Advanced Engineering Mathematics
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Lecture No # 11
Concept of Domain, Limit, Continuity and Differentiality

So in this lecture, we will discuss first the concept of domain, limit continuity and differentiability and later on we will proceed to a analytic functions and (( )) etcetera.

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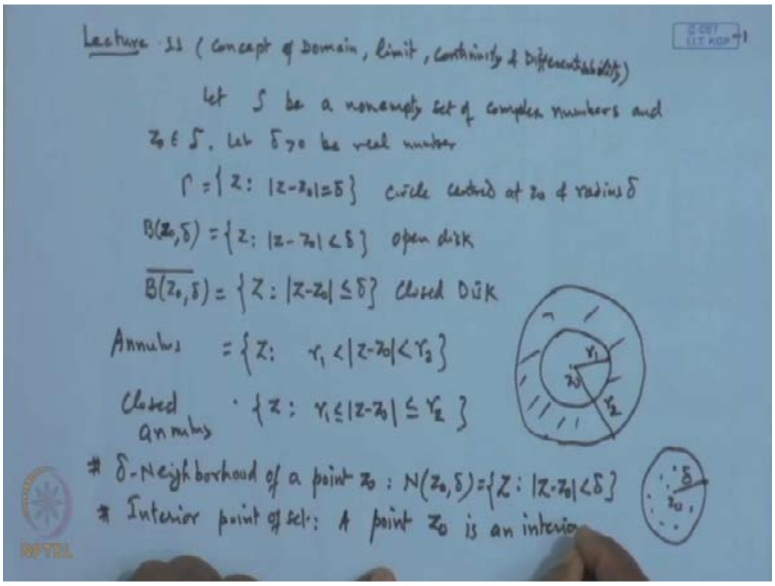

So let us first review the few concepts which are needed it in the (( )). So, let S be a non empty set of complex number complex numbers and a set of complex numbers; and suppose z naught be a arbitrary point in S, then the set of those point z such that not z minus z naught is equal to set delta, where delta is some positive number and delta greater than equal to 0 be a real number. Then this collection of the set is it denoted by basically some gamma and we called it is a circled centered at z naught and radius gamma and with radius delta.

So this is the similarly, when you say the set of those point z such that z minus z naught is less then, delta then this collection we will denote the b x not delta sorry z naught delta with the more centered at z naught with the radius delta. It is on open disk, it is called a open disk centered at z naught and with the radius delta with the radius delta and closed disk; the set of though z where z minus z naught is less than or equal to delta. This we denoted by say b z naught delta bar. It is the closed disk centered z naught and radius delta. So, you have then annulus b be the set of those points, the set of those complex number z such that r 1 less than mode z minus z naught less than r 2. Collection of all such points z centered z naught which lies between the two concentric circles;

This is one circles centered z naught with the radius r 1 and we have another circle which centered z naught and radius r 2. So this are the two concentric circles then set of those points is lies in between the two circles is known as the annulus annulus and if these are open and is the boundary is not in closed then we called it as the open annulus otherwise when the boundary is their then r 1 less than equal to z minus z naught less than equal to t two then this is called a closed envelope, a closed annulus closed annulus.

Now, the delta never hold neighborhood of this is defined as which is equal, now delta neighborhood of a point z naught. Delta neighborhood of a point z naught we denote this by n z naught delta and n is the collection of those complex number z such that mod z minus z naught is less than delta. It means, we are taking all such points inside a ball centered z naught with the radius delta then, this collection of such point, we called it as a neighborhood or delta neighborhood of a point z.

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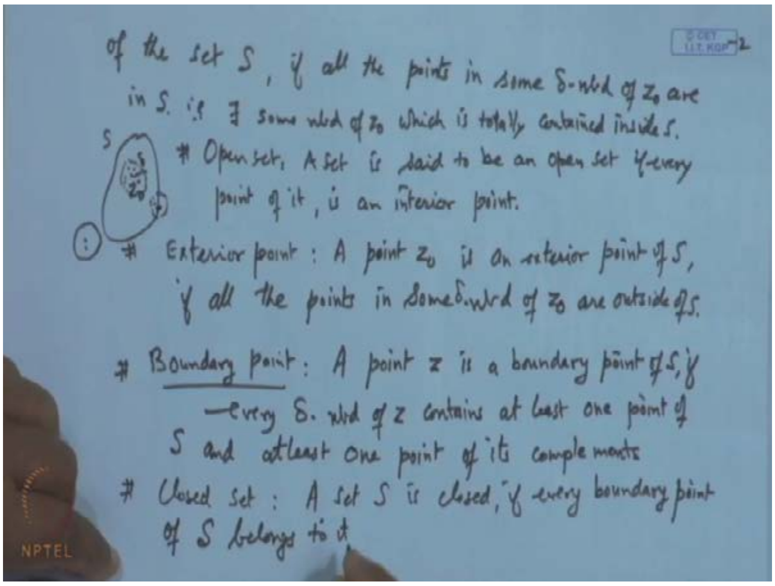

The interior point we mean, of a set means a point z naught a point z naught z not is an interior point is an interior point of the set S of the set s if all if all the points in some delta neighborhood in some delta neighborhood of z naught are in S. Or in other words we say that is there exist some neighborhood of z naught, which is totally contained which is totally contained inside S. So this is our S, a point z naught is set to be an interior point of the set if we are able to get one neighborhood around a point z naught with a positive radius a delta. Howsoever, small this delta may be but, they exits of some delta greater than 0, at least one delta we can find such that all the points inside this neighborhood all the point of S that is the, this neighborhood this neighborhood totally lies inside it then we point z naught is a interior point and collection of all such interior points of a point of z.

What is the open set of z is, a set is set to be open a set is set to be an open set if every point of it every point of it is an interior point, every point of it is an interior point. So, obviously the exterior point be defined as the set of those points the set of those point which are not the interior point or you can say a point z naught is an exterior point is an exterior point of a of the set S if all the points all the points in some neighborhood of z not, some delta neighborhood of z naught are outside of S. So, suppose this is a point here and we are able to get one neighborhood around the point this which does not contain any point of S, then we say this point is an exterior point of S.

So, let boundary point, a point z is a boundary point is a boundary point of S if every delta neighborhood of z contains at least one point of s and as well as and at least one point of its compliment complements that will not in S. So boundary point is a point, here this point will be a boundary point because if I draw any neighborhood along this point then, it will contain both the point of S.

As well as point outside of s that is at least one point is available here this is inside S and another point which is outside S. So such a point is said to be a boundary point. Now, closed set is, a set is closed a set is said to be a closed set if every boundary point of S point of S belongs to it which if contest means if it contains all its boundary points that all the complements of the open sides we also called it as closed side a set is said to be a closed, set when its boundary point all the boundary points are S bounded set.

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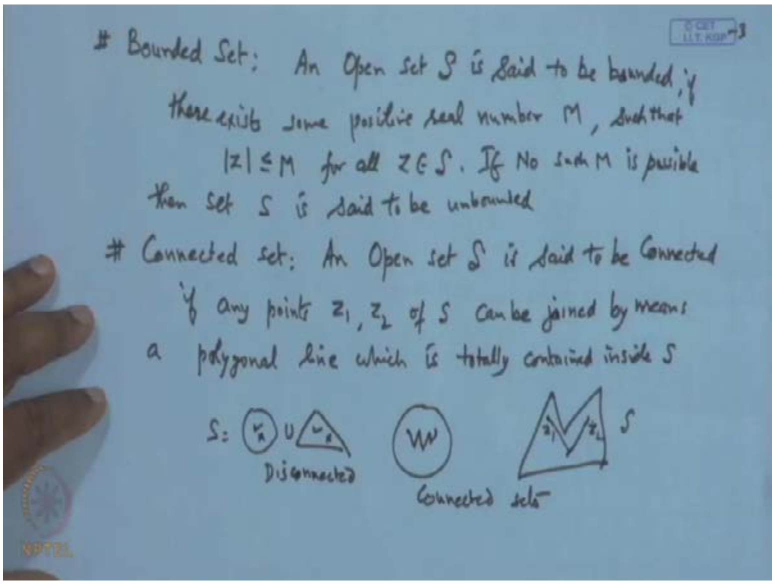

An open set s is said to be a set is said to be bounded bounded if their exists some positive real number m, real number m such that all the point of sets is less than equal to m in a absolute value mod of z is less than or equal to m for all z belongs to S. If we are unable to get such m then the z is said to be unbounded. If no such m is possible then, set S is said to be unbounded. This, we are unable to get such m for which this condition holds for all z then such a set is said to be a unbounded set. Then connected set; an open set s is said to be connected is said to be connected if any two points z 1 and z 2 any two points z one and z two of s can be joined can be joined by means of a polygonal line which which which is totally contained inside S, then the meaning is this.

Suppose we have a this set this is our set S we said this set is connected, because if we pick up two points z 1 and z 2 of a any arbitrary two points, then if I join this z 1 and z 2 by the straight line directly then, entire line does not belongs to it, because there is a gap, there is a portion which lies outside of it. However, one can drop the polygonal line like this. A polygonal curve means we are joined from z 1 to here and then this. So entire polygonal line lies totally inside this then, such a set is said to be a connected set.

If we take this say suppose I consider circle which are as a connected set, because take any arbitrary two point; one can always find the zigzag position, which is totally (( )) or may be a straight line also one can go for it, but there may be a set which are not an connected set. For example, suppose I take collection of this set; set of all point is which belongs to this as well as this in union of this, now this set S is the union of this two suppose then this set will not be a connected set because as soon as pick up any arbitrary point here or any point here; you are not able to get any polygonal curve which can join, connect this two point without leaving the set S. So such a set is disconnected set. These are the connected set. So this we need domain, we define now.

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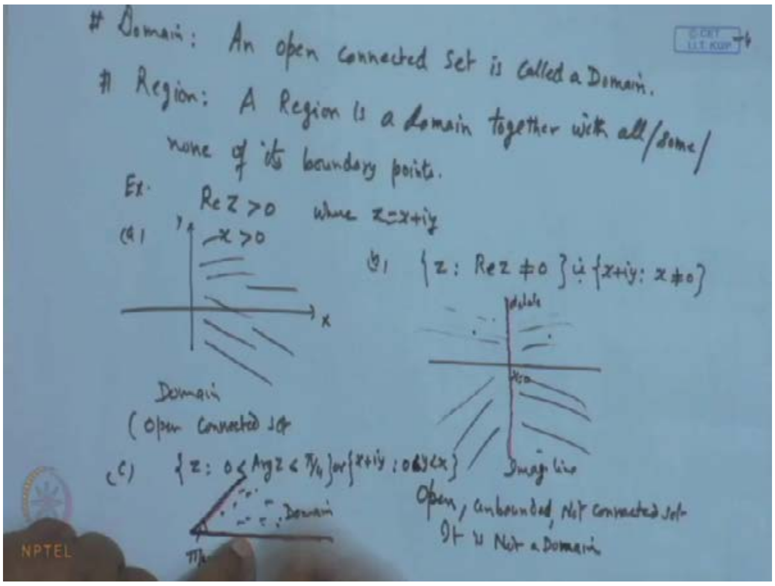

Domain, an open connected set an open connected set is called a domain. So domain is not an simply a set, but it should be an open connected set that is a collection of the points is said to be an domain, when it is an open set that every point is a interior point and connected means, if any two arbitrary point, we picked up from the set then one can able to draw a polygonal curve joining this two point which is totally concerned with itself. And region, a region is a domain together with all or may be some or may be none of its boundary point. So, when the domain is combined with elements of the boundary either all or may be some or made it none then this collection we call it as a region.

Now for example, if we take this suppose I say real part of z is greater than 0; then real part of z is greater than 0 means, x basically real part if z is x plus iy then real part of z is x, x is positive. So, this is our complex plane; this is x axis this one is y axis, now we want x to be greater than 0. So the all the points which are following here this will be that collection of this point will get thus z real part of z is greater than 0. Now, obviously this set s an open set because the boundary is not concluded, the second one is also connected one can join any two point so it is a Domain. Example of that it is an open connected set. However if we take suppose another example the set of those point where the real part of z is not equal to 0 then, in this case real part of z is not equal to 0. That is the x should be not equal to 0. That is the set of all point x plus i y where the x should be different from zero.

So, that is only possibility when you remove completely the imaginary line. If I removed this mark, this will be deleted delete this point imaginary line that is Bi-axis, if I delete completely the bi-axis then set of all points which are all here as well as which are here, except this line except this line this is deleted. So, this forms a set. Is this connected set? No it is not connected because if you picked up here or if you picked up one point here and one point here then, in order to connect this two points we have to cross this line but, this line is not an permitted because that x should not be 0 because at this point x become 0 so this set is not connected then will it be a region? Yes it will be a region because it will be a region is a domain.

So, it is not a domain or it is not an open connected set. It is the connection of the point set so, it is an open unbounded it is say a region open it is an open state unbounded not connected set that is all. And since it not an it not a domain it is not a domain while this one is a is a open connected set. Now, if we take this another example said c suppose I take the set of those points z, we are zero less than argument of z less than pi by 4 what happens? This is the argument of z is zero means this x axis argument of z is pi by 4 so this is angle pi by 4.

We are taking those point and which are line between zero and pi by four so basically this is the same as this set of all x plus i y we are argument is ten inverse by over x so 10 inverse 0 is 0. So we can get zero less than equal to less than pi less than x. This is the set. So this will be u by equal to z is this line because pi by four pi by four this is the pi by 4 and by is equal to z this line. So, we are getting only this points by this line, this two lines all not added it this line is dot this line is out. So, this set is a connected set; it is a open set so it will be a domain. That will be…

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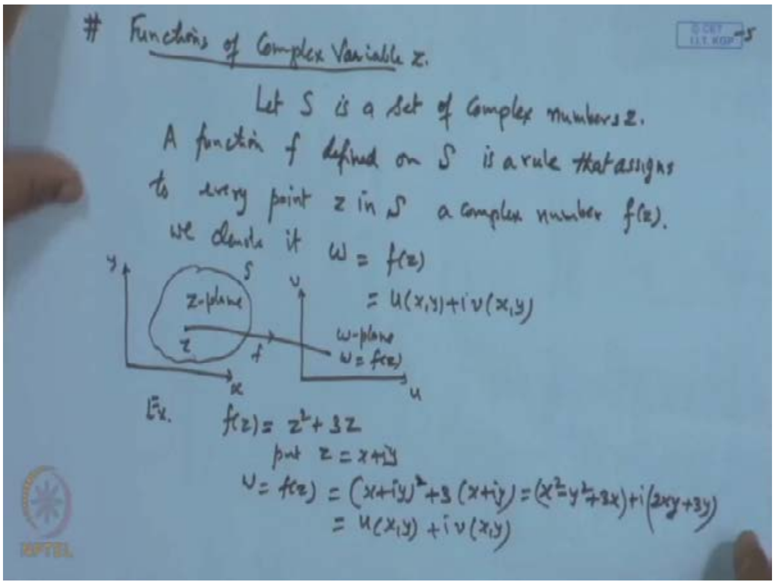

Now, let us come to the functions of a complex variable function of a complex variable z. So let S is a set of complex numbers z s is a set of all complex numbers z. A function f defined on s is a rule is a rule that assigns that assigned to every point z in S of complex number f of z; we denote this is denote it w equal to f z and z is a complex number, z is a x plus i y, then this mapping will also be complex number and we say it is u plus i v u x y plus i times v x y. So, if we have this z plane, here we have a w plane, this is x axis, this is y axis; this is real axis this is imaginary axis here u axis, this is v axis then, take any point here says small z and under this function f is assigned f is a rule which stand the point z of capital z, this is our capital S, a set of point on which the function is well defined. Then find the value of z under f.

Then this f will give a point f of z in the w plane and this will be a complex quantity. For example, suppose I take f z is equal to say z square plus three, z this is our problem. Then z is x plus i y so if I substitute z is equal to x plus i y and just open it then what happens? W which is f z this is equal to x plus i y whole square plus 3 x plus i y. So separate out the real and imaginary parts. So we are getting from here is that x square then minus ,minus y square plus three x is a norm this will be u plus i v i v be equal to x square minus x plus 2 x y 2 x y i times then plus three y. So, basically this comes out to be u x y and plus v as a function of x y. So, this gives you a function. The domain of the definition of the function f is the same as we defined in case of real that the set of those points we are the function is well defined a set to be the domain of definitions. So here, will be…

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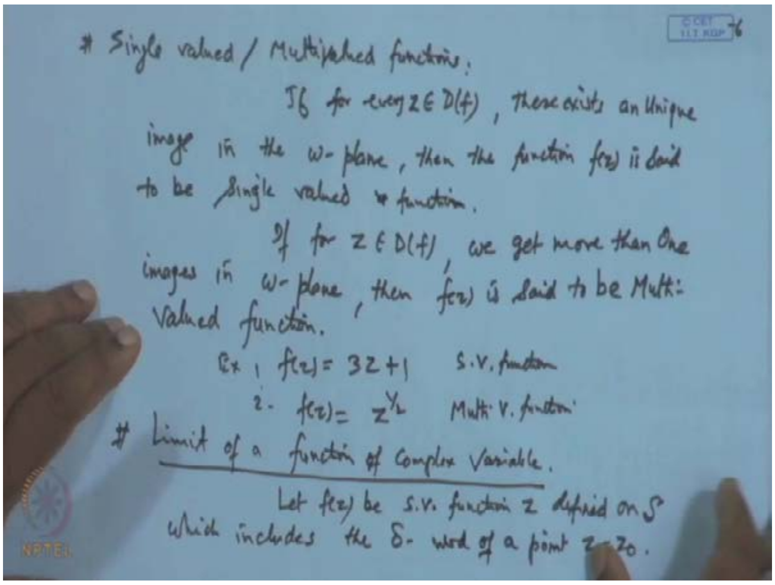

Now in case of the real we are looking for this single variable case single valued function only, but in case of complex also we get multi valued function multi-valued function. So, we divide in the single valued and multi-valued function so what are the single valued function? Multi-valued functions; if for every if for every z belonging to domain of f, d f i am writing domain of f or S we are the domain, there exists an unique image in the w plane in the w plane then the function f z f z is called is said to be is said to be single valued function function. Otherwise and if for z belongs to the domain f, we get more than one images. One images in the w plane then, we then the function f z is said to be is said to be multi valued function multi valued function.

For example, f z is equal to say 3 z plus 1, this is equal to a single valued function corresponding to each z, we get only one function however if we take the function f z, f z raised to the power half then for each z we have two images. So, it is a multi valued function multi valued function. So, this is will be, example we will see later on. those Let me finish first then, we introduce the concept of limits. The limit of a function of complex variables variables so let say how to define the limit? Let f z be a single valued function single valued function of z, of a complex point z defined on the set s which includes which includes the delta neighborhood of a point of a point say z is equal to z naught of a point z is equal to z naught.

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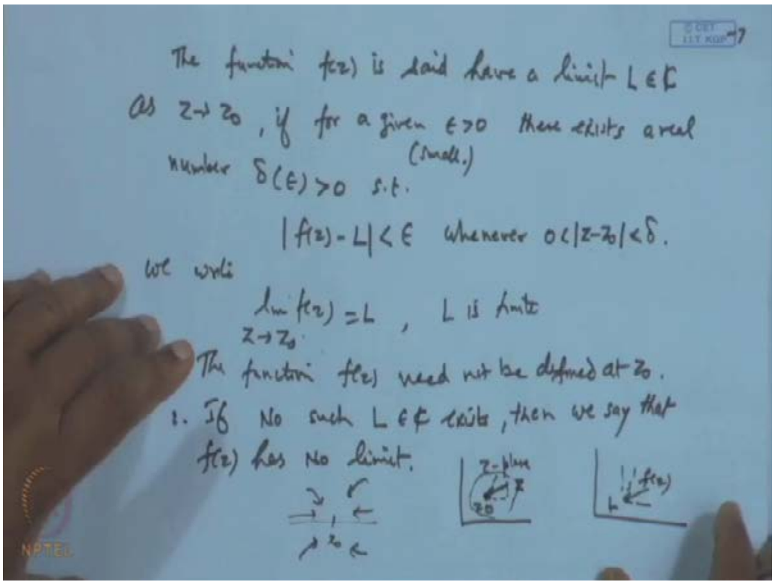

Now we say, then the function the function f z is said to have a limit limit say l which is also point in a complex number a complex number l as z approaches to z naught z approaches to z naught if for for a given positive number. If epsilon greater than zero howsoever small that may be there exists, if for a given variable there exist for a given epsilon greater than zero there exists a real number delta depends on epsilon greater than zero such that modulus of f z minus l less than epsilon whenever whenever less than mod z minus z naught less than delta where l is finite limit. finite where And we write this and we write it this as limit of the function f z when z is z naught where l is finite.

Now here, if you look we are taking the deleted number of z naught so here in case of the limit when we say the function f z is a limit f z limits z tells to z naught exist then is not necessary the function has to be defined z naught. So note it, in the function f z need not be defined at the point z naught is still you can say, the function f z has a limit and if this limit does not exist, if no such l is possible then we say the function does not any limit. If no such l if no such l which you see exists, then we say that the function f z, then the function f z has no limit. So, basically this concept of the limit depends on the part of the where the point is going.

When we say this is our z is placed z plane here is w plane, so suppose I take z naught here and I take any arbitrary point z in the neighborhood of z naught so that the z may approach to z naught in any ways; either this way or this way or may be some it is because it is in the neighborhood. Then corresponding images f of z if it goes to l; in whatever the path you choose whatever the part you choose if all the part is the function f z tells to l then only we say limit of the function f z, f z approaches to z naught exists. If they exist at least one point where the limit differs from another path the limit will not exist. So here this is something more complicated, then the limit of a function of a single real variable because in case of real, we picked up the limit only either from the left hand side or from the right hand side. And then we say, the limit of f z may extension to x not it exists, but in this case all the shorter direction the limit approach the path may be possible, you get choose any path which can approach the point z naught .

From any direction, in fact inside the circular disk centered at z naught within suitable idea delta, so here where looking the limit, we should be careful that the limit from all the paths should exist. But this there are infinite many part so how will you test the all part and find the, calculate? So, in order to avoid this thing we use the epsilon delta definition to conform whether the limit exist or not. If this condition is satisfied, then it will give the guarantee, then whatever the part is choose it the limit will always exist.

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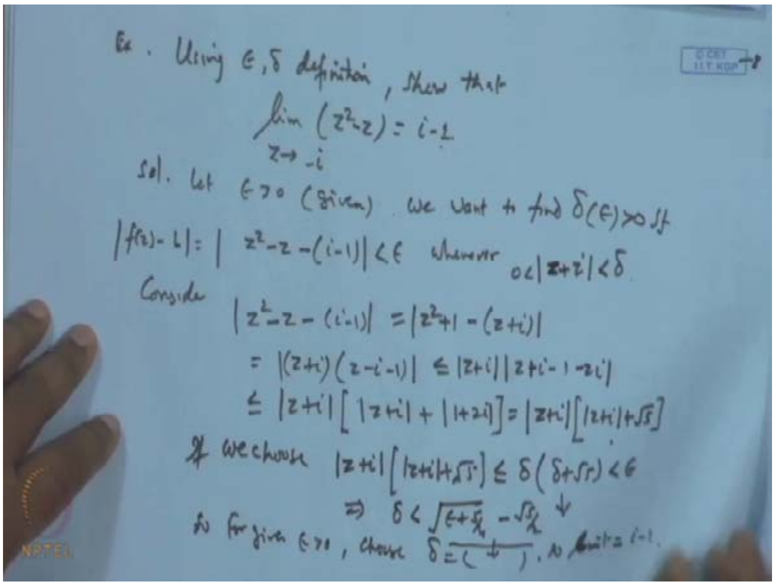

Let us see the example for example, let us say the function using epsilon delta definition show that the limit of this function z square minus z z times to minus i is i minus one. So let us apply the epsilon delta what the epsilon delta definition is said that for a given epsilon; epsilon is known, we have to identify a delta which satisfies the condition so let epsilon is greater than 0 is given. This is given. We want to find delta we should depend on Epsilon positive result such that this condition must work f z minus f z minus l that is f z is what? z square minus z minus i minus one this there should less than the epsilon provided whenever the z is zero less than z minus z naught, z minus z naught means z this is the point tends to I, they not it minus i so z minus z naught means this z plus i is less than delta.

So we have to identify delta how we start with this? Consider does this part mode z square minus z minus i minus one consider this. Now this will be equal to mod of z square plus 1 and minus z plus i. Now, this is same as z plus i we have z square plus one can be z plus i z minus i minus one is a mod this mode. So this is less than equal to mod of z plus of i in to the mod of z minus i minus one, but mode 1 minus i minus 1, because we want i again so plus i if I choose then it becomes minus i minus two i. That will satisfy the condition. Now this mod z plus i is again less than equal to z plus i and this will be equal to less than equal to mode z plus i and plus mode 1 plus 2 i just apply the mode z 1. Now, this is equal to mode z plus i and then with the bracket mode z plus i and what is that it is the square root of this so under root 5.

Now, suppose for a given epsilon we are able to get delta then we want to mod z plus should be less than delta. So, here we shall write that this portion this part is nothing but the right hand side of this right hand side of this part this. So, it is it less than or equal to, if we choose if we choose this z plus delta i z plus i mode mod z plus I plus root 5, which is basically less than or equal to delta in o delta plus root five. If I take this to less than epsilon, then it is satisfied that for this less than delta we are getting the differential less than the epsilon. So what is this?

If I solve this equation and then when get, you will find the delta we come out to be less than under root epsilon plus 5 by 4 minus under root 5 by 2. Why? Because we just got it equation z delta square plus root five delta is less than epsilon make the perfect square by adding this term epsilon five by four square and then subtracted we get this one. So delta it means, for a given epsilon greater than zero you have got the delta which is less than this value. So, if I choose this as a delta it depends on epsilon then whole thing is computed. So, this proves that so, for a given epsilon greater than we choose delta to be this number less than under root of this 1, this number then let so continuous a limit exists, limit is i minus 1. So this is the way. Please check it. This is the way we can do it. Limit i minus 1 is this, we check it again.

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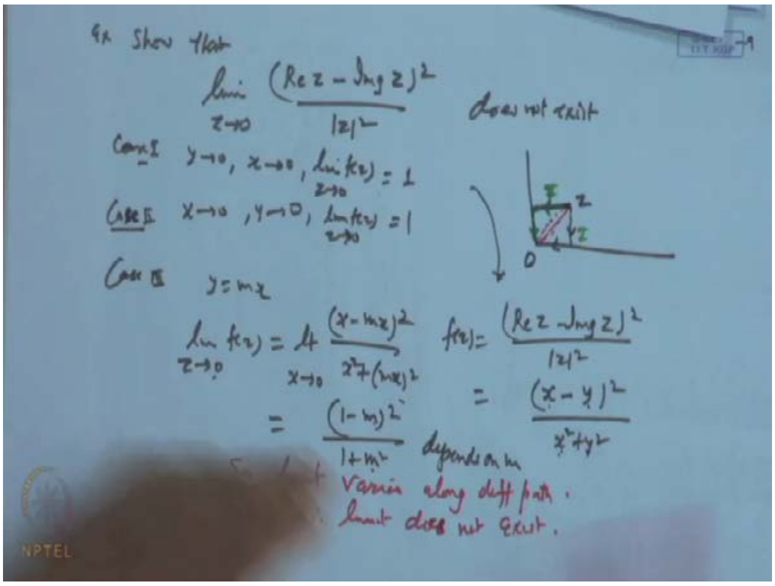

But if it is asked whether this limit exists or not or show that show that limit of this real part of z minus imaginary part of z whole square divide by mod z square z tends to zero so that, limit does not exist. Now here, we need not go for a external delta, we have to simply identify the part such that 2 pass along to 2 different part limit differs so what are the paths?

Now, we wanted limit when z approaches 0 from 0. So I can choose any path. Suppose I take point z, I take the first one this path. So, case one first part is when it goes down that is y tends to 0, x tends to 0. Later second case I can choose the path like this. This is our second case this second case this is first path, this is second path. Second path is say x is tending to 0 y tends to 0 and the third path may be any line, any curve it is a all line y is equal to m x y is equal to m x.

So I have taken, they may be many another path also but, it this three path if I choose suppose and in all three of between any two path, the limit does not an differs then we say it is a limit does not exist. So let us take this one; what is the function? Function f z is the function f z is real part object minus imaginary part object whole square divide by mod z square. That is the function is real part object is x imaginary part object is y by x square plus y square this is our function. So, we are interested in finding our limit so in the first case what is the limit of this function? Limit when z approaches to 0. So, first y tending to zero so when y is standing to zero then what will get x square by x square the limit comes out to one. You see in the second case when x 0, this is 0 this is zero and again the limit comes out to be 1. So, along these two paths the limit is coming to be one but, whatever the third path third part the limit of the function is z when z expose to 0 this is the same as the limit substitute y is equal to m x.

So, when z tends to 0 means if I take x tends to 0; y tends to 0. So here, substitute let x z tends to 0. Putting y is equal to m x so what we get? x minus m x whole square divided by x square m x square and from here we get x square is cancelling, we get one minus m whole square over 1 plus m square. So, limit of this system coming to be this but, it depends on n depends on in. So, it means if I check the slope of this line.

If I checks the slope of this line say this line, then the limit along this line and this line differs depends on m therefore, limit varies along various path. So, the limit does not exist, limit varies along different path. Therefore, limit does not exist does not exist so that is what.

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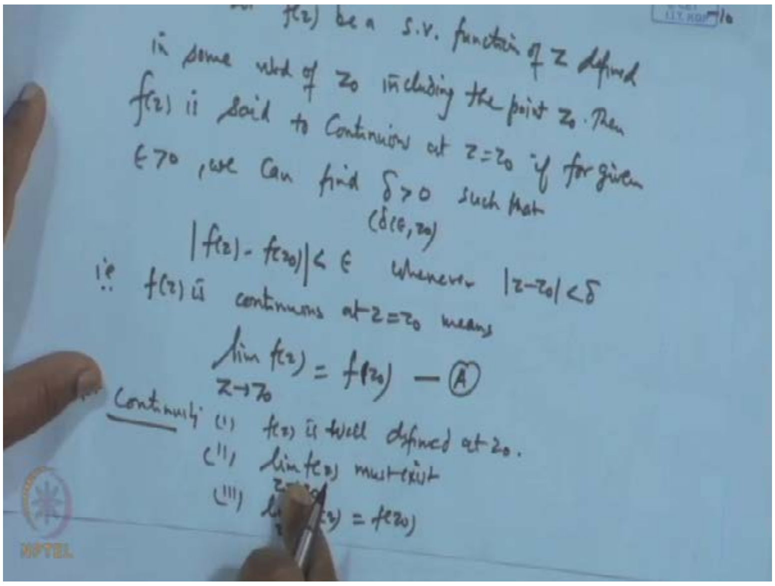

This a now let us come to the continuity a definition of the continuity. continuity A function f z; let f z be a single valued function single valued function of z defined in some neighborhood of z naught neighborhood z naught including the point z not. Here in case of continuity is must the function must defines at the point z naught then function f z is said to be continuous at a point z is equal to z naught if for given epsilon greater than zero howsoever small may be, we can find a delta, real number delta greater than zero with depends on epsilon as well as the point depends on epsilon as well as the point such that the mod of f z minus f z naught less than epsilon whenever mod of z minus z naught less than delta. That is the meaning is this; when f is continuous f z is continuous at a point z is equal to z naught means limit of this function f z, when z tells to zero exist and equal to the value of the function f z naught.

So, in case of the continuity; three point is very important first is the function must be defined at the point z naught, second is the limit of the function f z must exist and third is both to be identical. Now, if any one of the condition fills then the function will not be continuous at the point z not.

So what the possibility is for the continuity this one. So anyone else so we get for continuity three things are essential; continuity number one function f z is to define is well defined at the point z naught, second is the limit of the function f z when z approaches to z naught must exist and third is both should be equal part third is that both should be equal limit third is the condition is satisfied the third is limit exist must be equal to f z naught when z tells to 0, so third condition. So if suppose first two set satisfied but, the third does not satisfy, then the con point is said to be a removal discontinuity point.

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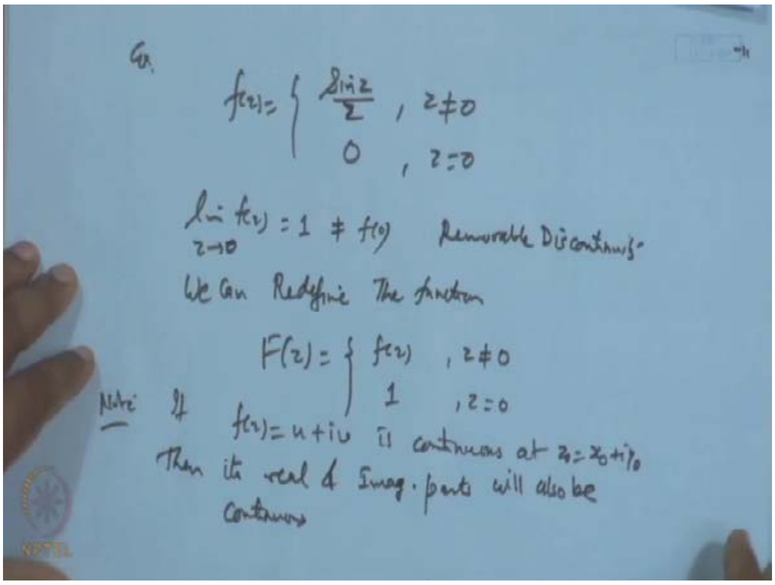

For example, if we take the function sign z y z and suppose I take the function f z, sin z by z when z is not equal to 0 and 0, if z is 0, then the limit of this function f z, when z tends to zero. Now if I take, limit of this function when sin z by z when z tells to 0 what will the limit? This is (( )) and thus one is it not, but it differs from the value of the function at the point 0, so the function has a discontinuity at 0. But this is known as the removal discontinuity. Why removable? Because we we can redefine the function. Function f kept at f z equal to this mode f z when z is not equal to zero and equal to 1 if z is equal to 0, then this function becomes continuous and that is all. Now, another result also note; if the function f is continuous is continuous at a point z naught which is x not i y not then, its real and imaginary parts its real and imaginary parts will also be continuous. This also will be continuous and that is all. Thank you very much. Thanks.