Advanced Engineering Mathematics
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Lecture No. # 13
Harmonic Functions

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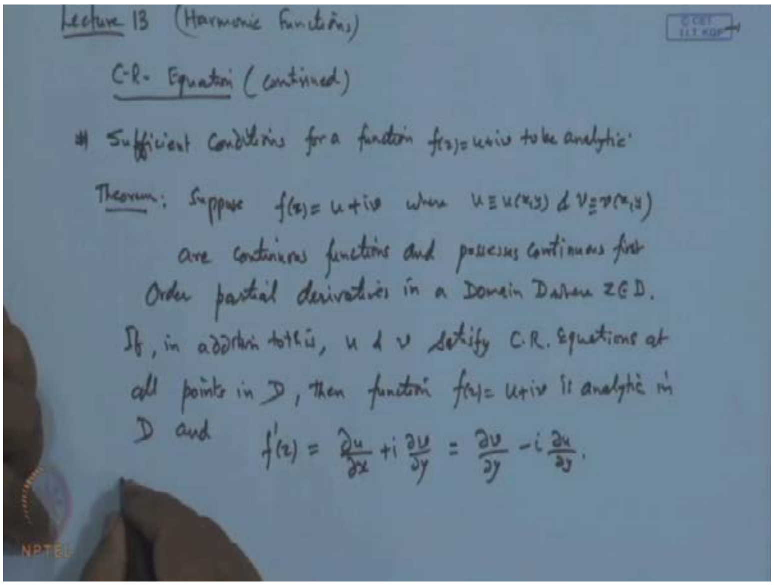

So, the last previous lecture, we were discussing about the C-R equations. So, first we will continue with C-R equations and then let on, we will go for harmonic functions. We have discussed during the analytic functions and we have seen the function is analytic at a point z naught, if it will not only differentiable z naught, but at every point in some neighbourhood of z naught and if the function is analytic everywhere in a domain D, then we say function at each point, then function is analytic at this point. function And we have seen the examples here the function is differentiable, but not analytic like more z square which differentiable at 0, but not analytic at the point 0, because is nowhere other differentiate the function is not differentiable in the surrounding of 0.

So, this and one more thing, which we have seen that C-R equations, these are the necessary condition for a function to be analytic or differentiable at the point z equal to z naught; however, these conditions are not sufficient which we have seen by an example f(z) is something which we have discussed. So, what should be the extra condition required over the partial derivatives, so that the C-R equations also we have as a sufficient condition for a function to be analytic. So today, first we will discuss the result the which says that under this restriction the C-R equations, we will also behave a sufficient condition for a function to be analytic.

So, let us see the first the sufficient condition, sufficient conditions for a function, sufficient conditions in terms of C-R equation for a function f (z), which is u plus i v to be analytic; this is our... So, we have the result in the form of theorem, the result says suppose function f, where even we are suppose the function f (z), which is u plus i v, u is a function of v is also function of f (z), where u and v are continuous functions and possesses continuous partial derivative continuous first order partial derivatives in a domain D, z in which the point z, where the z belongs to D.

Now, if even b if an addition to this if in addition to this u and v satisfy C-R equation, yet all points in D, then the function f which is u plus i v each analytic in D and the derivative of the function f (z) can be given by the formula del u over del x plus i times del v over del y all equivalently this same as del v over del y minus i del u over del y. So, if the function whose real and imaginary parts of continuous possesses continuous partial derivatives of first order in a domain D, where these points then all in addition to this if u and v also satisfy the C-R equation at every point in the domain D.

Then we say the function will be analytic in the domain d and to discuss the analyticity at a point z naught what we consider the neighbourhood around the point z naught and then we claim the that C-R equations must be satisfying at all points in the neighbourhood of the z naught, then we say this. So, proof now consider the delta neighbourhood of this consider N z delta that is the set of those points z dash, such that the distance from z dash to z limit less than delta that is the delta neighbourhood of the point z in D. Consider this now, it is giving that partial derivative of u and v both are continuous function.

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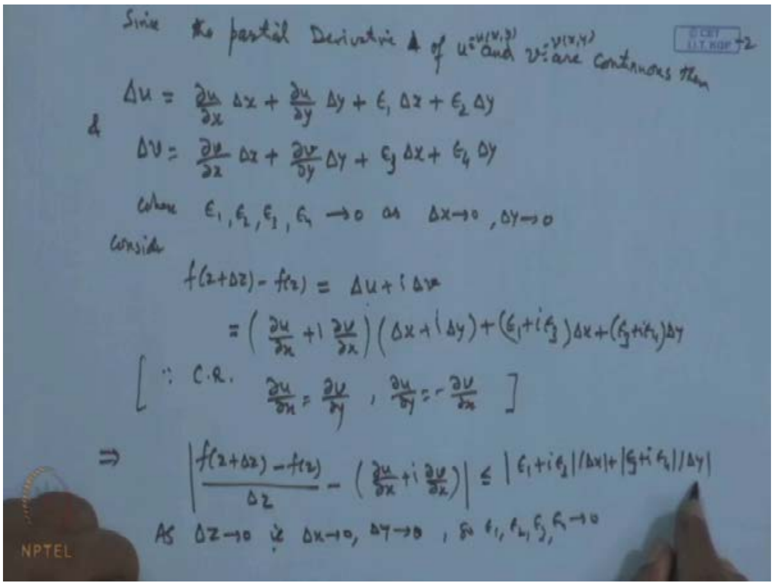

So, once it is continuous, then the total differential can be express in the fourier form since the partial derivatives of u and v are continuous and u and v are the function of two variable, u is a function of two variable, u is a function of u x y v is also a function of x and y. So, in case of the two variable, if function is continuous then we can expressed is the total incremental u can be expressed is del u over del x in to delta x plus del u over del y in to delta y plus epsilon 1 delta x plus epsilon 2 delta y and similarly delta v can be written as del v over del x delta x plus del v over del y delta y plus epsilon 1 or epsilon 3. You can change also delta x plus epsilon 4 delta y, this is the total increment in u and v in the neighbourhood of the point z where, epsilon 1 epsilon 2 epsilon 3 and epsilon 4 are very very small number and tends to 0 as del x goes to 0 delta y goes to 0. So, this is why definition of a function of the two variable, if you have a function of two variable, then total increment delta u and delta v can be expressed in to this form.

Now, consider f of z plus delta z minus f of z that is the total increment in f. So, that will be equal to the said delta u plus i delta v because f(z) is u plus i v which, f(z) plus delta u minus this that increment in delta u plus i time delta v, but delta u and delta v are giving by this. So, find out the delta u plus i time till. So, if I multiply this y i and this, then the entire thing can be expressed as del u over del x plus i times del v over del x. So, this delta x is out then del u over del y from here epsilon 1 delta x and epsilon 2 delta x. So, we can write it this as multiply by delta epsilon delta x plus i delta y plus epsilon 1 plus i epsilon 3 delta x plus epsilon 3 i epsilon 4 delta y. Now, this is given what is given is the partial derivative u and v are continuous and possesses a continuous and the satisfy the C-R equation. So, because of the C-R equation we can rewrite this everything in this form, is it not because this will be equal to del u over del x plus i times this 1, then when you multiply by i here, then what happen del u over del x into delta y.

So, here del u over del y i del will be the C-R equations mean, this is because C-R equations says del u over del x is the del v over del y and del u over del y is minus del v over del x. So, use this form and get this, if you just multiply you will get the same thing which you combine. So, you are not going in detail like this, now this part is nothing, but that f prime x, f prime z say del u over del x plus i times. So, we can say this implies that mod of f plus delta z minus f(z) divided by delta z and then delta z means delta x plus i delta y this is delta z divide by and minus f (z) divided by delta z. So, what you have getting is this term is free from this part bring it this side del u over del x i times del v over del x and then mod of this, now this part is less than equal to what modulus of this thing now modulus of this thing is mod of epsilon 1 plus i times epsilon 3 mod of delta x plus mod of epsilon 3 i times epsilon 4 mod of delta y.

Now, as delta x and delta y tends to 0 as delta z tends to 0 means that is delta x goes to 0 delta y goes to 0. So, the right hand side. So, epsilon 1, epsilon 2, epsilon 3, epsilon 4 will go to 0 therefore, delta x delta y already. So, right hand side will go to 0. So, right hand side will tends to 0 therefore, the left hand side will also limit of this is, nothing but this point.

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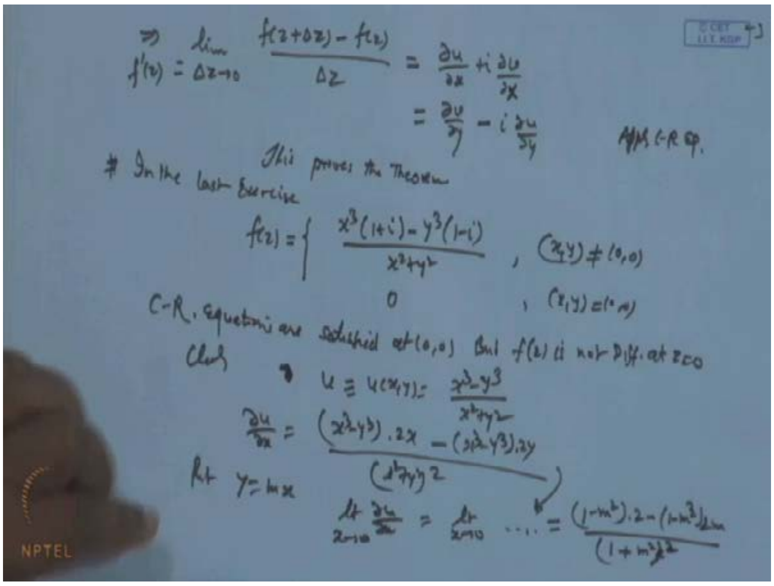

So, we get from here is, so the limit of this yet delta z tends to 0, f of z plus delta z minus f of z divide by delta z is nothing but del u over del x plus i time del v over del y and this limit, if it exist it is nothing but the what f prime z. So, the derivative of this is nothing but the value of this and now, if you apply the C-R equation, apply C-R equation then this same thing can be written in the form which we required it in the first one, that is we wanted this del v over del y minus i. So, use that is del u over del x is del v over del y and del and del v over del y C-R equations is the del u over del x is d1 del v over del x is these are the C-R equations, del u over del x its del v over del y and del u over del y is minus del v over del x. So, here we can write it this is del u over del x is del v over del y and del v I think this is x, mistake here somewhere may be del u over del y. So, somewhere this is x. So, this will be x therefore, this is x y this is x and now it is equivalent to the thing that is minus i del u over del y.

So, del u over del y minus del v over del x and del u over del y. So, this the same thing. So, this shows that that derivative exist, if the C-R equations are satisfy in the partial derivatives are continuous. So, this prove the results theorem now what went al1, when you discuss the previous result, if you say in the last in the last exercise that is will we go we have consider this function f(z) is x cube 1 plus i minus y cube 1 minus i over x square plus y square when x y differs from 0 0 and 0, when x y is equal to 0 0. Now, we have seen in this z this that the C-R equations are satisfied at the point 0 0, but the function f (z) is not differentiate yet z is equal to 0, this we have already estimates let us apply this result why this is shown it means the C-R equations are already the partial derivative are not continuous that is why this function is not analytic or differentiable exist 0 in the region what is the partial. So, trivially what is the del u the u becomes x cube minus y cube over x square y square this is u and 0 when it is 0.

So, when you write the del u over del x this becomes just x square of this, you tell now the numerator and denominator are having the same degree that is. So, if we put y is equal to m x then limit of del u over del x when x tends to 0 is equivalent to limit x tends to 0 of this thing and when you substitute this, it will depend on m it will depend on m limit will depend, because when you substitute y equal to m x x will be cancel out and we get 1 minus m cube into 2 minus 1 minus m cube m cube into 2 m y equal to m x and 1 plus m square m square whole square. So, basically this depend on by on m. So, limit does not exist limit does not exist. So, del u over del x is not continuous at 0 0. So continuity, therefore this function this just satisfying the C-R equation is not good enough to re justify the function is analytic or differentiate the 0. So, that is now on the other hand if the function does not satisfy the C-R equation.

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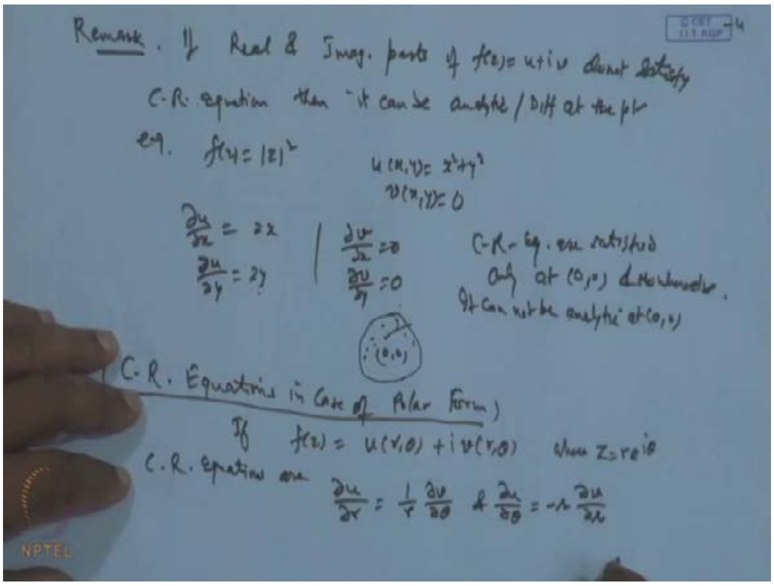

Then obviously, this function cannot be analytic the if the function f whose real and imaginary part of this if real and imaginary parts of the function f (z), do not satisfy C-R equation then it cannot be analytic or differentiable at that point cannot be analytic or differentiable at the point. For example, if you take the function f(z), which is mod z square. So, this is u x y is x square y square where the v x y is 0, now if you find the partial derivative of u with respect to x we get this thing del u over del x is this del u over del y is this where del v over del x is 0 del v over del y 0. So, C-R equations are satisfied only at 0 and nowhere else and nowhere else. Therefore it cannot be analytic at 0 0, because for the analyticity the function should not be only differentiable at this, but it should also differentiable at every point in this neighbourhood; however, it is analytic it is differentiable because C-R equations are satisfying at 0 0, the del u over del x and del v over del y they are all continuous functions.

So, this function is in continuous is differentiable at 0, but nowhere else is it know del u over del x and except at 0 0 we are. So, we get this is it not. So, we are getting 0 0 and this now the C-R equation in case of the polar form. Incase of polar form, this we will not try, but it is useful for sometimes when the function is not give in the in the form of x and y that u and v are not function of, but if a function of r and theta. Then they in order to just verify whether the function is an analytic or differentiable and if you want to apply this C-R equation then it is useful, if you know the formula for C-R equation in terms of the theta that is in a polar form. So, in the polar form if f(z) is a function which is u r theta and v r theta that is in the polar form, where z is r e to the power i theta then the C-R equations in the polar forms are del u over del r is 1 by r del v over del theta and del u over del theta is minus r del v over del r.

So, these are the form of the C-R equation in polar form and to drive this result, just what you do is z equal to r e power i theta. So, x plus i y is this. So, x becomes r cos theta phi becomes r sin theta. Now, find out the del u over del x and del u over del y keeping in mind this is the function of two variables. So, one variable constitute to this constant well doing the differentiation, but r is also a function of x y because r is becomes a x square plus y square and theta becomes ten plus y over x. So, r and theta both are the functions of x and y, hence in order to derive the we just go through.

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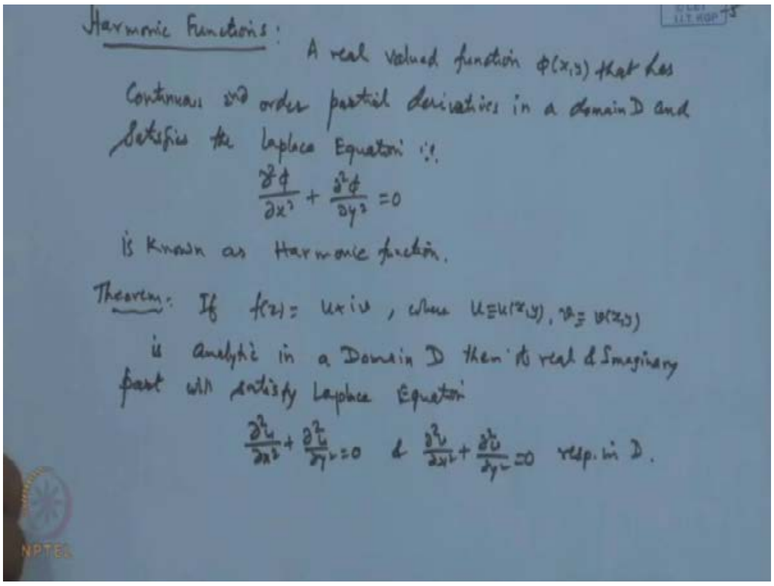

So, derivation we have just avoiding, because this not in very much, now it is come to our mind is harmonic functions, a real valued function phi which is function of two variable phi of x y a real variable function phi of two real valued function phi, which as phi x y that has that, has continuous second order partial derivatives in a domain D. And satisfy and satisfies the laplace equations that is, del 2 phi over del x 2 plus del 2 phi over del y 2 0 then a real valued function phi that has a continuous second order partial derivative in a domain D that is and satisfy the C-R equation is known as harmonic function is known as harmonic function. So, for the harmonic function what is known as the function must be the continuous equation, it should possess a continuous partial derivatives and perform this it must satisfies the laplace function. So, solution of the lap lace equations find which has a continuous partial derivatives up to or up to will be a harmonic functions now we want it to relate this concept with our analytic functions. So, the result is like this if a function f(z) where u and v is a function of x y.

Let f (z), if f (z) is analytic in a domain D, then its real and imaginary part will satisfy the Laplace equation then its real and imaginary part will satisfy laplace equation laplace equations that is del 2 u over del x 2 plus del 2 u over del y 2 is 0 and del 2 v over del x 2 plus del 2 v over del y 2 is 0 respectively, in the domain d that is u and v are harmonic functions. In fact, if you look or we will see when we go for further Cauchy integral formula and other thing, we will show that it function is analytic then it is infinitely time differentiable function that, we will function f (z) of complex variable, if it is analytic, then it is infinitely time differentiable function which is not true valid in case of the function of real variable, because your function may be differentiable once twice thrice, but may not be infinitely time differential function

So, what here if we have a very good results that for a analytical function, the infinitely time differentiable some is possible and will we get basically the expense and function deform of the series which call the power series tailor series. I had and then also in the similarity you know that series. So, because of that, if function is analytic, then it means u and v both are continuous function and will possess a continuous partial derivatives also and apart from this, if the real imaginary parts then since it is C-R equations are necessary condition. So, C-R equations are also satisfied at that point. Then what is that real and imaginary part will with the solution of the lap lace equations that is u and v both will be harmonic equations. So, if one can identify the one can say the real part is the harmonic imaginary part is harmonic see the proof.

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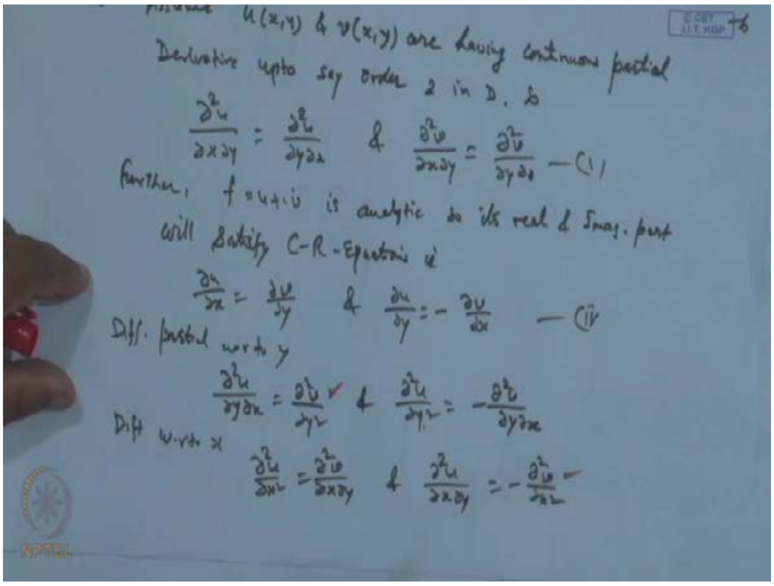

So, in order to prove this thing here, we are assuming of course but it automatically comes. So, assume u and v are continuous functions are having continuous partial derivatives partial derivatives up to say order 2 in the domain D. Of course, if we are not required because the result were not D 1 that is why we have assuming for this. So now, when this result in case of the two variables, if the u are function of the two variable is a continuous function and possess a continuous partial derivatives up to order as a 2, then next order partial derivatives will give the same value, that is whether we differentiate u first respect to x and then with respect to y, we will get the same result when we reverse the order the differentiation of u with respect to y first and the equation. So, the del u 2 u over del x del y will be the same is del 2 y over del y del x and similarly del 2 v over del x del y will be the same is del 2 v over del y del x then no difference, in case of if it continuous partial derivative of order 2.

So, this result is for a function of two variables and now further f which is u plus i v is analytic. So, it is real and imaginary part will satisfies C-R equation. So, it is real and imaginary part will satisfy C-R equations that is del u over del x is del v over del y and del u over del y is minus del v over del x let it be 1 or this be 1 this is 2, now you differentiate it. So, differentiate partial with respect to say y first equation. So, what we get from here is del 2 u over del y del x is del 2 v over del y 2 and this is del 2 u over del y 2 is minus del 2 u over del y del x. So, at this 2 del 2 u over del y 2 to del 2 u del 2 u this 2 are del 2 u this is del v over del y 2 and this del 2 u over del y 2 is del v over del x then with respect to x. So, what we get is the del 2 u over del x 2 is del 2 v over del x del y and del 2 u del x del y is minus del 2 v over del x 2 now you look this take this one take this portion and this portion, if these two are identical.

Therefore, this two will be identical. So, from here we get del 2 v over del x 2 plus del 2 v over del y 2 is 0, now if you look these two, this one and this one, then again these two are identical. So, minus of this meant by this implies this implies del 2 u over del x 2 plus del 2 u over del y 2 is 0. So, this shows therefore, this therefore, we get u and v are harmonic. So, this was the so it means we get this result that, if function is analytic function then its real and imaginary parts are harmonic function.

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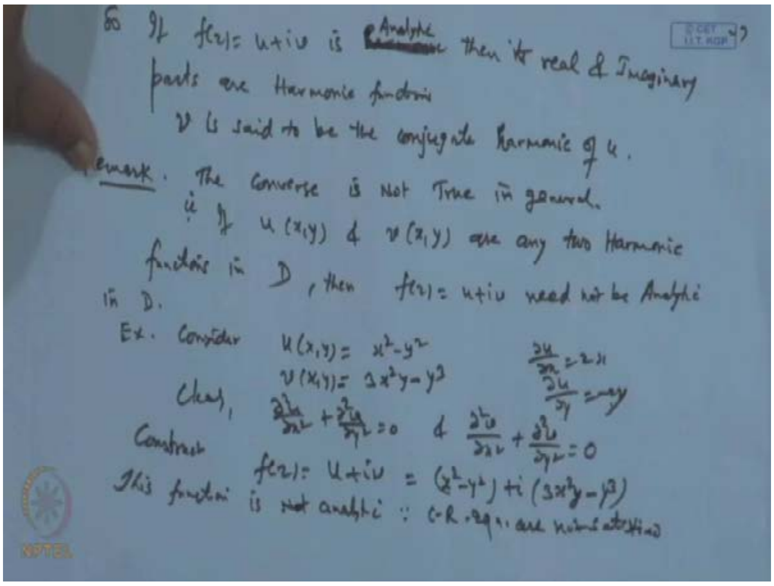

So, if f(z), which is u plus i v each harmonic, then its real and imaginary part are, this is analytic is analytic then its real and imaginary part are harmonic functions then in that case, we say v is said to be the conjugate harmonic of u. We said to be the conjugate harmonic of v and u said to be the conjugate harmonic of u v and vice versa each other that is the conjugate harmonic to each other. Now, the converse of this may not be true converse means if suppose we have the two functions u and v, which are harmonic and if I construct a function of f, which is u plus i v then it is not sure you cannot say the function is analytic. So, conversely for remark the converse is not true in general in general that is that is if u and v are harmonic functions are any 2 harmonic functions in the domain D then the function f (z) which is u plus i v need not be analytic in D.

Let us see the example, suppose consider the function u which is say, x square minus y square and v is suppose 3 x square y minus y cube now both these functions satisfy the harmonic laplace equation clearly del 2 u over del x 2 plus del 2 u u over del y 2 0 because del 2 u del u over del x is 2 x del u over del y is minus 2 y. So, del 2 u over del x 2 is 2 del 2 over del y 2 is minus z similarly del 2 v over del x 2 plus del 2 v over del y 2 is as a 0. So, they are nothing you can immediately say that this these two are harmonic function. Now, if we construct the construct function f z, which is u plus i v. So, u plus i v means this function x square minus y square plus i times of 3 x square y minus now, this function is not a this function is not analytic, why because C-R equations are not satisfy why because what is the C-R equations del u over del x the C-R equations are del u because del u over del x is nothing but 2 x del u over del y is nothing.

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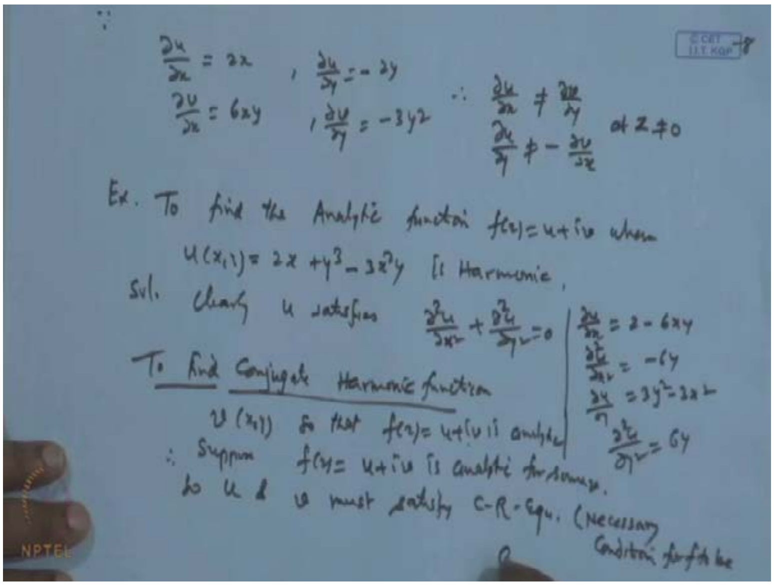

But minus 2 y del v over del x is del v over del x is six x by del v over del y is minus 3 y square. So, we say therefore, del u over del x the first form del v over del y and del u over del y the first from minus del v over del x, C-R's are not satisfying at the point this is not 0 0 at a point x y z different from 0. So, this function is not analytic functions therefore, the even the function u v all harmonic function, but the real imaginary part if you construct a function f with the help of those harmonic functions then the functions. So, obtain may not be a an analytic function. So, this is what now another is a once if the suppose a function u which given which a harmonic we want to find the conjugate harmonic of this. So, that the function object becomes harmonic this is and then let see the example to find the conjugate to find the analytic function f(z), which is u plus i v where u x phi, which is giving to be 2 x plus y cube minus 3 x square, where u y is harmonic this is the.

So, our aim is to find the analytic function with the help of this harmonic function. Let us see the solution since it is harmonic, so it must satisfy the lap lace equation and we want it the function. So, we want it the function f which is u plus i v, which is an analytic function it means the partial derivative of u and v must be a continuous function and satisfy the C-R equations. Now, let see what is our first whether it is a harmonic or not clearly you satisfies this equation del 2 u over del x 2 plus del 2 u over del y 2 is 0 why because if it take this del u over del x this is equal to 2 minus 6 x y del 2 u over del x 2 is minus 6 y. Similarly, if you go for over del u over del y by it is equal to 3 y square minus 3 x square and the del 2 u over del y 2 is 6 y. So some, so therefore, this is harmonic function now we want it this to be. So, to find conjugate harmonic function v, so that f(z) which is u plus i v is analytic is it know. So, necessary condition for the analyticity the C-R's must satisfy.

So, suppose v such a function. So, that u plus i v is analytic suppose this is analytic suppose, f is analytic for some v for some v. So, it C-R's equation must. So, u and v must satisfy C-R's equation because these are the necessary condition for the function f to be analytic is it enough for f to be analytic. So, let us apply this C-R's equation

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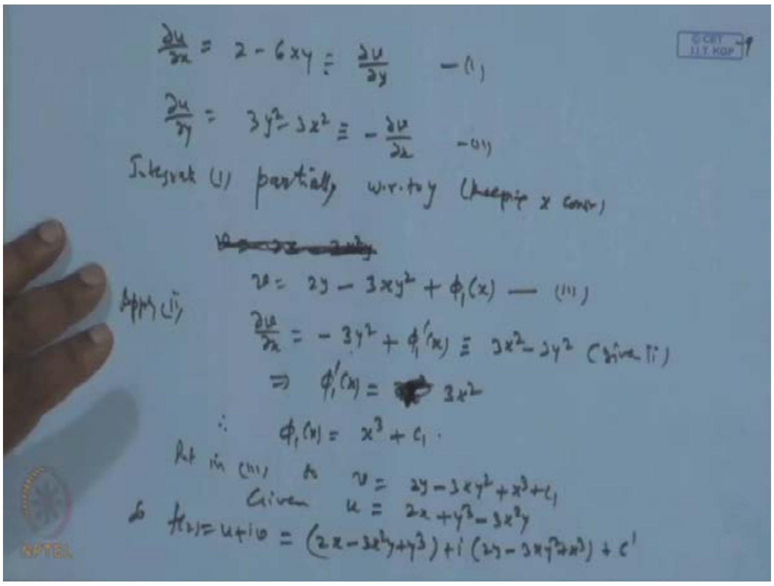

So, what is the C-R del u over del x must be equal to del v over del y, but del u over del x is what u is giving to with this. So, del u over del x is 2 minus six x y this is equal to del v over del y is it know del u over del x is del v over del y equivalent to this then del u over del y which is 3 y square minus 3 x square, this must be the same as minus del v over del x is it now let us take this 1 this is 2. So, from 1, if I integrate it partially with respect to y integrate 1 partially with respect to y keeping x is constant. So, what we get v becomes 2 x minus 3 x square y. this in we are integrating partially this. So, this v will be equal to 2 y minus then integrate y square by 2. So, 3 x y square is it not, and then constant of indication will be a function of f(x), why because when you are integrating partially with respect to y x constant.

So, I am keeping a constant I can choose constant as a function of x nothing now apply the second that is differentiate v with respect to x. So, when we differentiate v with respect to x, what you get del v over del x this is 0 here, we get minus 3 y square plus the derivative of this function, but this must be given equal to what minus of this part that is 3 x square minus 3 y square this is given. So, from here we get minus that is phi 1 x is 3 y square phi 1 x is 3 y 3 x square that is 3 x square are minus this cancel. So, phi 1 dash x. Therefore the integration of this phi 1 x becomes x cube that is equal to integrate with respect to x. So, we get x cube 3 plus a constant c 1 this constant of integration, we will come. So, we get from here phi 1 is there therefore, in this third is substitute in. So, we get v becomes x 2 y minus 3 x y square plus x cube plus a constant c 1 minus 3 x y square plus x cube and what was the u, u was already given this was the u 2 x.

So, u was given u as 2 x plus y cube minus 3 x square y. So, for function f becomes f (z), which is u plus i v we can just write u plus i v. So, 2 x minus 3 x square y plus y cube and I times of 2 y minus 3 x y square plus x cube and then constant of indication we can said that is all. So, this will be our function, which is analytic. So, that is why we can we can go we can find out the constants function, which are analytic, if I know the one harmonic function, any harmonic function we can identify the conjugate harmonic. So, that the this becomes now there are using of this C-R’s equation, we can apply this C-R’s equation to get this some results.

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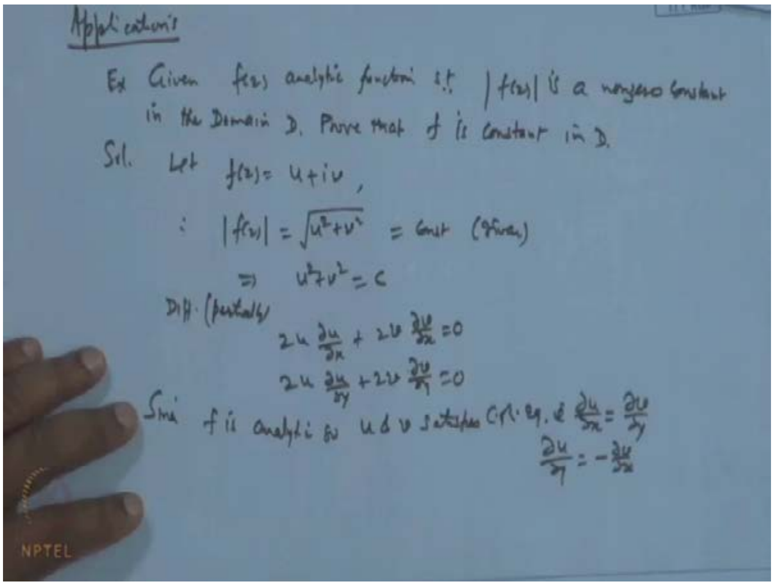

So, what is the application of C-R equations are, how to find how to use the C-R’s equation to get suppose, I ask this question given the function f is analytic given f(z) analytic function such that modulus of f(z), is a non 0 is a non 0 constant is a non 0 constant in the domain D, in a domain D given that so and then prove that was prove that f is constant function in the domain D. So, let us see the solution what is given the function is analytic, it means is real and imaginary part satisfy the C-R’s equations. So, let f z is u plus i v, where u and v are satisfy the C-R’s equation therefore, what will be the mod of z? Mod of z will be u square plus v square, is it know now this is giving to be constant this is give. So, we get from here is u square plus v square is some constant c that is all now differentiate it partially. So, we get twice u del u over del x 2 v over del x is 0 similarly twice u del u over del y twice v del v over del y is 0, this we get it now apply the C-R’s equation, if I take from your del u over del y, we know since f is analytic.

So, u and v satisfy C-R’s equation that is del u over del x is del v over del y del u over del y is minus del v over del x. Now, if I replace this del u over del y is minus del v over del x is it know del u over del y minus del v over del x and this del v over del y del v over del x is from your del v over del y is del v over del x. So, substitute it here, from 1 if I solve it using multiply the replace this thing in terms of x and we get twice u square plus v square del u over del x becomes 0 y, you multiply this by u this by v. So, when you multiply by u u square, you get u v then you multiply v and this to get cancel and this becomes using this 1 del v over del y is del v over del x they can be added. So, we get this 1, but this is constant this is constant. So, this cannot be 0 this implies the del u over del x is 0. So, what that u is independent of x, then similarly we can showed del u over del y is 0. So, u is independent of u.

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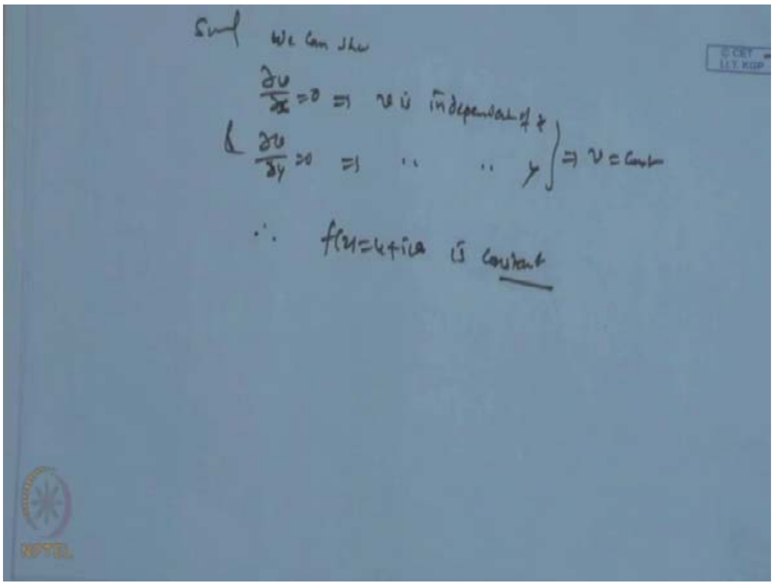

So, u will be a constant and similarly, we can proved for similarly we can show that del v over del x is 0 that implies that, v is independent of x and del v over del y equal to 0, will implies v is y. So, v must be constant. So, u must be constant u is constant from here v is constant therefore, function f(z), which is u plus i v is constant; thank you very much. Thanks.