Advanced Engineering Mathematics

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Lecture No. # 14
Line Integral in the Complex

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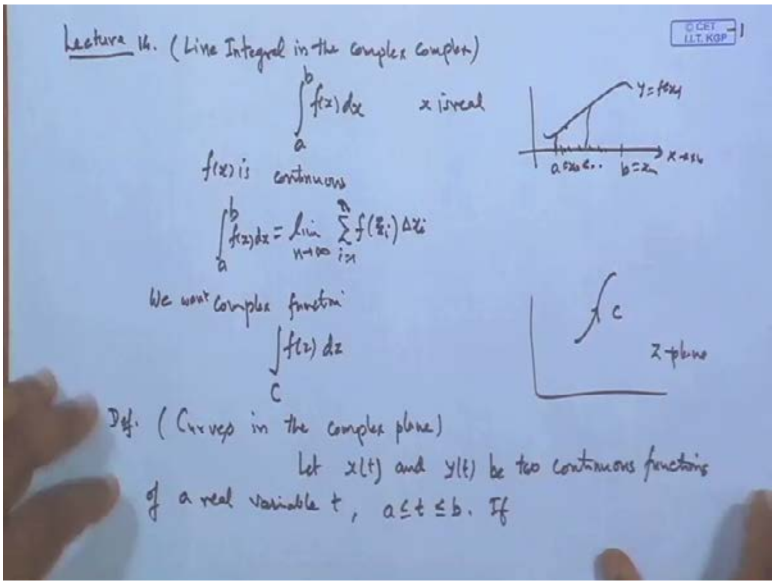

We know in case of a function real variable f(x) when x is real the integral a to b f(x) dx be called it as a definite integral the function f(x), and if you present the area bounded by the curve between the two ordinate x equal to a, x equal to b, y equal to f(x) and x axis. And when we define it we assume the function f(x) to be continuous function; then here the f(x) is continuous. Then this integral is basically is nothing but the limit of the sum, limit n tends to infinity sigma f of say xi i delta xi, i is 1 to n partition this interval into the points x1 x2 x n. And then in between these point find out the xi1, xi2 to xi n, consider the functional value at these points, and then multiply by the length delta xi take the summation i 1 to n, and n tends to infinity. Now, this limit if it is independent of the partition may be exist then this limit we denote by the integral a to b f(x) dx and will represent the definite integral. Now, without the limit a to b we get indefinite integral. We want to extend this idea to the function of a complex variable.

So, here today we want to integrate we want the; we will complex we want to integrate the complex function f(z) that is complex function f(z) be a complex function. We want to integrate this along the curve C; we have the C lies in the plane z plane this is our z plane. We want it to integrate this function f along the path C which lies in the complex plane. So, to extend this idea before, we need the definition of the curve in the complex plane. So, let us see first how to define the curves in the complex plane? For the let us suppose x(t) and y(t) be two continuous functions of the real variable C, of a real variable t, in where the t varies over the interval a to b.

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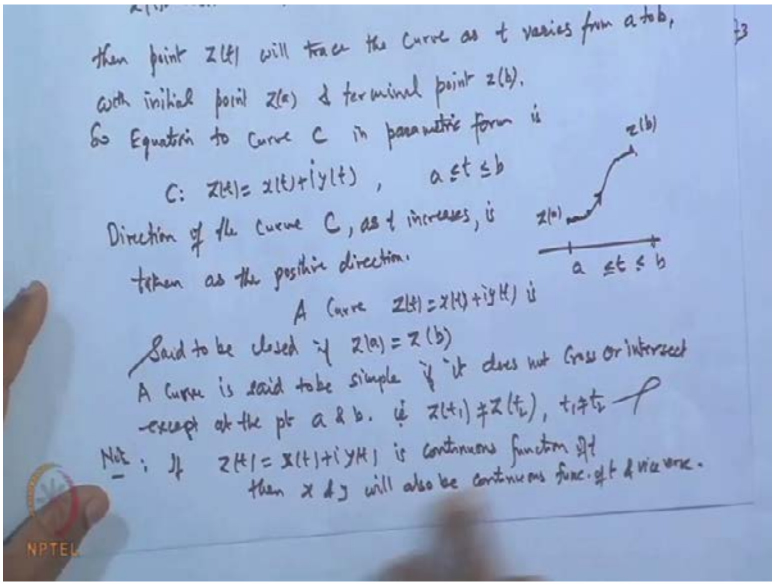

If z of t suppose x(t) plus i times y(t), where the t varies over the interval a to b, then this point z(t) will describe represents a point on the curve and will trace a curve. Then the point z(t) then the point z or z(t) will trace the curve as t varies from a to b. So, with the initial point as z with initial point z(a) and terminal point z(b) and z(b). So, the equation of the curve can be written in the parametric form. So, equation to the curve C in the parametric form is written as z(t) equal to x(t) plus i y(t) and t is varies from a to b. Now when we say the curve this is our curve; here is the interval a b, t is varying over this interval. So, corresponding to a, we have a point here say z(a) corresponding to b we have a point say z(b). So, as the t varies the point on the curve varies trace.

So, it will trace a curve, now this curve depends on the direction, because it can go in this plane on may be opposite direction. So, what we consider one of the directions as a positive direction. So, we take the direction of the curve C is taken as a positive direction, as t increases is taken as the positive direction. And opposite to will be the negative direction. Now when the curve traces here the point, if the terminal point and initial point coincide then the curve is said to be a close curve.

So, A curve z(t), which is x(t) plus i y(t) is said to be closed. If the initial point an initial point in the terminal point coincides. And a curve is said to be simple, if it does not cross or intersect, or does not except it the point a and b; Means in between a b if the curve is tracing t changing from a to b. Then the curve should not cross to itself, it should not go like this; for t line between a b, but at the point t equal to a and at the point t of b, because if these two are equal then the curve becomes closed which is simple closed curve in that case we will say simple closed curve.

So, that is the meaning of this is that if t 1 and t 2 are the two different point in the interval a b then z(t) 1 and z(t) 2 will give the different point on the curve where t 1 differs from t 2 that is all. Now we curve, a curve to be a smooth curve continuous dependent. So, if z(t) say note if z(t) is a continuous function if z(t), which is x(t) plus i y(t), if this is a continuous function of t, then its real imaginary part then x(t) and y(t), x and y will also be continuous function of t, and vice verse.

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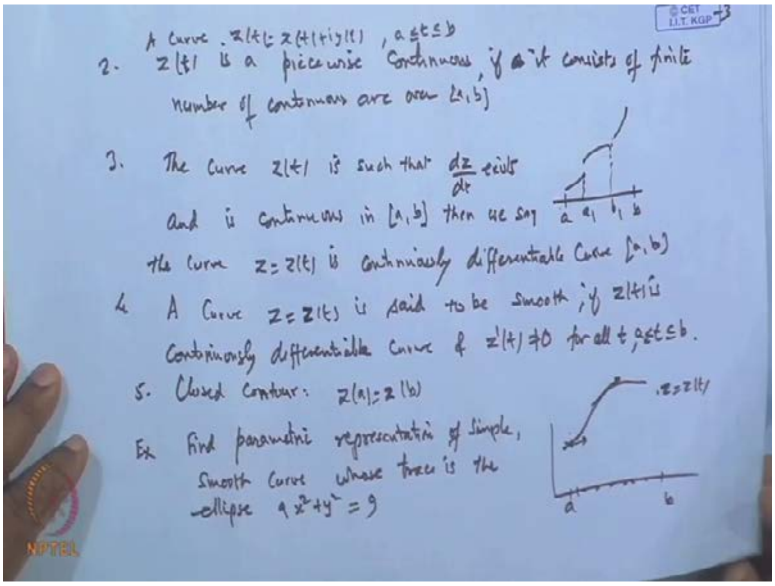

If(x) and y are continuous then the corresponding z(t) will also be continuous function of z. A curve is said to be a piecewise continuous or a function z(t) this is the second point z(t) the curve a is a piecewise continuous function or is a piecewise continuous. If it consists of finite number of continuous curve, if it consists of finite number of continuous arcs over the interval (a,b). That is a curve is said to be continuous, a curve z(t) represented by x(t) plus i y(t), t ranges from a to b is said to be a piecewise continuous curve. If it consist of finite number of continuous arc over interval, if suppose a curve is such which is not continuous throughout the interval a b. So, curve is like this may be like something like this, then there is a point of discontinuity here.

So, over the interval the function is not a continuous function of t, but if I subdivide this intervals a b into a, a 1, a 1, b 1 and then b 1, b then in each of intervals the function the curve is a continuous curve. So, as a curve as a whole from a to b will be considered as a piecewise continuous curve. That is similarly the function is differentiable its x and y both will be a differentiable function like this. Then third if you the curve z(t) is such that the derivative exist dz over dt exist, for each and is continuous in the interval a b. Then we say the curve z equal to z(t) is a continuously differentiable curve over the interval a b. Another which we a curve is said to be a curve z equal to z(t) is said to be smooth, if z(t) is continuously differentiable curve; and the derivative of this function z(t) is not equal to 0, for all t line between a b.

That is if the curve is such this curve is such which is continuously differentiable curve; it means at each point, the curve each for each point t naught t 1, t 2, t n the curve z equal to z(t) as a continuous is a continuous derivative at each point. And also the derivative is not equal to 0. So, in that case at each point we are having a tangent, and that tangent keeps on changing is direction. So, we have a turning tangent at point from a to b. So, such a curve is said to be a is smooth curve; When we have a turning tangent at each point on the curve that is now if z equal to a closed control, we mean if z(t) equal to if z(a) equal to z(b), then the curve is said to be a closed curve, then curve is said to be closed are we say closed control. Now we are interested in now the line integral of the function f(z).

So, before that let us see one example, where the curve say Ex find parametric representation of simple is smooth curve, whose trace is the ellipse, 9 x square plus y square is equal to 9.

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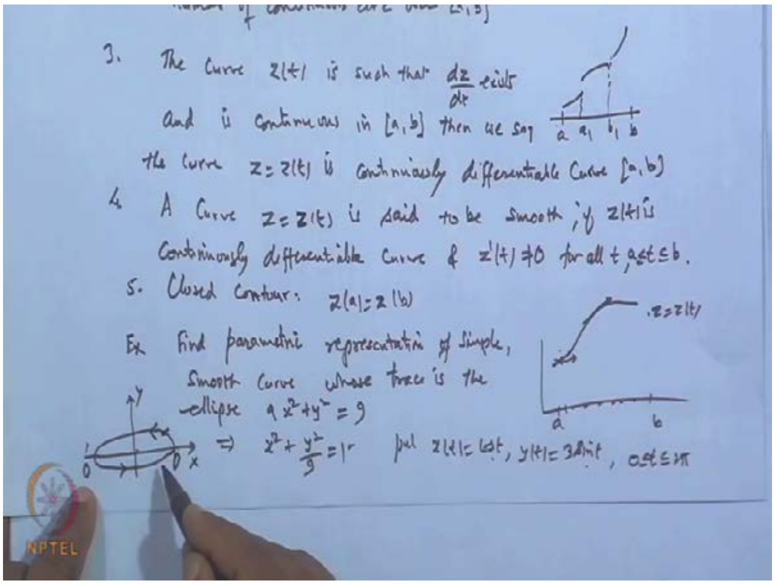

So, suppose we wanted to find the parametric representation of this curve, this smooth curve whose equation is given in the Cartesian form. So, obviously, if we look this curve we can rewrite this curve in the form x square plus y square by 9 equal to 1. So, if we take x(t) equal to cos t, y(t) equal to 3 sine t. Then it satisfies this equation and since it is a closed curve simple ellipse. So, if I take the t varying from 0 to 2 pi; that is we have this curve. This is our x axis here it is by axis. So, curve is moving in this side positive direction. What we are doing? We are taking a point, any point on this curve as cos t and 3 sine t. Then these points satisfy this equation, and since it varies from o to o dash, and then break here. So, the t will vary from 0 to 2 pi. So, the direction of the curve is returned.

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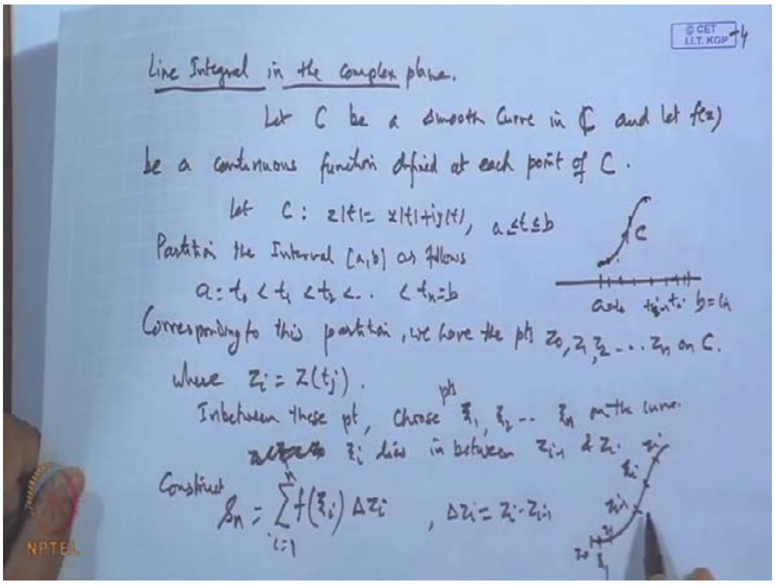

So, this will be the parametric representation of the curve. Now this will be enough for going for them as a line integral of this. Now we take the line integral of a function in a complex plane in the complex plane. Let us suppose C smooth, let C be a smooth curve in the complex plane C, and let f(z) be a continuous function defined at each point of C. So, this is our C is smooth curve C. Here this interval a b. The equation, the parametric equation of the curve C is x(t) plus i y(t). So, partition this interval, let the parametric of the curve. Let equation of the curve C is z(t) x(t) plus i y(t), where the t varies from a to b. Partition the interval a b, the interval a b as follows; a is t naught less than t 1, less than t 2, less than t n which is b. Now once you partition this interval into sub interval by choosing the point t naught t 1 t 2 in t n, then corresponding to this t naught t 1 we get a point on the curve. These are the point on the curve.

So, suppose we have a point l t and minus 1; here is the point. So, any point t i minus 1, and here is the t i. The corresponding point will be somewhere here. Now in between these two points say the point. So, corresponding to this partition we have the points z naught z 1, z 2 and z n on the curve C, where the z i means the value of the z at a point t j. Now in between these point, in between these points choose in between these point let us choose the pick up the zeta 1 zeta 2 zeta n at point choose xi1, xi2, xin it choose points on the curve. That is zeta 1 lies between z naught to z 1 z naught to z 1 that is, this is not because they are complex.

So, where the zeta i lies in between z i minus 1 and z i. So, here is z i minus 1. This is our z i and here I am taking zeta i. Similarly, here is z naught, this is z 1 here is somewhere we are taking zeta 1 a point one this. Now construct this, the value of the function at the point zeta i multiply by the delta z i we means, this is the z i minus 1 z i. So, delta z i will be equal to z i minus z i minus 1. This is our delta z i. And then take the summation when i varies from 1 to z i minus 1. So, I will vary from 1 to n. Let this denote by s n. Now once you had this partition you are getting this, now since the partition you take another some and so on.

Now, once you keep on increasing the points; in between a b partitioning point, then correspondingly this point on the curves are much closed. And take the limit when n tends to sufficiently large. All when the mod of delta z i goes to 0, in that case of the points are almost closed.

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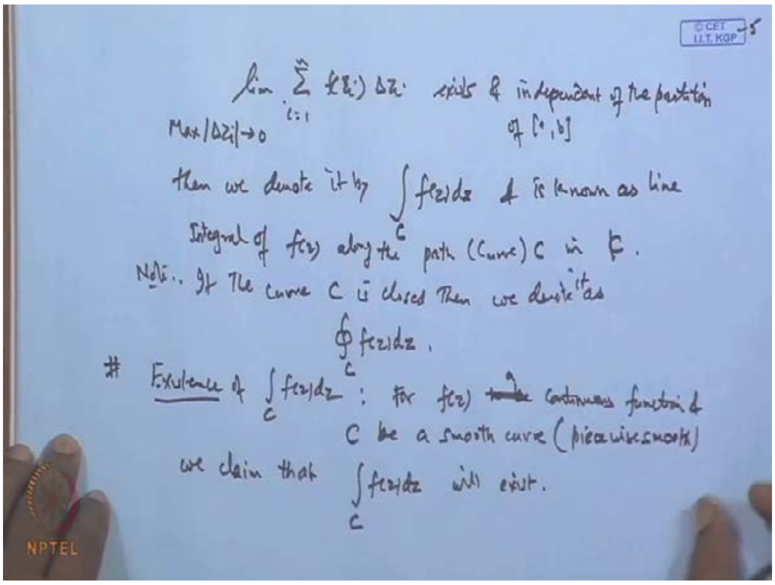

So, now take this limit of this sigma i equal to 1 to n, f of i delta I when the mod maximum of mod delta i goes to 0. And this limit maximum of this part tends to 0. So, if this limit exist, when max mod delta z i means the points are basically very closed each other and if this limit exists and independent of the partition of interval a b. Then we denote this limit y, then we denote it by integral f of z dz along the curve c, and is called and is known as line integral of the function f(z) along the path C; along the path or along the curve C in the complex plane. So, that is what is now if C is closed, if the curve

C is closed curve, then we denote this thing as, we denote it as integral along the path C under f(z) dz. That is this shows the integration is taking along the path C which is a closed curve. Now whether this function this integral will exist under what condition this integral will exist.

So, let us see the existence of this compression here, because what we have assume is , if the function if C be a smooth curve and f be a continuous function. And then we are saying that if this limit exists, and independent of path then we say the integral is line here. But basically our claim is that under this restriction the integral will definitely adjust. So, let us see existence of the line integral along the path C. So, we assume for f(z) to be continuous function, and C be a smooth for a f(z) continuous function, not for f(z) are continuous function; and C be a smooth curve is smooth curve. All may be a piecewise smooth curve is smooth curve, we claim that the integral f(z) dz along the curve C will exist.

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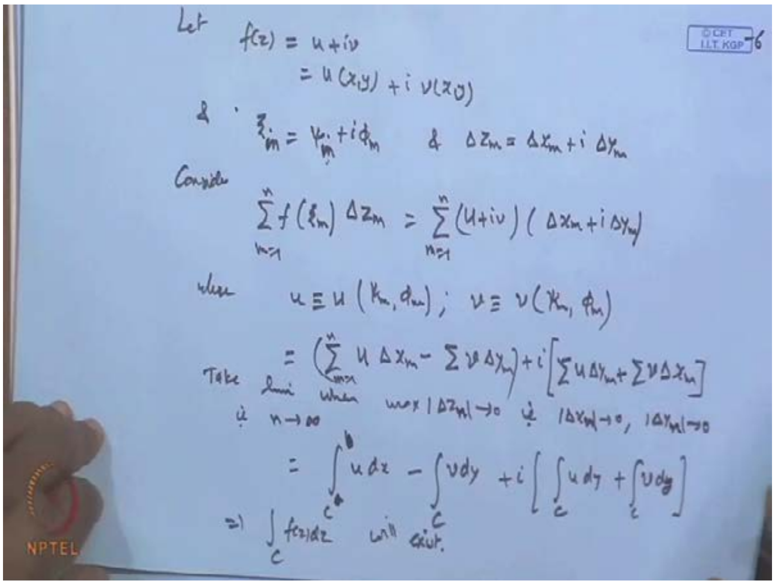

So, let us see why it is show? Let us suppose f(z) is u plus iv, well basically u is u function of x, y, v is as a function of x, y. And let us take zeta n we are choosing the point what point we have taking the point is it not. So, let us take the point xi i. The point say xi i plus phi i times of all let us, because this I should not be confuse. So, let us say this i m, xi m, i m, and delta z m. Let us take delta x m plus i times delta y m. In fact, this will be and we get. So, let us said this, now construct this. Consider the some sigma f of xi m into delta z m. m is 1 to n. So, this will be equal to sigma m is 1 to n, f of n is. So, u and v, u x and y is this by. So, it will be function we will be function of xi m and phi m both. This will be equal to u plus iv, and then we can write this thing as delta x m plus i times delta y m.

Where u is basically a function or depends on this value xi m phi n and v also depends on psi m phi m this substitute now separate out the real and imaginary part. So, when we separate out the real imaginary part, we get sigma u delta x m, minus sigma v delta y m plus i times of plus i times of v sigma u delta y m plus sigma v delta x m. And the limit varies from these. Now, take the limit, take limit when maximum of mod delta z m tends to 0, that is delta x m will go to 0 delta y m will go to 0, maximum of this thing maximum of delta x n z n . So, this will go to 0 delta n now im is 1 to n. So, that is the same that is the n tends to infinity. Now u f is given to be a continuous function we have assumed f to be a continuous.

So, even be both are continuous function. So, continuous function this u delta x m the m is 1 to n and limit tends to infinity or maximum of delta x m goes to 0, this basically integral definite integral as a some limit of sum. So, it will give the integral a to b u this will be the integral a to b u illustrate of this, because integral not in. We can write u dx without, then I will see, and this integral taking over the curve C. So, let us C this is integral v dy along the path C, and then I times this will be integral u dy along C plus integral v dy along C. Because of this, the function of single real variable we know that this can be this limit will exists and gives the integral of u dx integral of v dy integral of and so on.

So, if function f(z) is analytic is continuous and this is a smooth curve then we have this limit exist and we get this value and that. So, complete the shows the existence of the line integral. Therefore, the integral f(z) d z(a)long the path C will exist. That is now there are certain properties, which this line integral as we case of a real variable gets and that like a definite integral. We have some of the definite integral some of the integral the constant times are one integral of this is constant of the integral f dx and like the.

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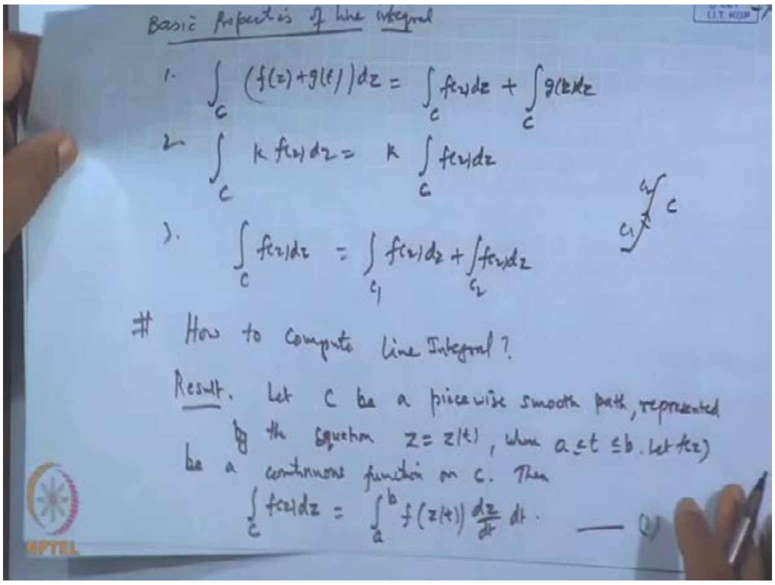

So, similarly they are few properties which is the basic properties of line integral, and the first property is; the they are two function set f in g, f t f(z) plus g z(t)hen this integral or even if you multiply by constant dz is the line integral of this function f(z) dz plus line integral of gz dz along the path C. Second a constant times of f(z) dz is the same as k times in line integral of the function f(z) dz along the path C. Then also this is interesting one. Suppose, we have a curve C and the integral of this function f(z) dz is there, If I wake up this curve into two C parts c 1 and c 2, retaining the same direction. Then this integral will be the same c 1 f(z) dz plus integral c 2 f(z) dz where c 1 and c 2 are the partition of the curve C and.

So, let see line integral, how to compute this line integral the question is now? So, how to compute line integral? Because we know the definition, by definition is line integral is nothing but the limit of this some. Is it not? That is the line integral you take partition of this, take the limit of this, and then limit exist we say the line integral may exist. But this is the basic by basic definition of a line integral, but every line integral it cannot write it, it cannot do every times this of calculation partition. So, how to computing the there is one result which is very interesting and will give the clue. How to calculate the line integral? The result is this let C be a piecewise is smooth path represented by the curve by equation z equal to z(t), where t varies from a to b.

Let f(z) be a continuous function on the curve C, then the result such the line integral f(z) dz along the path C is nothing but a definite integral a to b f of z(t) into dz by dt into dt. So, this is very good result, because directly the line integral limit transfer to definite integral, and we are solving such a definite integral is it not that is. So, let see first how to this applied first? In order to bring it to line from line integral to the definite integral of t, what we have to do? We must know the parametric equation of the curve C, once we know the parametric equation of the curve then calculate the dz by dt, which is very simple. And then find out the value of the function at each point on the curve, that is the f of z(t), and this the substitute the limit. So, that the entire path is covered, t varies from a to b. So, once it is they are then you can cover entire path get in this way.

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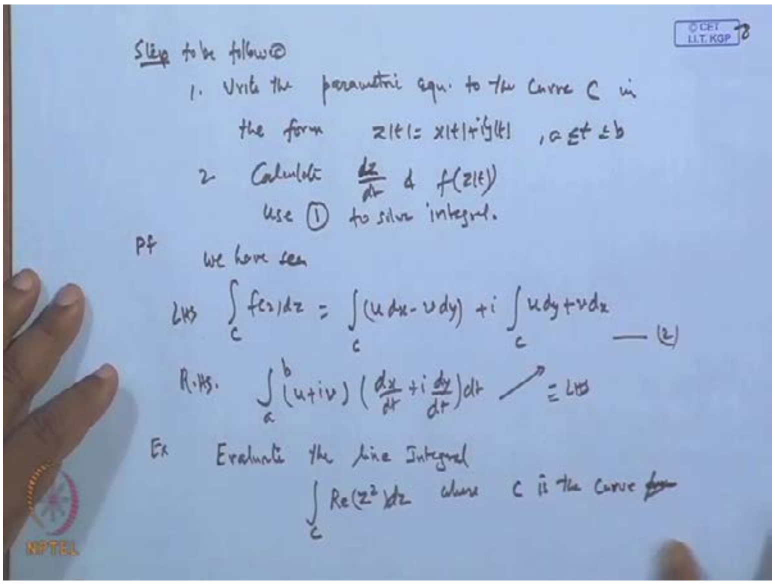

So, in order to what are the steps is taken? The step is to be followed number one step is the write the parametric equation to the curve C, in the form z(t) equal to x(t) plus i y(t) where the t varies from a to b. Then second one is, once you get this substitute it, calculate the dz by dt in value of the function at a point z arbitrary point z(t). And then use the formula, use formula one to solve the integral, and integration of limits would be existing. So, let us see why the proof of this is very how does it follow? The proof is like this, we have seen the integral f(z) dz along the path C is nothing but what this was the integral? Remember u dx minus v dy along the path C plus i times integral u dy plus v dx just as shown of the line integral we have come across about this.

So, this is the left hand side. In the right hand side will also give the sample value, what is the right hand side? Right hand side is integral a to b u plus iv f(z) is u plus iv dz means dx by dt plus i times dy by dt and then dt is it know. So, basically this comes out to be the same as left hand side. So, this way we can easily get let us take an example now this one.

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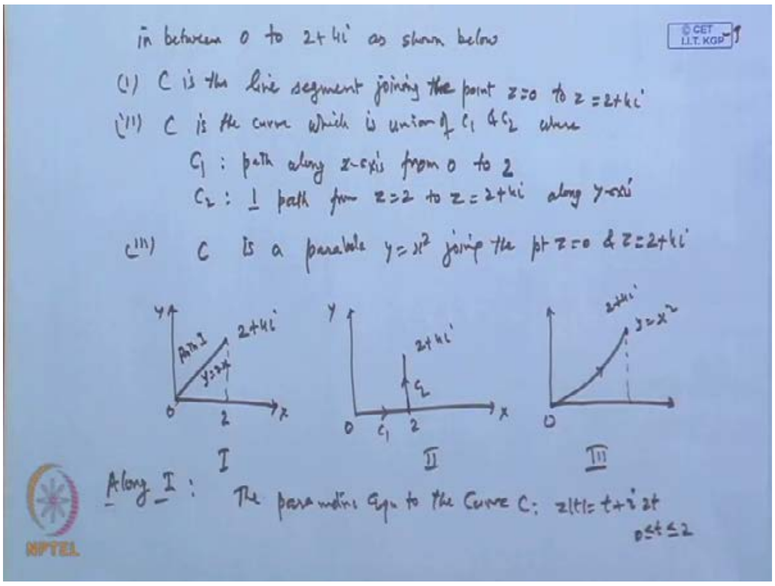

So, suppose I take evaluate this line integral evaluate the line integral real part of z square dz along the path C, where C is the curve is the smooth curve, is the curve form is the curve as shown curve in between 0 to 2 plus 4 i as shown below. The first is C is the line segment, joining the point z equal to 0 to z equal to 2 plus 4 i, second path the C is the curve which is union of c 1 and c 2 where c 1 as shown in the figure c 1 exist from c 1 is the path along z axis from 0 to 2, where the c 2 is the path is a vertical path from z equal to 2 to z equal to 2 plus 4 i along y axis.

Then third is the C is a parabola, y is equal to x square joining the points z equal to 0 and z equal to 2 plus 4 i. So, we have seen three cases. This is the first case z equal to 0 this point here is z 2 plus 4 i this is the. So, this is the first, this is first path; here this is x axis, this is y axis, x axis y axis. Second path is C is the curve which is the union of c 1 and c 2 joining path along x axis from 0 to 2, so 0 to 2 this is c 1 and then 2 to 2 plus 4 i in a vertical direction, so this is c 2. So, c 1 in c 2 give and third path is this to x square. So, it is that is y equal to x square is a parabola. So, this will be above parabola y equal to x square from 0 to 2 plus this is the point where 2 plus 4 i. So, these are the path which we want the value of this line integral along these three various path.

So, let us the case one this is case two this is case three let us evaluate this integral along various. So, first path so along one, the line integral C is given by this. So, what is the parametric equation of the curve? The parametric equation to the curve C is z(t) will be what this is the point is it not joining 0 0 and 2 4 is straight line. So, basically the equation of this line we cannot y equal to 2 xis it not. So, if I take x to be t then becomes 2 t must very such that this entire thing is covered. So, if I take t x varies from 0 to 2 and t I am taking x to be t.

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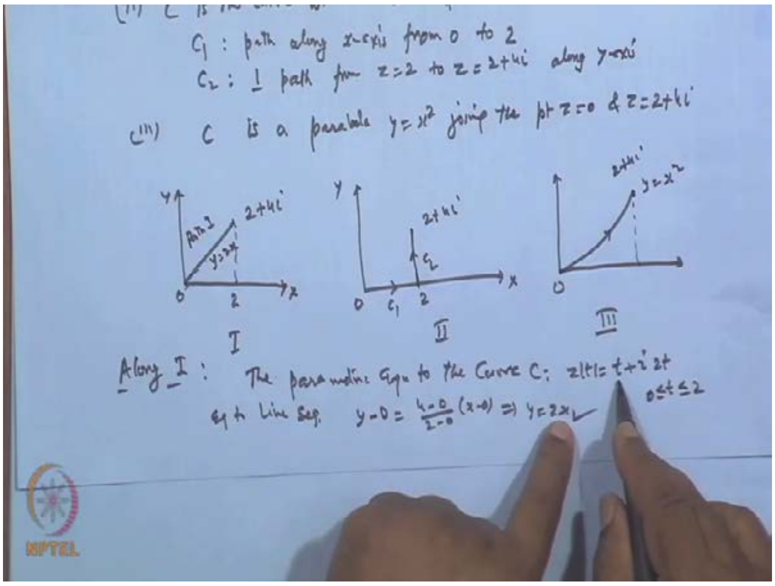

So, t must vary from 0 to here what I did, I have calculated then segment joining these two line segment equation to line segment is y minus by this y minus y 0 equal to double dash minus 0 over x minus 0. So, that comes y equal to 2 x is each point on this curve satisfy this condition. So, if I take x to be a t, then the y will have 2 t, but x the t must varies such a way, so that the entire path is covered. So, if I take t equal to 0 here and t equal to 2 here, then t varies from 0. So, it will cover from here to here in a. So, this is the parametric equation of the curve, once you get the parametric equation then you just calculated, what is our integral was integral C real part of z square dz.

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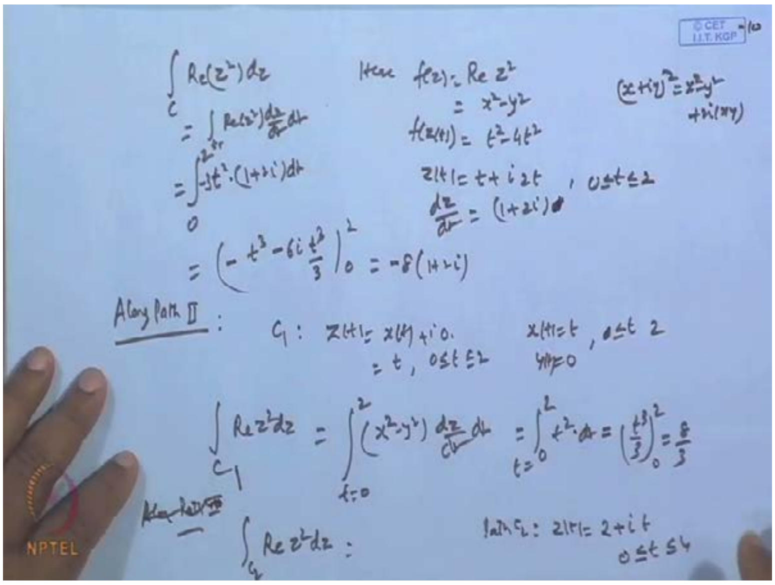

So, first you function what is our function here? f of z is the real of z square means x square x plus i y whole square. So, this will be equal to x square minus y square plus 2 i times x y, so real part of this means, x square minus y square. Now, compute this value at the point of the curve. So, this value when you take this function what is the show integral the curve is x square minus y square, x is what? x is t y is 2 t. So, we get from here is the value of this f z(t) is nothing but t square minus 4 t square is it because that. So, z(t) we comes and z(t) is t plus i times 2 t. So, d z by dt becomes 1 plus 2 i dt 2 i that is all. So, substitute this. So, integral real part of this means t square minus 3 t square and then dz by dt. So, this is equal to integral real part of z square dz by dt and dt and t varies from this. So, dz by d t is 1 plus 2 i and d t and what should be the limit 4 t t varies from 0 to 2.

So, we get t is 0 to 2 and then the line. So, we compute this values the value will come out to be what that calculate is whatever the value is coming minus 1 plus 2 i minus 3 t square. So, we compute it. So, t cube by 3 that is minus t cube and then minus 6 i t cube by 3 is it, and then take the limit 0 to 2. So, finally get in to be minus 8 1 plus 2 this will now along the path two is that the first path is c 1 then c 2. So, along c 1 the parametric equation of the curve z(t) by 0. So, x coordinate only. So, it will be the x(t) only y 0 i time 0. So, what is the x(t) here that x becomes t. So, here is t andy becomes here along this path x(t) is t where t varies from 0 from here to here 0 to 2 andy 0.

So, y is always 0 along this. So, this will be 0 path and here we t lies between 0 to t. Hence integral of this real part of z square dz along path C is real part z square x square minus y square. So, x square minus y square dz by dt dt and t varies from 0 to 2 substitute the value x in y x is t, so 0 to 2 exist. So, t square and then dz by dt becomes one dt and that value will come out to be t cube by 3 and then 0 to 2. So, it is the 8 by 3. Then third path along third the path sorry along the c 1 along c 2 along this is along c 1 then along c 2 compute the calculate the path c 2 is c 2 z(t), if I take c 2 the x coordinate always two by changing. So, it can 2 plus i t where the t varies from 0 to 4, then compute this enough. Similarly we get the other stop here, next time we will continue from here. Thank you.