Advanced Engineering Mathematics
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Lecture No. # 15
Cauchy Integral Theorem

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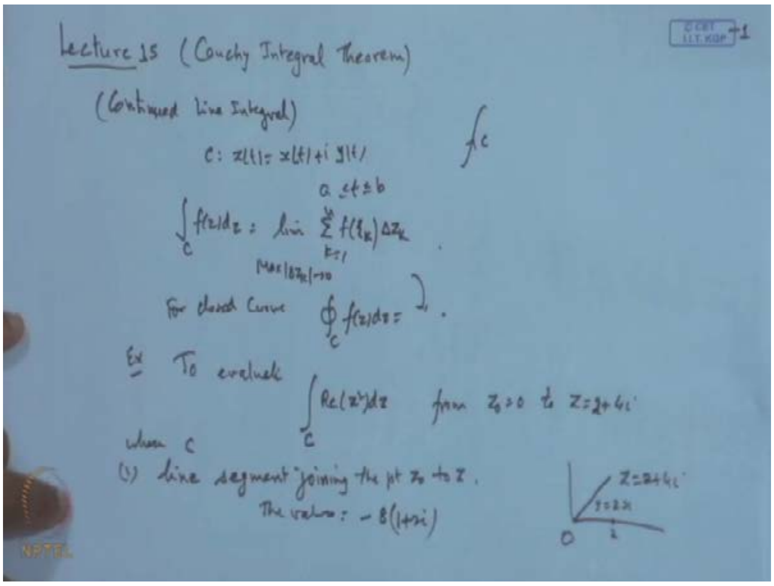

So, the last lecture we have discussed the line integral of a function of complex variable j, and integral C fj dj, which we have defined as continued line integral. That if a curve C is a curve whose parametric equation is given as x(t) plus i times y(t), where t ranges from a to b and c is a smooth piece wise curve; and function f which is define at continuous define at each point of the curve C. Then integral C f(z) dz this is the limit of the sum sigma f of k into delta z k is 1 to n and the maximum of this mod delta z k goes to 0. So, if this limit exists, we denote this by integral f(z) dz, and then C is a close control, then we use the notation for a closed curve, for closed curve we use this integral C f(z) dz as this.

And then we have taken a one example, that to evaluate this integral real part of z square dz along the path C from the point 0 z naught is 0 to the point z, which is a 1 plus 2 plus 4i, where C is 3 ways is define, one is the line integral the line segment C is the line segment joining the points z naught to z, and in that case we have seen the value has come out to be the value of this integral. We have computed it because it is a curve this is line, and here is 0 this is the point z 2 plus 4i. So, basically this line become y is equal to 2 x. The line is y is 2 x and then x will vary from 0 to 2, and after computing the integral we have seen the value of this integral was coming to minus 8, 1 plus 2i.

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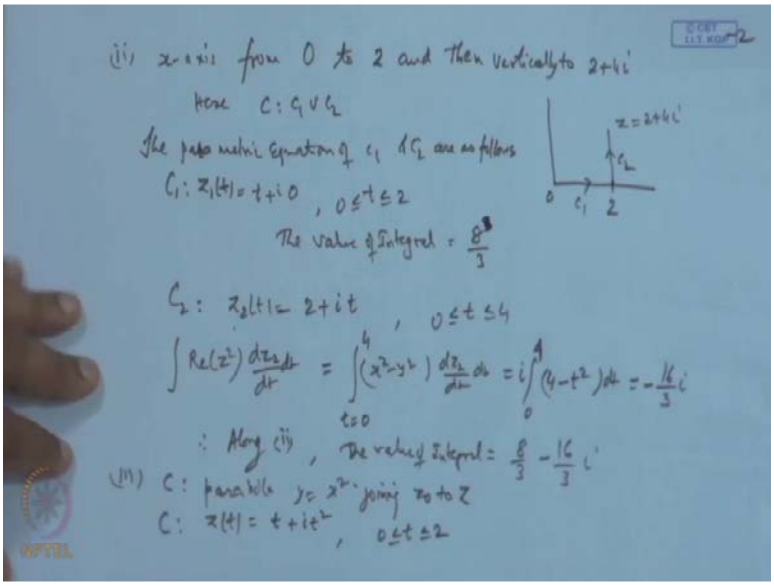

In the second case, so this because we have done it already. In second case we have see is the x axis is, C j x axis from 0 to the 0.2, and then vertically to the 0.2 plus 4i. That is this is of a point zero here is 2, this is the here this is 2 plus 4i. So, we are going along with this along this. So, basically C here, C is c 1 union c 2 two curves are there. We have as parametric equations of c 1 and c 2 are as follows; c 1 you can write it the equation, because along the path c 1, what happen it? The x equal to t by 0, so z 1 t which I am writing for c 1 as t i plus i times of 0, where t varies from 0 to 2; and then if you substitute all these in the value of this integral that also we have taken, the value of integrals along this path was coming to be 8 by 3 this was the value along c 1.

Then c 2, c 2 is the path vertically this. So, x coordinate is 2. So, z 2(t) is 2 plus y is varying so 2 plus i t, i t where the t ranges from 0 to 4. So, what happen in the integral? When you go for the integral, real part of z square dz by dt dt. So, here z is this equation d 2 z. So, dz 2 we will write along this path. So, we get real part of x f z square x square minus y square x is t x square minus y square dz 2 by dt, and then t varies from 0 to 4.

So, that will come out to be 0 to 4 x is 2. So, 4 minus t square dz by dt dz 2 by dt is this is 0 this is 1 only that is i only, so it is i dt i dt. So, it will come out to be this i dt, and when you compute this value for this the value will come out to be minus 16 by 3 i.

So, c 1 along the values coming this c 2 along. Therefore, along the second path along second the value of the integral is c 1 in a c 2 that is 8 by 3 minus 16 by 3 i. Earlier the value was something different it was minus 8 1 plus 2i and here. And in third case also, when you take the path is a curve parabola, the path if you remember this path parabola C is the parabola, y is equal to x square, joining these two point joining z naught to z. So, in this case the parametric equation of the curve C will be x(t), if I take t x is t then y becomes t square. So, i t square and t varies from because it will go from 0 to the 0.2 is it not 0 to 2. So, it is 0 to 2 and then find out the dz by dt.

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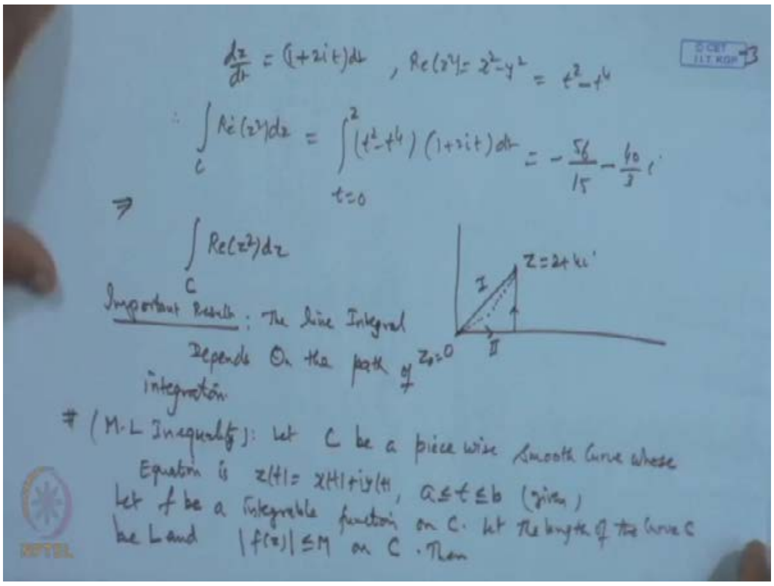

So, if you take the dz by dt, this comes out to be 1 plus 2 i t d t and then real part of z square is x square minus y square x is t by t square. So, minus t 4 this is and therefore, the integral of this along the path C real part of z square dz is equal to integral t square minus t 4, 1 plus 2 i t dt, and t varies from 0 to 2. So, compute this value, the value will come out to be minus 56 by 15 minus 40 by 3i. So, what we conclude is, that this is our two points z naught and z, is 2 plus 4i. We want the integral of this function real part of z square dz along the path C, where the C varies in the first case. It is a straight line joining these two points, this is the first path.

In the second path we are going along this, and this is the second path, and third path will be like this a parabola. So, in all the three cases what we have seeing the values are different; they are not coming to be the same. It means the line integral of the function f, though it is continuous function will exist, but it depends on the path of integration. So, the important result is the line integral depends on the path of integration that it. So, our aim is to because once it depends on the path of integration then the problem is there; because we cannot take this value all that value as subsequent work, because it depends on the path of integration. So, we have to identify certain condition on the function. So, that when we get the integral, the integral should be independent of the path it should not depend on the path.

So, once you have that type of the function all the condition on the function. So, that it independent of path then this value will be taken up for our subsequent work. And in this the Cauchy integral theorem plays the vital role. In fact, that is the key point from where you can justify, that the integral will be independent of path provided certain conditions on the functions are satisfied. So, let us see now prior to the Cauchy theorem let little bit just one more concept about the M-L inequality. And that M-L inequality says, what is this M-L inequality? If f is an integrable on a curve C, let C be a piece wise smooth curve, a smooth curve whose equation is say suppose z t is x(t) plus i y(t), t varies from a to b is given whose equation is this given.

And let f be a integrable function on C, integrable means it is continuous basically. So, once you take a continuous t will give the function; there are some functions also which are even not continuous, but it can be integrable. So, let it be integrable functions. And let us suppose let the length of the curve C be L. Let a length of the curve be L, and function f is such which is bounded by M, bounded function for on C the function is well defined, but the value of the function does not exist by a some constant say M on C. Then this M-L inequality says that, modulus of the integral f(z) dz over the curve C is less than equal to M times L. And this is known as the M-L inequality, because definition of the line integral f(z), when function f is integrable.

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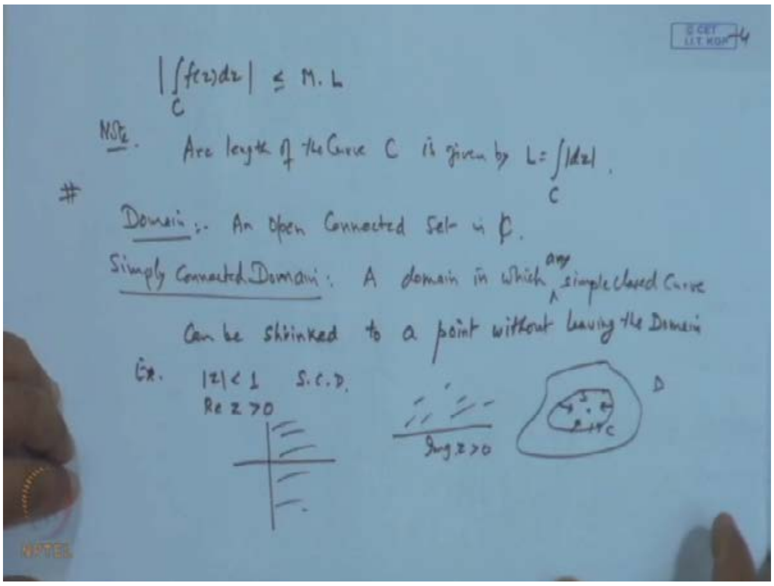

Then we can write it this in the form of the this way, in the form of this series. And then when you take the mod of this basically mod of this is less than equal to this which is less than this mod of dz is nothing but L. So, this is very now if the arc length of the curve C is given by this formula, which is also used will be used mod dz over the curve C. So, these two things will be needed in future whenever we go for let us come. We have discussed the domain is an open connected set. Now for the Cauchy integral theorem when we consider a function f which is analytic in a domain do not simply take the domain; we do not simply take the domain what we do we require something more, and that we has a concept of simply connected domain which is required for this.

So, first domain is an open connected set is it not in the complex this is a domain. Then simply connected domain, we define the simply connected domain; a domain in which any simple closed curve can be shrink to a point without leaving the domain. So, what is the meaning of this? Suppose this is the domain D we say this domain is a simply connected domain, if we picked up any closed curve C, any simple closed curve C, and try to excuse it. And if it shrinks to a point single point to a point without leaving the domain means; no times these shrinking the points are outside of the domain then such a domain is said to be a simply connected domain.

For example, our circle mod z less than 1 this is a simply connected domain, simply connected real part of z is greater than 0; real part of z is greater than means x this is the plan x is real part. This is the imaginary part of z. So, this is a imaginary part of z is greater than 0, because this is positive part, and real part of z is this one this is the real part z.

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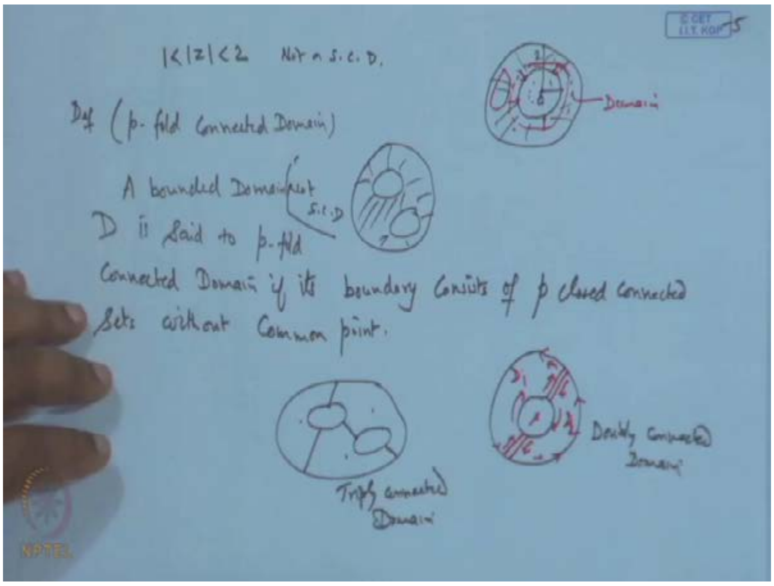

So, in this real part if I bring it any closed curve then it can be shrink to a point without leaving the domain; however, if we picked up the domain like this, suppose I take mod z 1 is less than mod z less than 2. Then this domain is analysis centred at 0, between the two circular disc, circle between mod z less than mod z 1 and mod z 2. So, if this portion is analysis; now in this analysis if we take a close curve, then always you cannot get the condition to be satisfied. That is we cannot always shrink to a single point without leaving the domain. For example, if I take this domain in this domain if I take this circle, and then try to shrink it to a point then as soon as it comes to this boundary up to here there no problem. But as soon as it crosses the boundary these points are not available in the domain because this is the only domain.

So, it cannot be shrink to a single point; however, if you take this curve it can be shrink to a single point. So, it does not mean the domain is a simply connected domain. What the condition is any simply closed curve the condition is any simple closed curve can be shrink to a point without leaving the domain? So, that is why this is not a simply connected domain. Similarly, a domain of this type, if I leave this two holes and only this portion are taken, then this is also not a simply connected domain not simply connected domain. So, such a domain which are not simply connected domain are said to be a multiply connected domains. And then the domain which has a two boundaries, we defined as a doubly connected which has a three boundary which has a triply connected domain and sign. So, in general we define a domain as a p-fold connected domain, a boundary domain D is said to be p-fold connected domain if it is boundary consist of p closed connected set, connected sets p closed connected set without common point.

Say for example, in this portion I am not taking this is out. So, this is not a simply connected, but if I join this thing like this; then once you start the moving from here going towards this, and then coming here finally it comes here. And like this now this becomes a simply connected domain. Similarly, when we go from here, up to here and then come from this going this, and coming like this way and again this side. So, this will be again it means this domain D is divided into two sub domains D 1, D 2, which are simply connected domains. So, it will be a doubly connected domain. This is the doubly connected domain now here you remember there is a one hole in that. So, if a domain has a single hole it is a doubly connected. If it is two holes it will be a triply connected.

For example, this supposes we have these three holes then again we can do like this suppose we get this one and then like this. So, we can get this. We can go through this way and then three domains 1, 2, 3. So, three we are getting this domain, which has an two holes, so this will be a triply connected domain. So, that is very important for this, now come to main result which is the Cauchy theorem.

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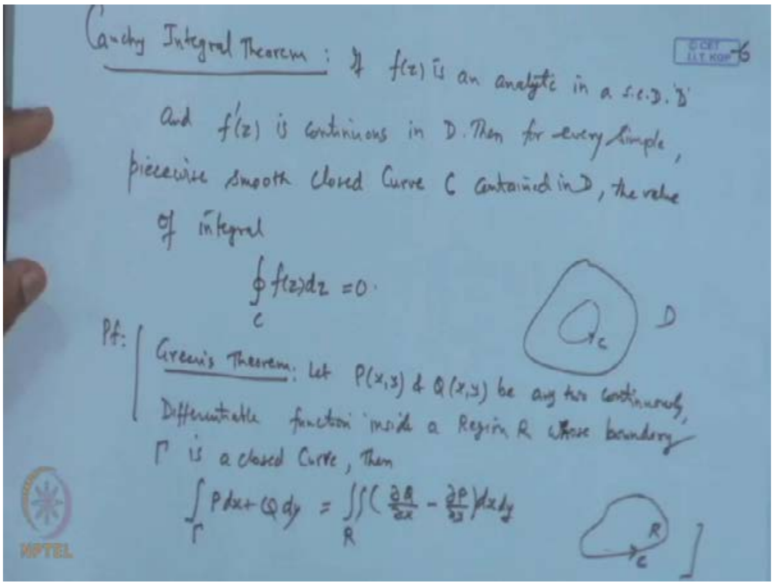

Now Cauchy integral theorem says if f(z) is analytic, if f(z) is an analytic function in a simply connected domain. In the simply connected domain D, and the derivative f prime z is continuous and the derivative f prime z is continuous in the domain D in this domain D. Then for every simple piece wise is smooth closed curve C contained in D; then for every simple piece wise is smooth closed curve contain in D the value of the integral C f(z) dz will be 0. So, what it shows? This shows that if a function is analytic in the simply connected domain D, means function must be analytic as domain should be simply connected domain.

And this is the next condition here chosen Cauchy, but without even with this condition one can prove that is proof is given by the goal said, and then even this condition can be dropped. The reason is means the function is analytic as, I told earlier also then all of its derivatives will exist. So, all derivative exist means automatically the derivative the f prime z will comes continuous. So, basically this condition is need not be required in proof, but that proof itself is very wrong. So, Cauchy did it so Cauchy use this restriction, and then prove it with the help of greens theorem. So, what this says is, suppose f be a analytic function in a domain D, and derivative is continuous.

Then if I pick up any closed curve C, which is simple piece wise closed curve totally contain C; and its point also function is analytic at each point as well as inside C also continuous, then the value of this integral will be zero. Then the proof of this based on this green's theorem. So, before going prove what is the green's theorem is? Let us see first the green's theorem, because this will in proving this let f and P let P and Q P, which is a function of x y. And Q which is also a function of x y be any continuously differential function, any two continuously differentiate with functions means P and Q both are continuous. They are differentiable partial derivative with respect to x and y of P and Q continuously.

Differential function inside a region R, inside the region R whose boundary C gamma is a closed curve. Then this theorem says the value of this line integral of the function P dx plus Q dy. This is equal to the double integral of the function del Q over del x minus del P over del y dx dy over the region R. So, this is our region R. R is this region, boundary is this C is a closed control curving the R, function P and Q both are continuous. A continuous partial derivative at is point on this domain D, with the help of the green's theorem. So, we make use of this result in establishing improving our Cauchy theorem.

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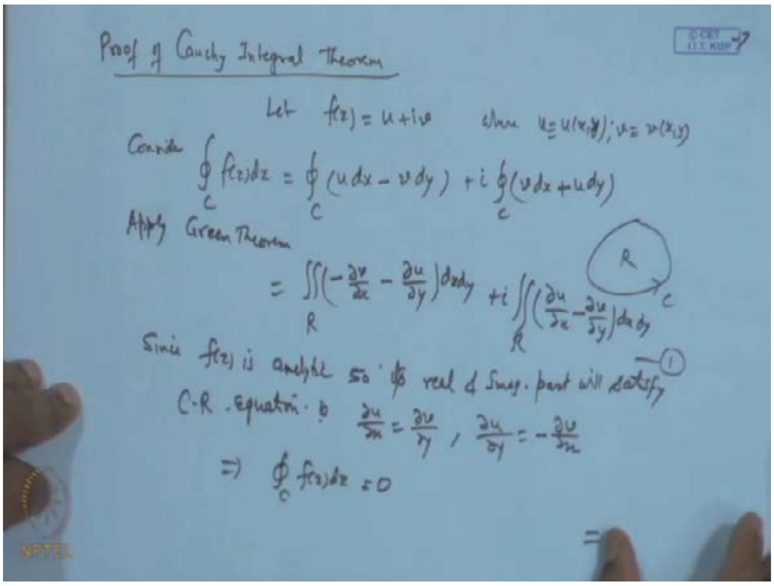

So, proof of the Cauchy theorem; proof of Cauchy integral theorem let us suppose f(z) is u plus i v, where u is a function of x y, v is also a function of x y, and f is giving to be analytic and continuous. So, even be both are analytic and continuous. Consider integral f of z dz, now when you substitute fz is u plus i v and dz equal to d x and dy then it can be break up into 2. Parts as we have seen in case of line integral u d x minus v d y plus i times integral v dx plus u dy. Now apply green's theorem, because function is analytic derivative is continuous. So, even be both are continuous functions a continuous partial derivatives and C is a closed curve. So, the region this is our C this will be the region R. So, if we apply the ah theorem green's theorem; then this can be written as the double integral over the R double integral over R del Q that is equal to P dx plus Q dy the integral is a del Q over del x.

So, minus del v over del x, then minus times del P over del y, del Q over del x minus del P over del y dx dy ; then plus i times double integral R here also this is del Q over del. So, del u over del x minus del v over del y d x d y. Now since the function f(z) is analytic. So, it is real and imaginary parts will satisfy C-R equation. So, u and v are the real. So, while C-R equation del u over del x is del v over del y del u over del x it del v over del y. So, second part vanishes. That this del u over del x is del v over del y, and del u over del y is minus del u over del x. So, first vanishes. So, this implies that integral C f(z) dz is 0.

Now Cauchy has proved this, but the Cauchy has dropped the idea of that continuity, and then also proved the thing. So, we are not interested in the now let us take few example which will show the application of the Cauchy integral theorem. But some we also see it does not contradict that the Cauchy integral theorem though the integral is coming to be 0.

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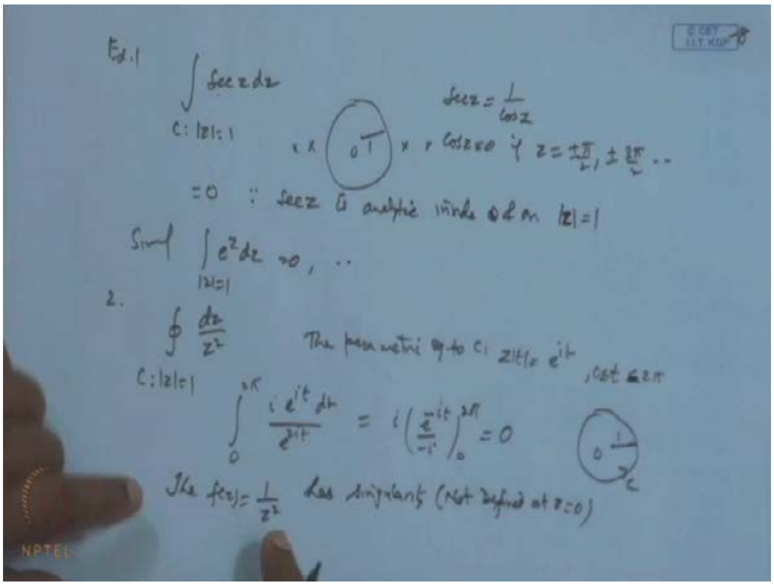

Let us see the example. Suppose I want the integral of the function z dz over the curve say C which is mod z equal to 1. Now what is z? z is 1 by cosine z. So, cosine z becomes 0, if z is the point plus minus pi by 2 plus minus 3 pi by 2 these are the point we have become 0. So, function is not well defined at this point; but if you look the circle centred at 0 with the radius 1 then all these points lies outside here somewhere they are lying. So, function z is totally analytic inside as well as on the boundary of this C. So, this value this is 0, because sec z is analytic inside an on mod z equal to 1 no similarity. So, by Cauchy integral theorem the value will be 0.

Similarly, if we take the integral e to the power mod z equal to 1, it is 0 and like the any analytic function, if you picked up the value will come out to be 0. Then let us see some suppose I take the integral dz by z square, we want it to evaluate it along the circle mod z equal to 1. So, write down the parametric equation of the curve to the circle C is, z(t) equal to e to the power i t cos t plus i times n t mod z is 1 and t varies from 0 to 2 pi 1 side 0 to 2 pi that is. So, if you substitute it then dz becomes i e to the power it dt, and then twice i t 0 to 2 pi, and if I compute the value of this the value will come out to be e to the power minus i t by minus i 0 to 2 pi now e to the power i 2 pi cos 2 pi plus i times n. So, it is 1 and minus this 0.

So, basically the values come out to be 0. So, value of this integral along this close path is 0 this is the circle C centred at 0 with radius 1 the function the value of this integral is coming to 0. What is the function? f z square has or not defined we can say, because we have not introduced the concept of not defined f(z) t equal to 0. So, function C is to be analytic there it means the function is not analytic; inspite of this, the value of this integral coming to 0 that it contradicts the Cauchy theorem? No, it will not contradict the Cauchy theorem, but the Cauchy theorem says if the function is analytic then the value of the integral along any closed path must be 0. By it does not say the other around. That if the integral of the function is 0, then whether the function is analytic that does not the conclusion this does not the meaning of the Cauchy theorem.

That is Cauchy theorem is only one based to, that is if the function is analytic then the integral of the function along any closed path in a simple domain will be 0; that is a thus analyticity is the sufficient condition, but not the necessary condition.

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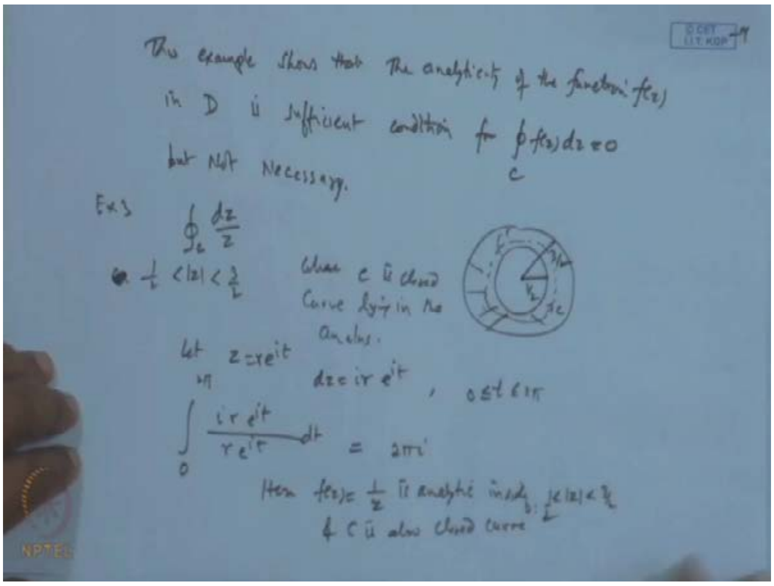

So, this shows so this example shows that the f is analytical that the analyticity of the function f(z) in a simply in a domain D is sufficient condition; condition is a sufficient condition for integral f(z) dz along the closed path C to be 0, but not necessary. This is the one let us take the example another example. Suppose I take the curve integral evaluate this integral dz by z along the path mod z equal to say mod z less than half and less than 3 by 2, let us take this; that is we are taking an half and 3 by 2. So, this is the portion C is a curve, C is any closed curve we are choosing any closed curve this is our closed curve C, this is C analysis boundary.

This is the domain and C is this boundary lies in this C lies no C, where C is a closed curve lie lying in the analus in this. So, circle lie in the analus. Now what is the value of this? Let us see. So, let z equal to e to the power i t, r e to the power i t because you take the any r e to the power it then what will be this dz becomes r e to the power i t into i. So, integral t varies from 0 to 2 pi circle. And then this becomes dz means i r e to the power i t z is r e to the power i t and dt and then t is 0 to pi. So, what you are getting is? This is coming 2 pi i is it not and here the function f(z) which is 1 by z is analytic inside the analus one is less than this less than 3 by 2 half less than mod z less than.

So, function is analytic in the domain this is our domain where the function is analytic, but and C is also closed curve is still the value of this integral is not equal to 0. Again it does not contradict the Cauchy theorem, because we cannot conclude from the Cauchy theorem the reason is all the conditions of the Cauchy theorems are not satisfied. But the condition of the Cauchy theorem is the function must be analytic in a simply connected domain, but D this domain is not simply connected this inside the domain D so is not.

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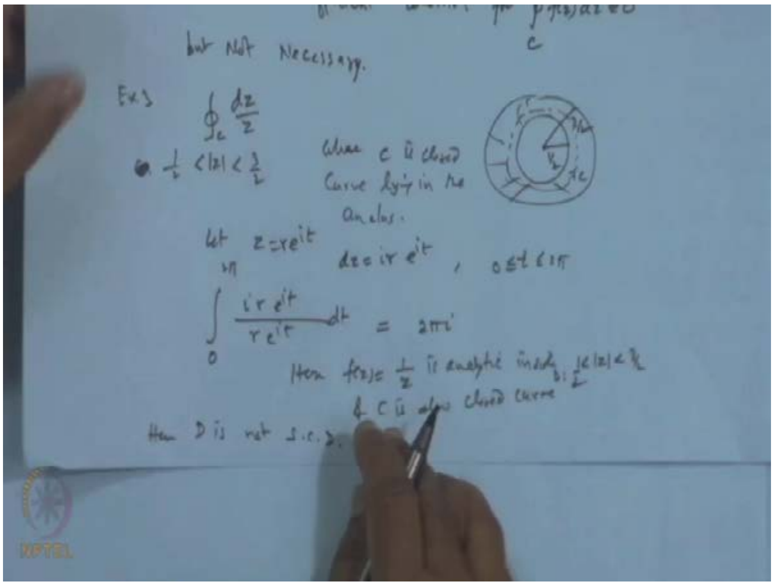

So, here D is not simply connected domain that is why we cannot say the (( )). So, what we...

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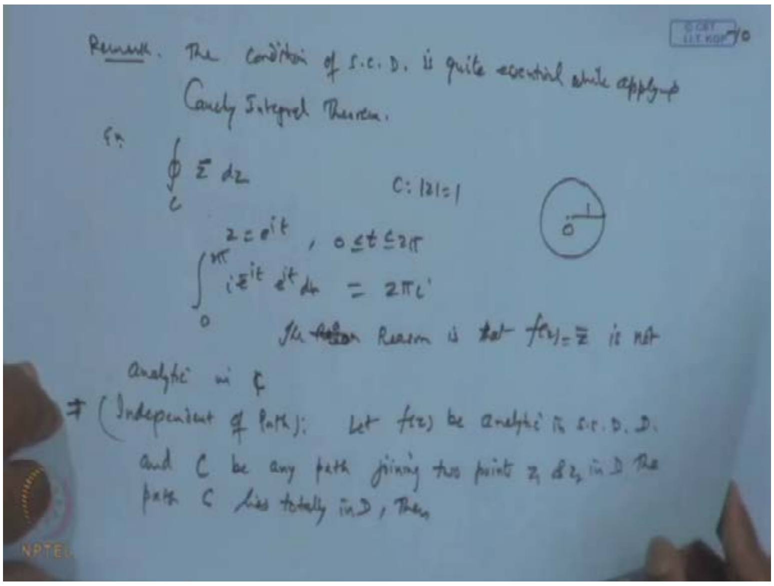

So, in the Cauchy theorem what remark we can say that the condition of this simply connected domain D is quite essential, in while applying Cauchy theorem, Cauchy integral theorem. So, that is must in that let us one more example let us the suppose I take the integral of this curve z bar dz. Now this close curve C is a circle mod z equal to 1. This centre 0 radius 1 C is a closed curve.

Now if I compute the value of this integral, the value will come out to be t range from 0 to 2 pi, and that z value is minus i t dz is e i t d t 0 to 2 pi. So, the value will come out to be 2 pi, and d t is i 2 pi i, so value will come out to be 2 pi. Now here again the curve is closed, but the value is not coming to be 0; the reason is that the function f(z), which is z 1 is not analytic is not analytic in any (( )) in fact, this we have already seen that this function is only differentiable at 0, and is not analytic to domain.

Therefore, whatever the closed curve is choose, the function is not analytic at all therefore, so this will. So, we get this, so thing now let us come to the slightly we as we have told earlier that line integral since it depends on the path of indication. So, there is no use, unless are you sure that the line integral computation of the line integral in independent of the path. So, for this the Cauchy theorem helps. So, what is the result is independent of path. What this says is, let f(z) be analytic in a simply connected domain D. In simply connected domain D, and let us and C be any path any path joining two points z 1 and z 2 in D 2 path joining z 1 and z 2 in d the path C is lies totally in D.

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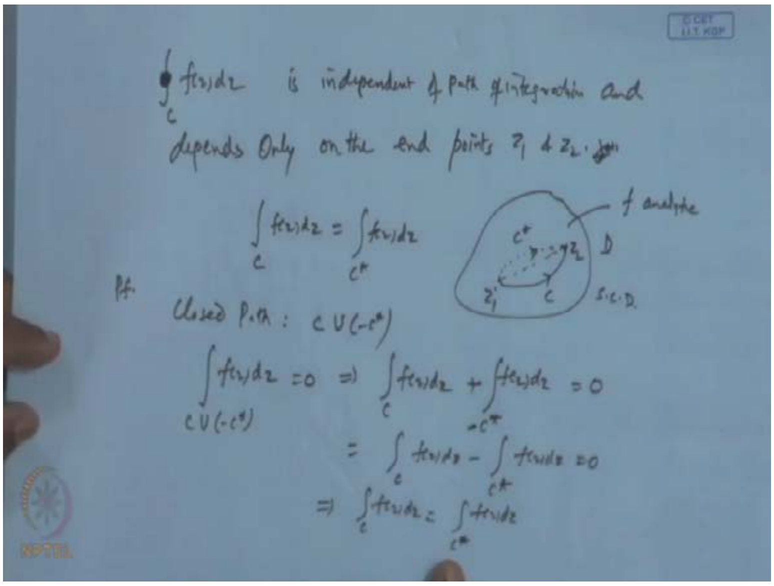

Then the integral f(z) dz along the path C along this is not curve, along the path C is independent of path, this line integral is independent of path of indication. And depends only on the end points z 1 and z 2 joining, joining the curve end points on the curve. So, what the meaning is? Suppose D is a simply connected domain, D is a domain and a simply connected domain, f is a function which is analytic in the simply at each point in the simply connected domain. This is a curve C joining the point z 1 to point z 2. Now this theorem says that if I integrate this function f(z) dz along the path C; then this integral will be independent of path. That is if I take another curve say suppose I take this curve C is then the value of the integral along C, and the value of the integral along C star will be the same path is whether you join this or this.

So, this can be followed immediately from the Cauchy theorem. Let us take suppose two path C and C star joining the point z 1 and z 2. So, when we start from z 1 go along the direction of this C up to z 2 and then come back to z 1. So, this will closed path. So, we get the closed path will be C union minus C star because the direction is. So, this is the closed path. So, and the function is giving to be analytic throughout this domain. So, it is also analytic inside l continuous on this boundary also. So, integral of this function f(z) dz along C union minus C star will be 0, but this is the same as the integral C f(z) dz minus integral f(z) dz C star, because this or we can write minus and then minus C star then minus of this becomes the minus because the order means this will be negative value. So, we get this and this is coming to be 0. So, integral C f(z) dz minus integral along C star from here to here f(z) dz is 0. This implies integral C f(z) dz and integral C star f(z) dz both are same both will give the same value.

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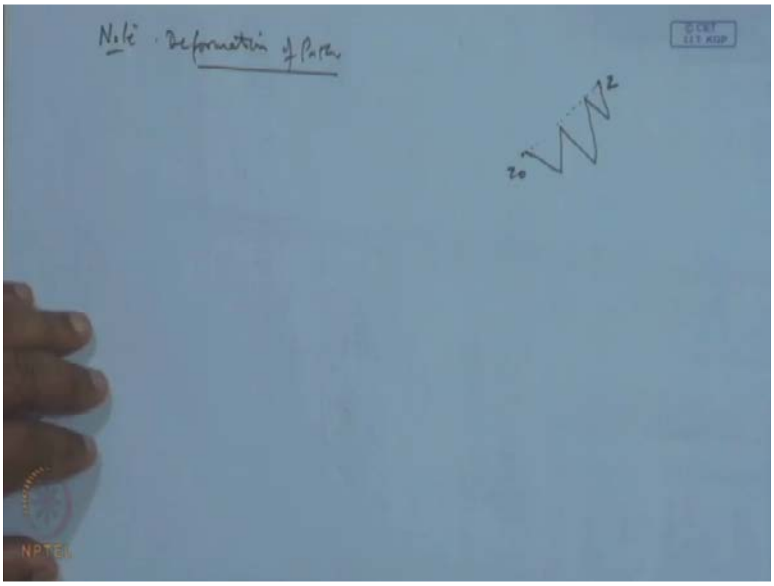

So, it is independent of the path, and this is the one of the testing result which is a consequence of the Cauchy integral theorem. Another consequence of the integral theorem is, the second note suppose we have a two point z naught and z. And a very zigzag path is giving same just this is simple piece wise smooth curve. We wanted to find the integral of the function along this then instead of this. If we bring it to a path, which is much simpler, then the value of this integral along this path and this path will be the 0 will be the same, and that give the deformation of the path, that gives an idea of the deformation of path; means, we can deform the path whatever the path is there. We can try to we can think that C is the deform path of C star, slowly we are just reducing as coming down and so that C star in sequence height, and always in this process the integral will remain the same. Thank you very much. Thanks.