Advanced Engineering Mathematics

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Lecture No. # 12

Analytic Functions, C-R Equations

So, we will continue the our previous thing, you are discussing the continuity of the functions.

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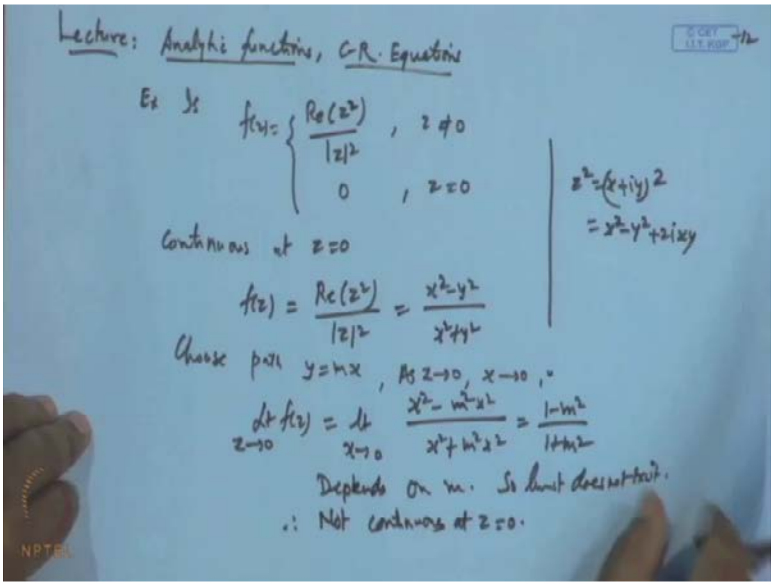

So and we have define a function is continuous, if the limit of the function f (z) when z approach to z naught, exist and coincide with the function value of this. Now, in case if a limit does not exist or the function limit exists, but it differs from the functional value, then the function will not be continuous or the function itself is not defined at the point z naught, then also it will be a discontinuous function. So, let us take an example one for the continuity, before you starting for the analytical function, suppose is this function is f (z) define by this, real part of z square divided by mod z square, when z is not equal to 0 and 0, if z is 0. Is this function continuous at z equal to 0? So, first thing is the function is well defined at the 0, now we have to calculate the limit of this function f (z) when z approaches to 0.

So, if you look the function f (z). The f (z) is basically coming to be real part of z square divide by mod z square, that is the same as z square is x plus i y whole square and that is the same as x square minus y square plus 2 i times x y. So, real part of this will be x square minus y square over z square is x square plus y square. So, normally when we have a function in the rational forms and if the degree of the numerator and denominator is the same, then in that case the limit may not exist. So, that (( )) clue, it means the limit of this function may not exist, so function may not be continuous. So, let us start with a various path. So, I just start choose the path y is equal to m x. So, as z tends to 0, it means x will go to 0 y will go to 0, so we say x will tends to 0 y will tends to 0. So, when you substitute y equal to m x if you reduce to a function of x and we say limit of this function f (z), when z tends to 0 is the same as the limit of this function x tends to 0 x square minus m square x square divide by x square plus m square x square.

So, that is 1 minus m square over 1 plus m square, now depends on m. So, limit varies as the m changes y equal to m x changes, the corresponding limit changing. So, limit does not exist, therefore, it is not continuous at 0, so this will be (( )). Then, we go for this differentiability of the function; this is the example of a continuous function, continuity continuous function continued.

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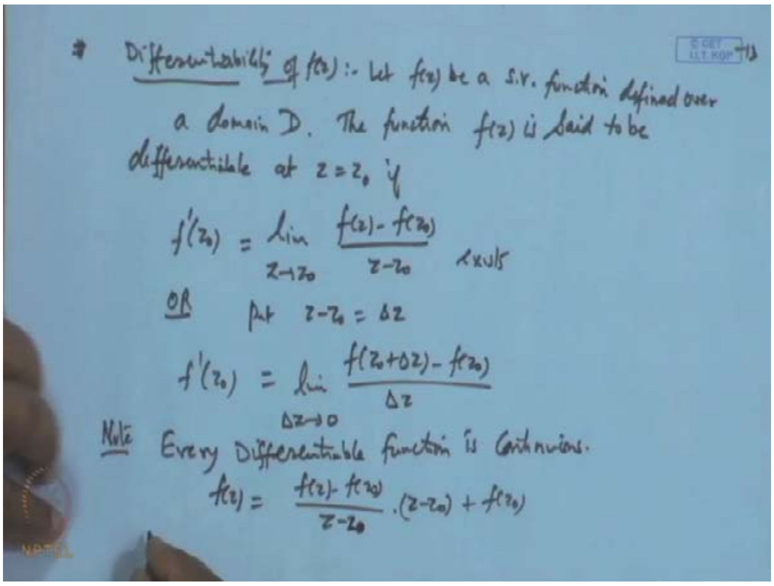

So, examples of the continuous… Now, definition of differentiability of the function; so differentiability of a function f (z); let f (z), we have single (( )), now we will always look the single valued functions. Let f (z) be a single valued function defined over a domain D, the function f (z) is said to be (No Audio From: 04:49 to 04:56) differentiable at a point z equal to z naught; if the limit of this function, limit of this ratio f (z) minus f(z naught) divide by z minus z naught, at z approach to z naught exists that is whatever the path you choose limit should exist or with the epsilon delta definition limit must exist, satisfy the epsilon delta (( )) then this limit exist we denote this as f prime z naught and we say the function has a derivative at a point z naught equal to this, that is what.

So, when we say the limit exist, all this can also be written like this. Substitute z minus z naught is delta z, then the same thing we can put it else f(z naught) plus delta z minus f(z naught) over delta z and limit delta z goes to 0, this is the same as the limit and derivate of the functions, either we use this formula or may be this expression both will give the same. Now, obviously if the function is differentiable must be continuous, every differentiable function is a continuous function. The reason is this because we start with f (z), this we can write it as f (z) minus f(z naught) divide by z minus z naught into z minus z naught plus f of z naught, the same thing now f (z) tends to z naught. From the one the limit exists. So, limit of this ratio f (z) minus f(z naught) over z minus z naught, this will come out to be f prime z naught exist and this will give z 0 then plus f (z) naught.

So, what we get limit of f (z) when z tends to z naught is a f (z) naught. So, this implies that limit f (z) when z tends to z naught is f (z) naught. So, is it not a continuity condition? So, f is continuous, f is continuous at z naught. So, every differentiable function which is differentiable it has to be continuous, they have get that point. If a function is continuous at every point in the domain, then we say it is continuous throughout the domain and similarly, the differentiability also.

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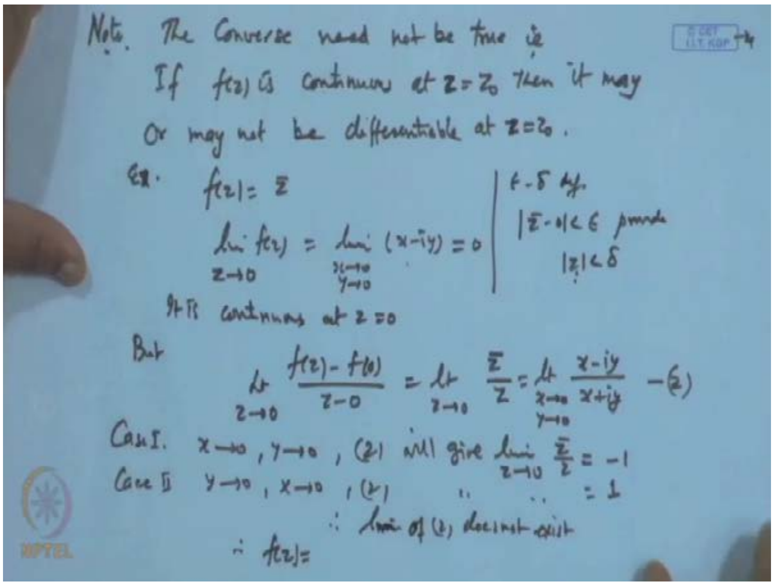

But the conversing is not true, the converse need not be true that is if a function f (z) is continuous at a point z equal to z naught, then it may or may not be differentiable at a point z equal to z naught. That is continuity does not implies always differentiability, but differentiability always implies the continuity.

For example, if you look the function f (z) at z bar, then the limit of this function f (z) when z tends to say 0, z tends to 0. This limit is what? Limit of this x tends to 0 y tends to 0 z bar means x minus i y, so it is 0. And, in fact if I apply the epsilon delta definition, same thing will go. So, for epsilon and delta definition also, we can say mod of z bar minus 0 can be made less than epsilon, provided mod of z is less than delta, mod z means mod x is tending to 0, x y tending to 0, so this will go to 0. So, it is continuous at z equal to 0, but if I look the differentiability then f (z) minus f 0 divide by z minus 0 limit z tends to 0, what is this? Limit z tends to 0 f (z) is z bar, f 0 is 0 and divide it z. And, this is the same as limit x tends to 0 y tends to 0 x minus i y over x plus i y. now, again I choose the different path.

So, if you take the path one when x tends to 0 first y tends to 0 later on, then this equation 2 then 2 will give limit z tends to 0 of z bar by z of this (( )). When x is tending to 0, then what happened this i by i gets cancel and the value will be minus 1. Case second, if y tends to 1 first later on x tends to 0, then 2 will give the limit by z 0. So, it is x over x, the limit will be 1, so limit varies then the path changes. Therefore, limit does not exist; limit of 2 expressions does not exist. So, it is not differentiable. So, f (z) equal to z bar is not differentiable at z equal to that is what, so the conversion need not be true always. Now, there are functions which are differentiable only at a single point and nowhere else, they are all the functions which is differentiable everywhere throughout the complex plane and there are the function which are differentiable only inside certain domains and outside is not there.

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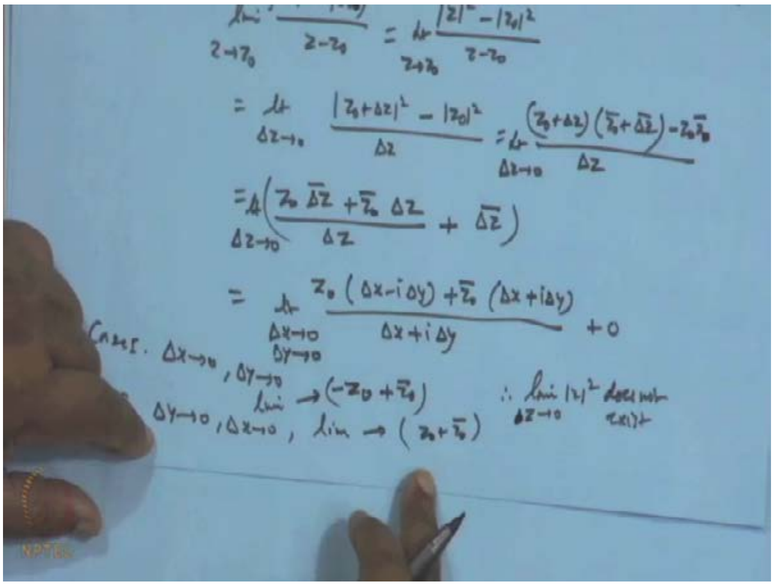

So, let us (( )) this. So, suppose I take this function f (z) is equal to say mod z square. Now, we claim that this function is differentiable at z equal to 0 and nowhere else. Because what is the limit of this, f (z) minus f(z naught) over z minus z naught limit f (z) tends to z naught, if I conclude this it will mod z square minus z naught square over z minus z naught limit z tends to z naught. Or this is the same as limit delta z tends to 0 z naught plus delta z whole square minus mod z naught square divide by delta z, is it not. But this will be equal to z naught plus delta z into conjugate of this, mod z square z z bar minus z naught z naught conjugate divide by delta z and limit delta z tends to 0.

So, if we simplify z naught, z naught get cancel and what you are getting is, that z z naught into delta bar z plus z naught bar delta z divide by delta z and then another term which you are getting is delta z bar, limit of this z delta z goes to 0. So, if delta give this to 0 delta bar z will go to 0, say it is no problem and here it can be written as z naught, this is what? delta x minus i delta y, this will be z naught bar delta x plus i delta y over delta x plus i delta y limit delta x tends to 0 delta y tends to 0 and this is 0 bar. So, there is no problem.

Now, again the limit when I choose the limit, suppose delta x tends to 0 first and then delta y goes to 0. So, when delta x goes to 0 first this is 0 this is 0, delta x is 0. So, what you are getting is here from say delta x 0 is z naught minus i delta y z naught plus i delta y and then divide by delta y, so delta y gets cancel. So, what you are getting here? z naught will minus i, i delta y is out plus z naught plus this, that is all. This is the value of the limit. So, limit will be, this is the limit, this will be their limit, will be this, is it not y i delta y i get cancel from here and minus z naught or here the z naught.

And, when delta y goes to 0 first and then delta x goes to 0, then the limit will come out to be, now delta y goes to 0 first this part is 0 delta x out, so limit will be z naught plus z naught bar. Now, here we have getting z naught plus, here we have getting minus z naught plus, these two differs. Therefore, limit does not exist; delta z tends to 0 f of z means this mod z square does not exist. So, z tends to does not exist. So, it is not differentiable at z naught, but if I take z naught to be 0, what happened, if I take z naught to be 0 z naught bar is also 0. So, limit exists.

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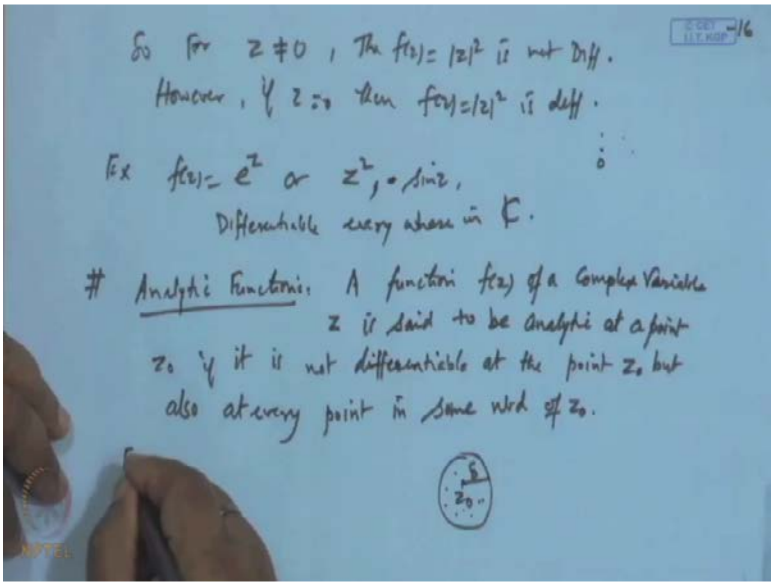

And, in fact, with the help of the epsilon delta definition you can proof that this limit exist. So, for z different from 0, limit the function f (z) with mod z square is not differentiable. However, if z equal to 0, then the function f (z) is differentiable, because this limit will exist, whatever the path you choose if you get the same thing epsilon delta definition will help you in getting, so that I am leaving to you, so this part. It means this is the only; this is the function which is differentiable only at single point 0 and nowhere else and let us starts which is not different. But if you look the function another function f (z) this is say e to the power z or z square or may be z cube and so on, these are the function sin z cosine z, these are the function which are differentiable everywhere in the complex plane. And, in fact, the derivative will come e to the power z, 2 (( )) z, cosine z and so on and so, far like this.

But there are the functions which are differentiable only inside the disk and nowhere else also that also we can get it, like 1 over 1 minus z, it does not, it is not define beyond z equal to 1. S0, this function is analytic (( )) only inside it, outside only mound it as singular point. So, we are not going for this. So, this different type of the thing functions you can get which are differentiable at single point nowhere else and also (( )). We now come to our analytic functions. (No Audio From: 19:06 to 19:13) A function f (z) of a complex variable z, z is said to be analytic at a point z naught, if it is not only differentiable at the point z naught, but also at every point in some neighborhood of z naught.

What is the meaning of this? Suppose, we say function f is given and z naught is given, a function f is said to be analytical at this z naught. If function is differentiable at z naught and there exist a neighborhood around that point (( )) delta may be, such that at each point of this neighborhood function is also differentiate. So, when the function is analytic then differentiability of the function at that point is not only necessary, what is necessary is that it should also behave properly, it should be differentiable around the point z naught also, that is why. So, the difference between differentiability, analyticity; when we say the function is differentiable at point z naught we are not bothering much whether the function is differentiable at other point or not, simply continuity is important.

But when you go for the analyticity of the function at a point z naught, then we took here the differentiability of the function in the surrounding point of z naught, that is in some neighborhood of z naught the function must also be differentiable at each point apart including the point z naught itself. So, this is the way. So, for example, just now we have seen f (z) is equal to mod z square; now this function is differentiable at the point z equal to 0 and nowhere else. So, it cannot be analytic, so it is not analytic at the point z is 0.

However, if you look the function f (z) equal to e to the power z or sin z, now these are the functions which are differentiable throughout the complex plane. So, whatever the point you choose and whatever the neighborhood you choose around that point, at every point the function remains differentiable.

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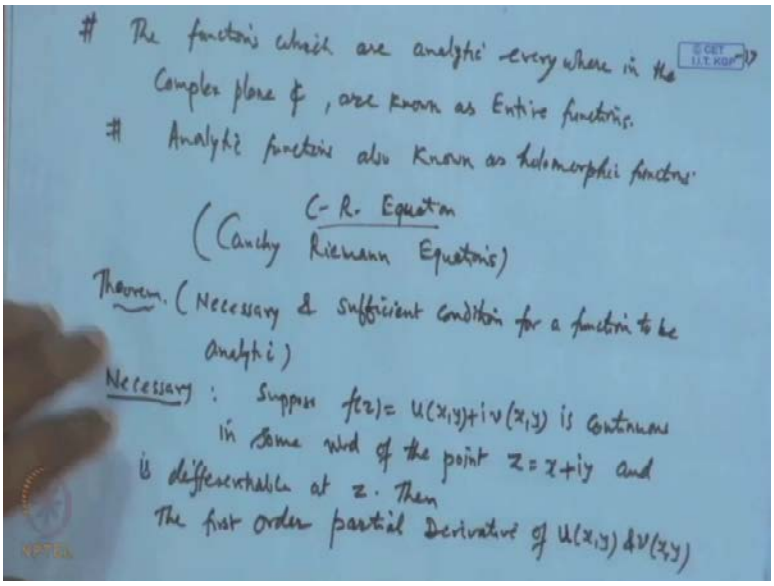

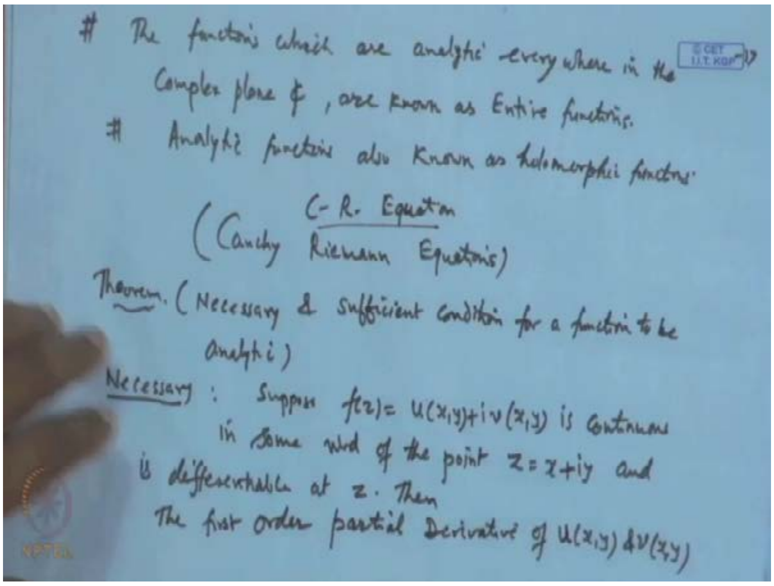

So, it will be a analytic function though analytic functions. The function which are analytic everywhere in the complex plane is said to be entire function. The function which are, for the function which are everywhere in the complex plane complex plane c is, are known as entire functions. Now, analytic functions are also known as polymorphic functions. So, this in new terminology which we (( )) use it, polymorphic function or analytic function. Now, as we have seen that functions there are the function which are differentiable, some function which are differentiable, but not analytic, some function which are differentiable only at some point and nowhere else or differentiable at every point, similarly analyticity also there.

So, when it go for the concept of this analytic functions or to test the given function is analytic, what we need is that the derivative of the function must exist in, at the point z naught and in the surrounding of the z naught, but to test those at those point which are infinite number is not a (( )) task. So, what we do, we develop some conditions which are known as the necessary or sufficient conditions for testing the green function to be differentiable or to be analytic and that will need to the concept of the C-R equations. So, let us see first the conditions and then C-R equation. So, let us say C-R equation, these are known as the Cauchy Riemann equations. In fact, it is given by Cauchy Riemann and the equations are in the form of the partial derivatives of a function u, it satisfies certain condition.

So, we will discuss what is the Cauchy. So, this Cauchy Riemann equation comes out when we go for the differentiability of the function, the necessary condition for the differentiability of the function or analyticity of the function. So, let see the result, theorem this time saying the result of theorem, the necessary and sufficient conditions for a function to be analytic. So, first is necessary part, what this (( )), suppose the function f (z) is continuous which is u (x, y) plus i v (x, y) is continuous in some neighborhood of the point z x plus i y and is differentiable at the point z, f (z) same point. Then, what are the necessary conditions says, then the first order partial derivatives of u which is u (x, y), partial derivative u which is a function of x y and v which is also a function of x y, first order partial derivative of u and v exist.

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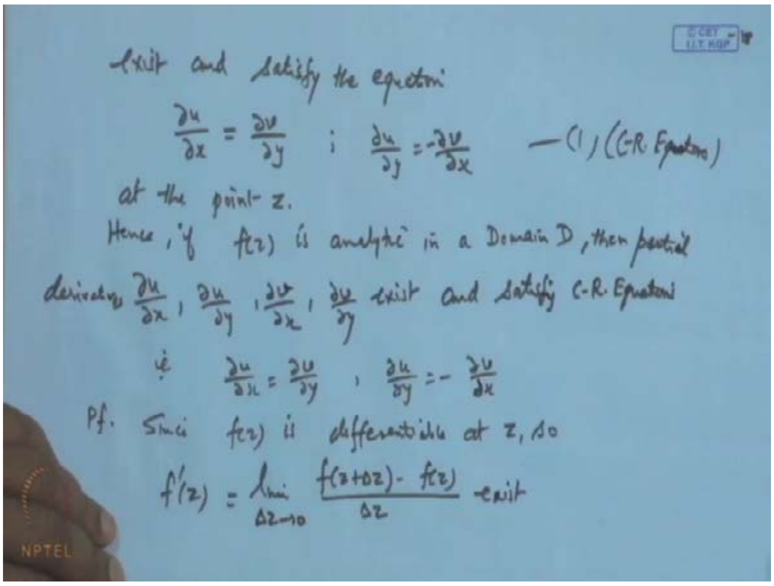

And, satisfy the equation, del u over del x equal to del v over del y del u over del y equal to minus del v over del x, let it be 1. At the point z, so what is this z says if a function is differentiable at the point z naught, then it must is real imaginary part must satisfy this equation, now this equation is known as the C-R equations, del u over del y. Now, similarly, if a function hence you can try because if you differentiable then analyticity for the same analytic condition as because function is analytic it has to be differentiable. So, then this, these are the necessary condition for the differentiability, so it will be the necessary condition for analyticity of the function also.

So, hence if z is analytic in a domain D, then the partial derivatives then del u over del x, del u over del y, del v over del x, then partial derivatives. del u over del x, del u over del y, del v over del x, del v over del y exist and satisfy C-R equation, Cauchy Riemann equations. That is del u over del x is del v over del y and del u over del y is minus del v over del x, so this condition (( )). See the proof this result first and then we go for the sufficient later.

Now, what is given is, the given is the function is differentiable this given function is differentiable. So, it means that derivative at prime that exist. So, limit f (z) minus f(z naught) z f (z) plus delta z minus f (z) over delta z exist. So, since the function f (z) is differentiable at point z, so the limit of this function f (z) plus delta z minus f of z divide by delta z, at delta z tends to 0 exist and the value we write f prime z, this will be there.

Now, f (z) we are taking to be the u plus i v, this is our f (z), u plus i v. So, if I use this thing z plus delta z, delta z is x delta x plus delta y.

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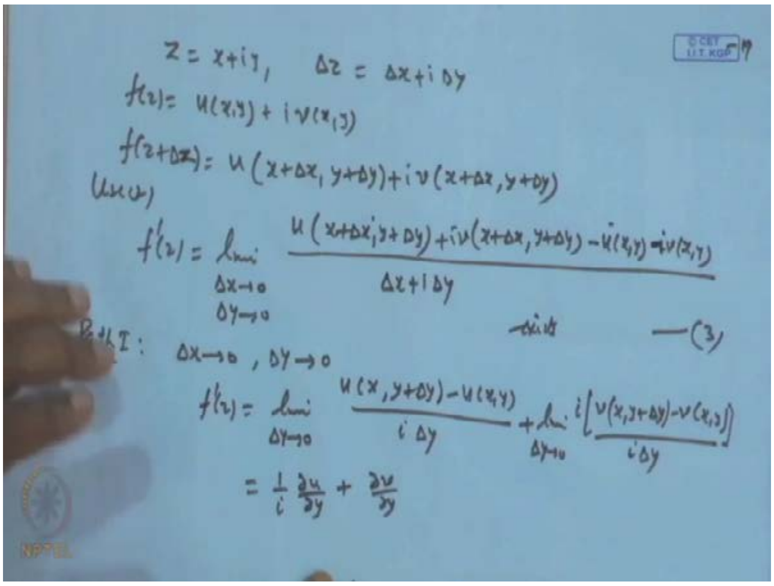

So, z is x plus i y delta z is delta x plus i delta y. So, f of z is u (x, y) plus i times v (x, y) and then f of z plus delta z. What will be this value is u x plus delta x, y plus delta y plus i times v x plus delta x comma y plus delta y just in a similar way. So, now let us substitute it. So, this equation is said to put it in 2, f prime z which is limit delta z tends to 0 means delta x goes to 0 delta y goes to 0, f of z plus delta z means it will be the same imagine u x plus delta x y plus delta y i times v x plus delta x y plus delta y minus f (z), that is u (x, y) minus i times v (x, y) divide by delta x plus i delta y, let it be third.

Now, case 1 or path 1 because this limit exist, this exist this is given. So, whatever the path you choose the limit comes out to be the same, so I choose the first path is delta x tends to 0 first and delta y tends to 0 later on. Then, as soon as you put delta x to be 0, then what happened is, this x u x y plus delta y minus u (x, y) by i delta y. So, we are getting from here is limit f prime z is equal to limit delta y tends to 0 u x y plus delta y x delta is 0 minus u (x, y) this term, these two terms I am choosing then divide by i delta y and then from here we get limit delta y tends to 0, i is out common, so i times v (x, y plus delta y) minus v (x, y) divide by i delta y.

Now, let see this delta y is tending to 0, there is the change in the second coordinate by changing to y plus delta y, there is no change in the x coordinate. So, it means you are taking the partial difference; this is partial increment in u keeping x as constant and when this partial increment divides by delta y, so this is ratio of the, this changes delta partial differentiation by divide by delta y, so limit delta y give the partial derivative of u with respect to y. So, you are getting this is 1 by i del u over del y. Similarly, when you go for this i get cancel here, there is the partial increment in v divide by delta y. So, it will give the value del v over del y del v over. Now, this is 4, now second path you choose.

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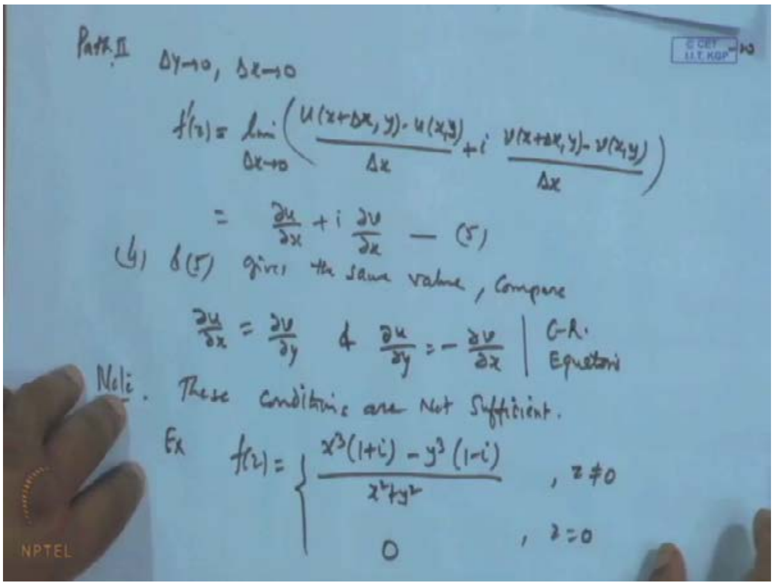

Second path, suppose delta y goes to 0 first and delta x goes to 0 later on. Then, in that case, we get f prime z equal to limit delta x tends to 0 u (x plus delta x, y) minus u (x, y) divide by delta x plus i v (x plus delta x, y) minus v (x, y) divide by delta x. So, again when you take the limit of this, then you are getting this is again the partial increment in x. So, it will give del u over del x and this will give i times del v over del x, fifth. Now, four and fifth give the same value because derivative exist, so limit exists therefore, compare it, so compare it real imaginary to imaginary. So, what you get is del u over del x is equal to del v over del y real to real and imaginary to imaginary, so minus i will come here, so del u over del y equal to minus del v over del x and these are called the C-R equations.

So, if the function is differentiable at the point then that particular point the C-R equations must satisfied. Similarly, if the function is generated at the point z naught, then the C-R equations must satisfy at this point. So, these are the necessary conditions which means same, it means if a function f whose real imaginary part does not satisfy the C-R equations, then the function will not analytical or will not be differentiable, is it not? Because these are necessary condition, these are not sufficient I am not saying it is a sufficient condition. So, basically we say these conditions are not sufficient, means C-R equations are satisfying but the function is not analytic function at that point where not differentiable at that point. The C-R equations, this condition are not sufficient means that for a function f for means the C-R equations are satisfying at that point, but the function is not differentiable at that point, then obviously, this will not be sufficient. For example, let us see the function f (z) which is x cube 1 plus i minus y cube 1 minus i divide by x square plus y square, when z is not equal to 0 and equal to 0 if z is 0.

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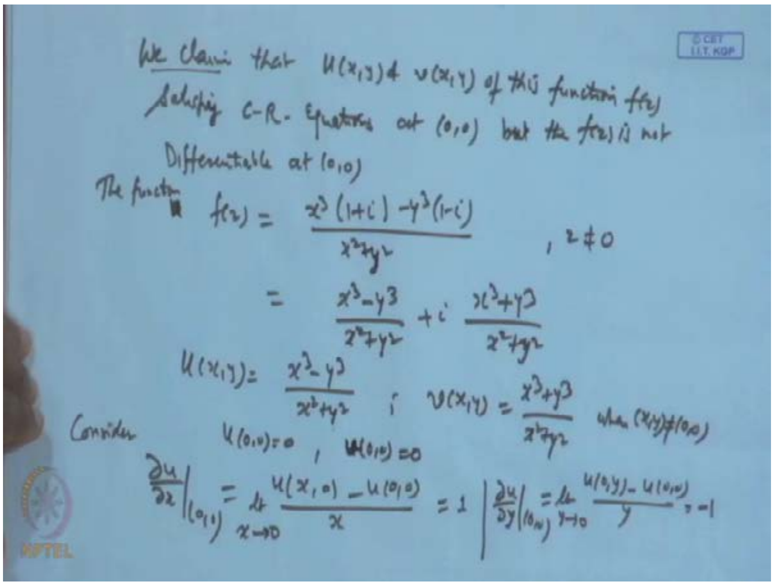

So, suppose I consider this then what we claim is the function u and v of this function satisfy C-R equations, but the function is not differentiable. So, that we claim, that the real and imaginary part u and v of this function f (z) satisfy C-R equations at the point (0, 0) but the function f (z) is not differentiable there, this be a claim. So, if we prove this claim, then obviously, this condition C-R satisfying the C-R equations is not good enough to say the function is differentiable. So, let us see one, what is the function u? The function f (z) is x cube 1 plus i minus y cube 1 minus i over x square y square, is it not, when z is not equal to 0. So, this function we can write like this x cube minus y cube over x square y square plus i times of x cube plus y cube over x square plus y square so, but u (x, y) becomes x cube minus y cube x square y square where the v (x, y) is x cube plus y cube when (x, y) is not equal to (0, 0) and 0.

When u is 0, v is 0 (0 0). Now, we claim this function u and v satisfy the C-R equations at 0, it means the partial derivative of u and v must satisfy that condition. So, what is the partial derivative of v? So, let us see what is the del u over del x at the point (0, 0). Consider this by definition; we are differentiating u partially with respect to x at 0. So, there will be change in x variable only, so u (x, 0) minus u (0, 0) divide by x and limit x tends to 0, this will be there del u over del x at the point (0, 0). Now, u is this, so value take by 0 and then u becomes what? x, this is 0 by x, so it is 1. Similarly, if you go for this del u over del y at the point (0, 0) then what we see here, this is the limit y tends to 0 u (0, y) minus u (0, 0) divide by y. Again, when you take x is 0 it is coming to minus y divide by y 1 so it is minus 1 and del u over del v over del x at the point (0, 0) similarly, you can prove.

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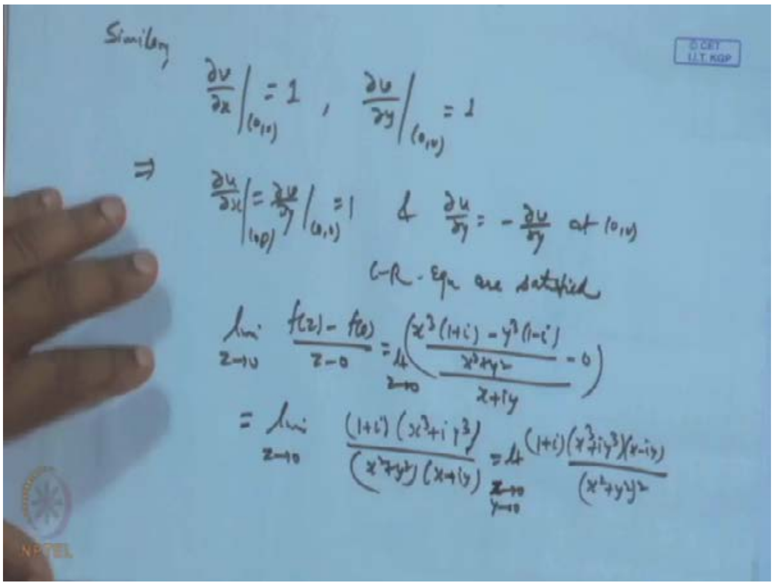

Similarly, you can verify the del v over del x at the point (0, 0) comes out to be 1 where the del v over del y at the point (0, 0) will also be 1. So, we are seeing that del u over del x is del u over del x is 1, del v over del x is 1, del v over del y is minus 1, del v over del y into del v over del x into del v over del y. So, we get del u over del x is del v over del y at the point (0, 0) at the point (0, 0) which is 1 and del u del y is minus del v over del x at (0,0). So, C-R equations are satisfied, but the function is not differentiable, but what is the derivative of this function? But limit of this z tends to 0 f of z minus f 0 over z minus 0. Now, this is nothing but what is our f? f is this function is it not, now this function f (z) can be written as all x cube 1 plus i minus y cube 1 minus i divide by x square y square f 0 0 0 over x plus i y and limit of this at z tends to 0.

Now, this expression we can write like this then we say this is same x limit 1 plus i x cube plus i y cube divide by x square plus y square and then x plus i y, is it not, just you can manipulated and get this z, so there is not a main problem. Now, multiply this conjugate, limit z tends to 0 multiply by conjugate. So, when you multiply by conjugate your getting this limit z tends to 0 means x tends to 0 y tends to 0 then or z by y equal to m x also. So, 1 plus i x cube plus i y cube x minus i y divide by x square plus y square whole square. Because this x minus i x plus x square plus y square, so it give as y square.

Now, substitute y equal to m x and let x tends to 0, then what happens? When you substitute y equal to m x, then x is out x is canceling and what you are getting the limit comes out to be 1 plus i 1 plus i 1 minus i divide by 1 plus sorry i equal to m, so it is x m. So, here is m cube, this is m, this is m cube, m cube I will write again; so I will write this. So, take this substitute m x.

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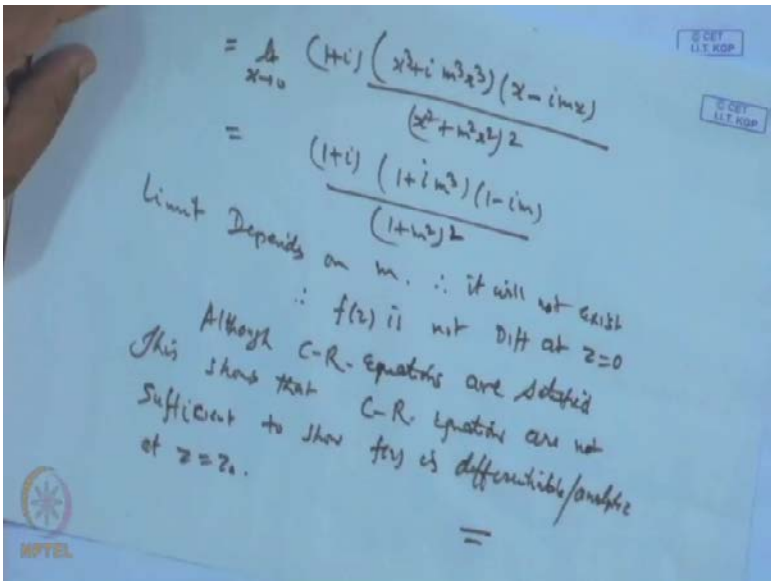

So, what we are getting is limit y equal to m x means limit x tends to 0 1 plus i x cube plus i m cube x cube x minus i m x divide by x square m square x square x as it tends to 0. So, this x square, x goes to out i x (( )) gets cancel and we get these value 1 plus i 1 plus i m cube 1 minus i m divide by 1 plus m square whole square. So, this limit depends on m. Therefore, it will not exist. So, function f (z) is not differentiable at z equal to 0 although C-R equations are satisfied. So, this shows that C-R equations are not sufficient conditions or not enough to show or not sufficient to show the function f (z) is differentiable or analytic at the point z naught, z equal to z naught, (( )). Now, if you look the converse part of it, converse means the non satisfying the C-R equations will give the condition.

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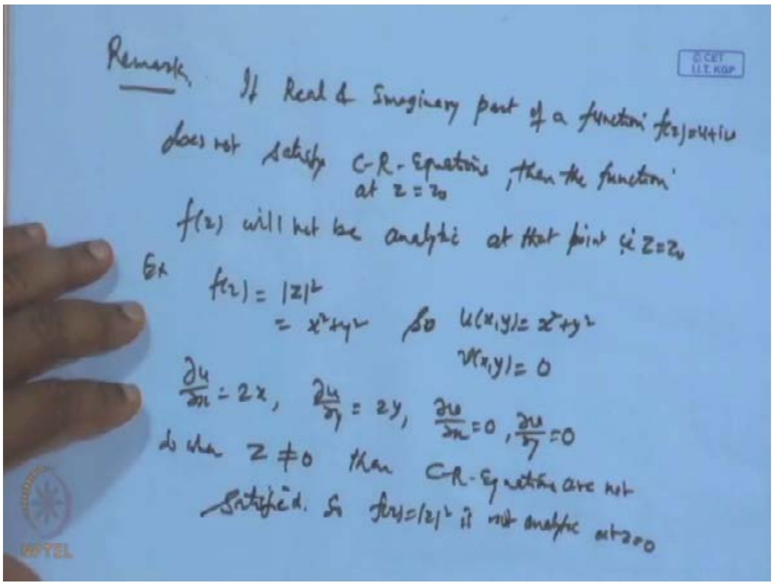

So, remark if a function f if real and imaginary part of a function f (z), which is real and imaginary part of this function f (z) which is u plus i v does not satisfy C-R equations. Then C-R equation at some point, C-R equations at z equals to z naught, then the function f (z) will not be analytic at that point. That is z equal to z naught, because C-R equation is necessary condition for a function to be analytic or to be differentiable. For example, if we look the function f (z) which we have seen already seen that mod z square this function is not analytic at the point 0, is it not? It is not analytic at 0, because the function is not differentiable at the point around 0. The reason is when you take the f (z) what is this is nothing but the x square plus y square. So, our u becomes x square plus y square v becomes 0.

So, when you take the del u over del x it is 2 x, del u over del y it is 2 y, while del v over del x is 0, del v over del y is 0. So, except when x and y are 0, the conditions are not satisfied. So, when z is different from 0, z is differentiable either x will be non zero or y will be non zero, so all both by non zero. So, if it not zero, then C-R equations are not satisfied. So, the function f (z) which is mod z square is not analytic at z equal to 0 that is all. C-R equations are satisfying at 0, but that does not mean that function is differentiable, because that is not sufficient condition unless you prove that these partial derivatives are also continuous that we will take next time that what are the sufficient condition, what restriction, what extra condition you can put it on the partial derivatives, so that while satisfying the C-R equations will give the guarantee the function is analytic at that point. Thank you very much.