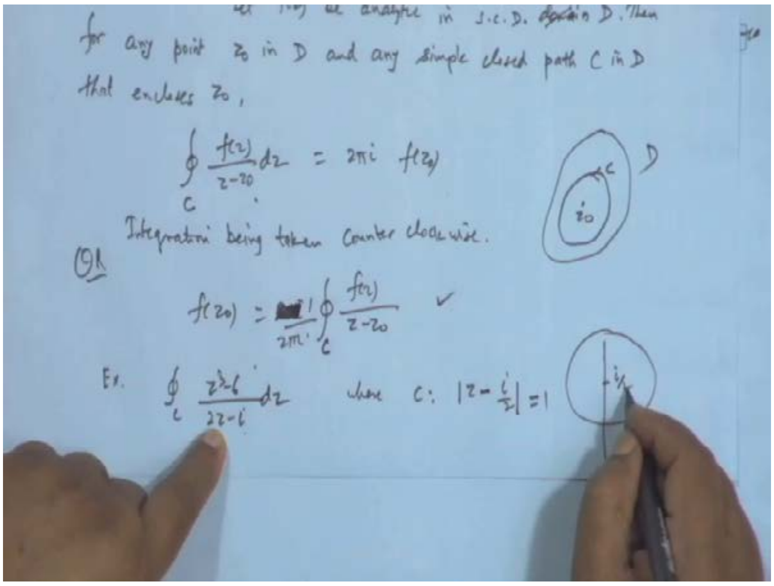

Then what we do is, where C is the circle mod z minus half this is the i by 2 z naught i by 2 a centre is i by 2 with the radius say one. So, if we considered this circle centred at i by 2 with radius say one. Then what we get is, this function z cube 6 over 2 z minus i ceases to be analytic, so but the function z cube by 6 here the function f(z) f(z) cube by minus 6 by 2, this function is analytic the inside the path C at an point i by 2 inside.

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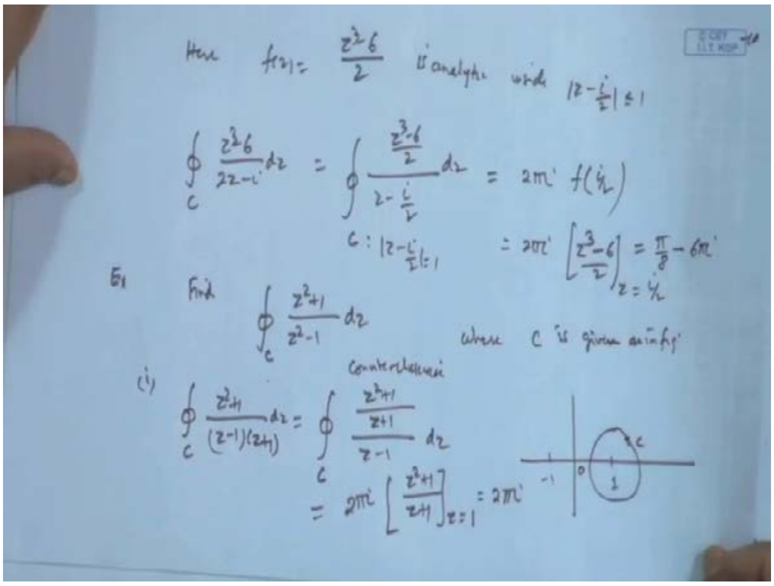

So, whole the function is not analytic f(z) equal to i by 2, but if we look this function is analytic inside the curve z minus i by 2 is less than equal to 1. And what is if the function f(z) analytic inside the domain D or inside the path C then the f(z) over the z minus z z naught the value of this integral along curve C will be. So, basically this is can be written like this the integral C z cube minus 6 2 z minus i dz can written as integral z cube minus 6 by 2 over z minus i by 2 dz the path C and C is the circle.

So, C is the circle z minus i by 2 is one therefore, by this Cauchy integral theorem the value of this integral will be 2 pi i times 2 pi i times. The value of this integral is value of the function z naught value of the function f at the point z naught is what point z naught is i by 2 i by 2. So, this comes out to be 2 pi i z cube minus 6 by 2 at the point z equal to i by 2 and if we compute by this the value will be come out to be pi by 8 minus 6 pi i. So, this is another examples also let us consider the curve find the integral C z square plus 1 over z square minus 1 dz along the path C, where the direction is second counter clockwise; where C is given as in figure. Suppose we have of first case C is this is the 0 here is minus 1 this is one suppose this C is this curve closed curve surrounding the point one. So, if we take the curve C surrounding this point then what function z square plus 1 over z square minus 1 this is to be analytic at z equal to 1, but if we consider this function f.

The integral z square plus 1 over z plus 1 divide by z minus 1 dz suppose function into this form along the path C then this path of the function the new vector is analytic this analytic. So, and this function which is analytic inside the C and having the point one inside this c c encloses the point one. So, apply the Cauchy integral formula the Cauchy integral formula say the value of this will be 2 pi i time the function f(z) f(z) equal to z naught that. So, you can find out the value of this integral and get this comes out to be 2 pi i the 1 plus 1 get this 2 pi i now in the second case.

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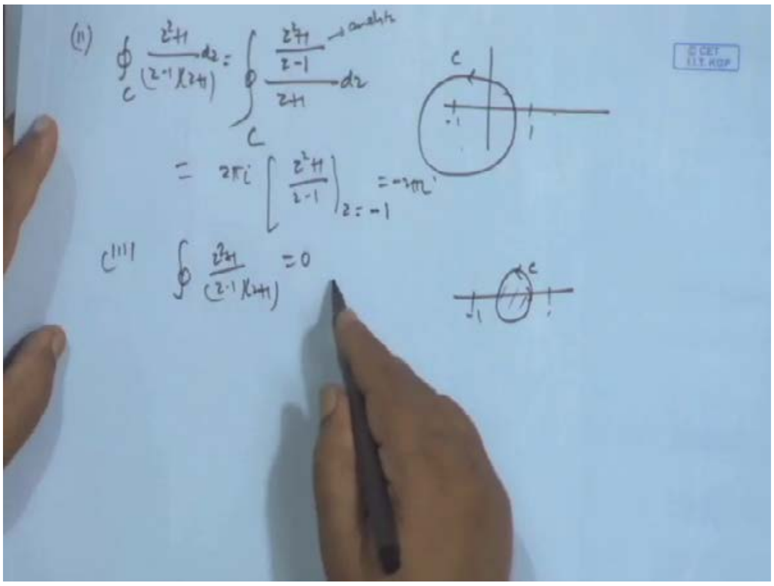

If suppose I second path if I substitute this curve and we get this curve, suppose I substitute here say minus 1 1, and I substitute like this say like this curve C this is the curve C, where minus 1 include it and one is outside. So, this same integral along this curve z square plus 1 z minus 1 z plus 1 dz we can rewrite like this z square plus 1 z minus 1. So, z minus 1 divided by z plus 1 dz along the path C, because the function is throughout analytic except that point z equal to minus 1. So, that point I am taking below. So, it is f(z) over z minus.

So, this function is analytic inside this curve C. So, integral Cauchy integral formula 2 pi i times the value of this f(z) is equal to minus 1 and that value will come out to be minus 2 pi i, and suppose it encloses both these points then you apply Cauchy integral formula. Integral theorem for multiple connected domains and the or if we take this integral minus 1 to 1 and this is the curve C, then the value of this integral f(z) will be 0, because the function is analytic. So, the value will come out to be 0. So, that is the advantage of computing by the integral by just manipulating the whole function.

Thank you very much.