Advanced Engineering Mathematics
Prof. Pratima Panigrahi
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture No # 10
Spectrum of Special Matrices
Positive / Negative Definite Matrices

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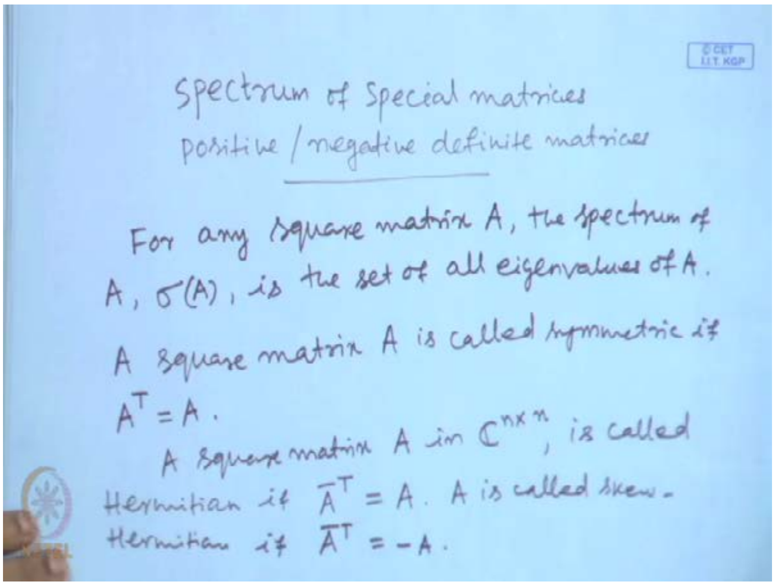

So the final lecture in this series is on this spectrum of special matrices spectrum of special matrices and positive definite and negative definite with matrices, are this is positive or negative definite matrices. So here, we shall discuss about this spectrum of some special matrices; spectrum of a matrices means that is a set of all Eigen values of the matrix. So, for any square matrix for any square matrix A the spectrum of A usually it is denoted by sigma A, is the set of all Eigen values of A.

So, here we shall discuss about the nature of spectrum of some special matrices like Hermitian matrices, unitary matrices etcetera. So, first recall that a symmetry matrix this square matrix, a square matrix A is call symmetric if a transpose is equal to a and a square matrix with complex entries or in other words a square matrix A in c n cross n that is a is a matrix of size n by n with complex entries is called Hermitian if a conjugate transpose is equal to A that we have already seen before remaining on this conjugate transpose that we take conjugate of complex conjugate of entries in a and then we take it transpose.

Further this A is called skew Hermitian a is called skew hermitian if this A conjugate transpose is equal to minus A. is Also we are again other kind of matrices that will discuss, but before that let us see on one example that real matrix is Hermitian if and only if it is symmetric but the this is not true for complex matrices.

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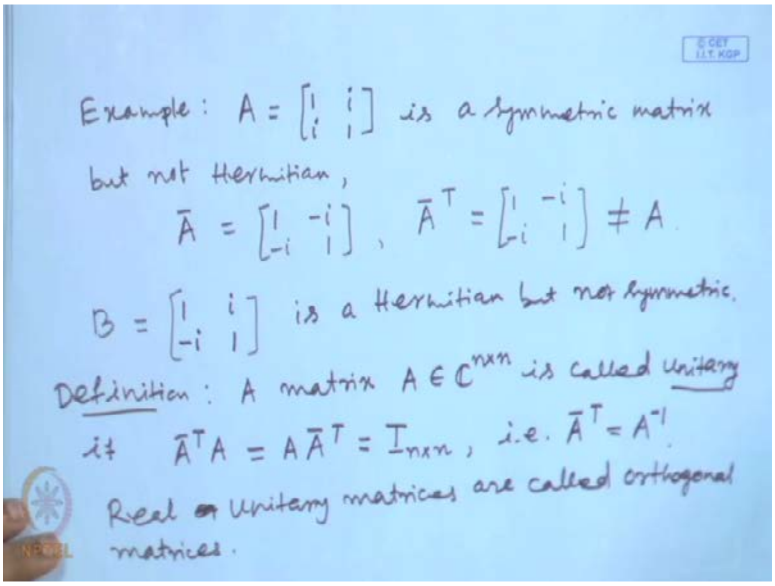

So let us see one example that this matrix A with entries is that 1 i i 1. This is symmetric matrix this is a symmetric matrix, but not Hermitian that one can check easily, because this A conjugate if we take, then we get 1 minus i minus 1 1 and A conjugate transpose that is also we get 1 minus i minus i 1. So, this is not equal to matrix A, therefore it is not a Hermitian matrix. On the other hand, let us take this matrix b with entries that 1 i minus i 1. So, this is a Hermitian matrix this is a hermitian Matrix. Hermitian, but not symmetric so in in complex case matrices may be symmetric but they there not Hermitian and matrices may be Hermitian some matrices and then not to be symmetric. So, we will have also different kind of special matrices that that is called unitary matrix. So a matrix A in c n cross n that means n by n square matrix with complex entries is called unitary; unitary matrix if a conjugate transpose times a is equal to a times this A conjugate transpose is equal to this identity matrix n by n identity matrix or that is same as that is same s a conjugate transpose is equal to a inverse.

Further, these real unitary matrices there called orthogonal matrices. real unitary matrices unitary matrices are called orthogonal matrices. So, here we discuss the property of Eigen values or spectrum of Hermitian matrices and unitary matrices. So, in this case matrices we consider this since we are dealing with Hermitian matrices they are complex matrices with complex entries and unitary matrices are also with complex matrices. So, we consider the inner product the standard inner product this c n.

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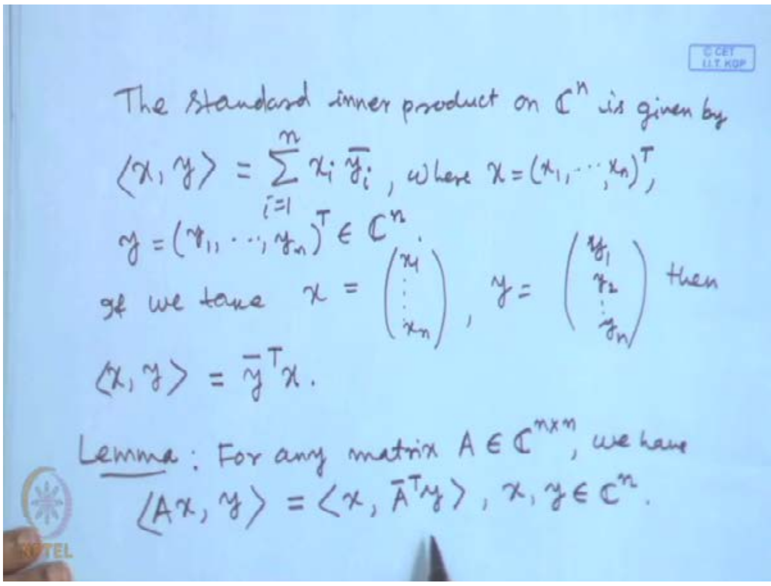

So re recall that standard inner product on c n the standard inner product on c n is given by this x y inner product of x y is equal to summation i equal to 1 to n x i y i bar, where this x is consist of this x 1 x 2 to x n and this y its components are y 1 y 2 up to y n are vectors n c n. So this standard inner product can also be expressed in this way. So here we write this x and y we can take their less transpose or in other words let us take, so if we take x as this vector, x one column vector basically x 1, x 2 to x n and y as this column vector y 1, y 2 to y n then this inner product of x and y can be represented x y conjugate transpose times this vector x. So, we shall use this inner product in this form. So, let us have n result for matrices, so that we shall use frequently.

So consider this for any matrix A in c n cross n we have a x inner product of A x with y that is that will be equal to inner product of x and conjugate transpose of a multiplied with y. So, this is true for every x and y in this c n. So, this is quite useful of course, not difficult to check this easily one can verify like this.

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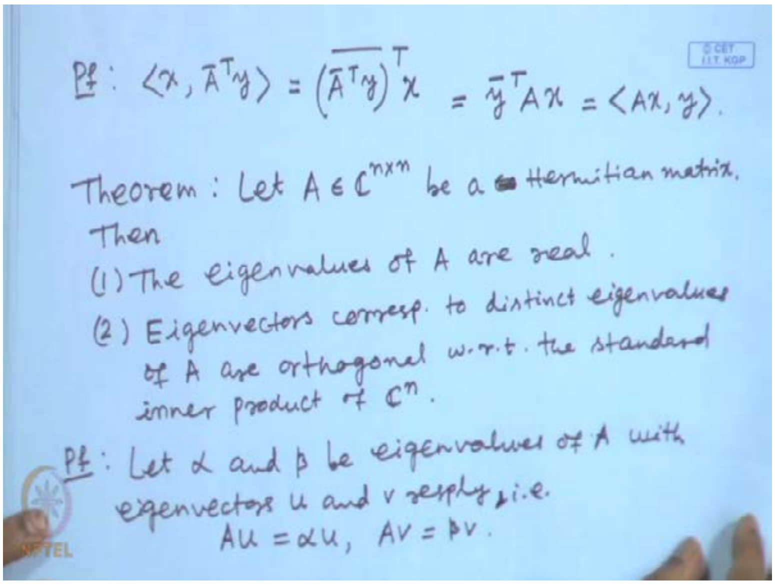

Let us start with this inner product x and a conjugate transpose times y so from the definition of standard inner product in c n we can write this a conjugate transpose y this whole conjugate and transpose this multiplied with x. So, if we consider conjugate transpose of this quantity, then we get conjugate transpose of y and this conjugate transpose of conjugate transpose of A, this will be a again. So, we get this A x and this exactly the inner product of A x with y. So, this is quality useful. That means it says that if we consider this inner product of A x y and mean a is multiplied with x this matrix A is multiplied with x and inner product of A x y this will be equal to that we like to take the matrix a to the second component then, we have to consider its complex conjugate transpose, conjugate transpose of this matrix A.

So using this property we shall studied about the nature of spectrum of Hermitian matrices, and that is given in this theorem. So, it says that we consider a Hermitian matrix A, belongs to this c n cross n c n by n square matrix with complex entries be a Hermitian matrix be a hermitian matrices, then we have the following results; first one is the Eigen values of a are real. the eigen the eigen values of values of a are real (( )) with the matrix a is a complex matrix all its Eigen values are real, so the Eigen vectors corresponding to distinct values of A are orthogonal with respect to the standard inner product in c n. Earlier we have seen that Eigen value corresponding to distinct Eigen vectors corresponding to distinct values are linearly independent but here Eigen vectors corresponding to distinct values are Hermitian matrices are orthogonal.

So this second is this Eigen vectors corresponding to distinct Eigen vectors, distinct Eigen values of A are orthogonal with respect to the standard inner product in c n. inner product of c n So, to prove this, we consider Eigen values. Let alpha and beta where Eigen values of A with Eigen vectors u and v respectively.

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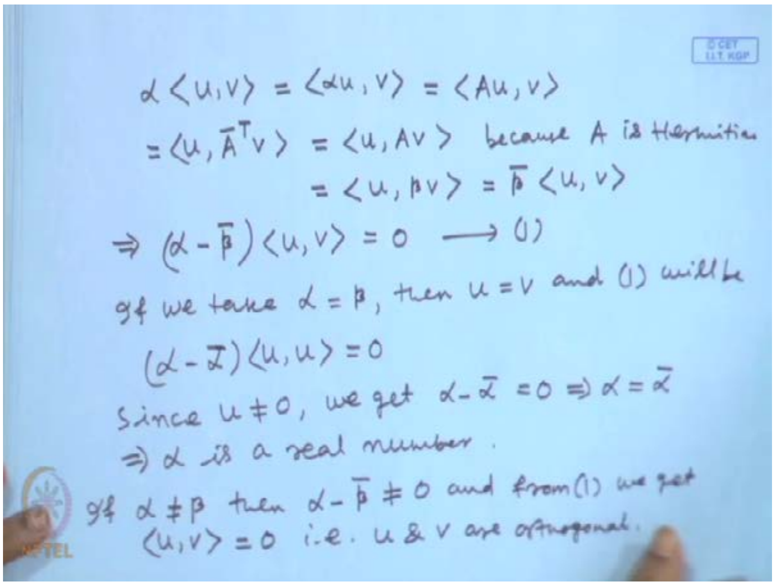

So, the next we consider here this inner product of course, this means that that is this a times u is equal to alpha u and this a times v is equal to beta v. Next we consider here this inner product say alpha times inner product of u and v that is from this property of inner product we can write inner product of alpha u and v and this alpha u is equal to a u and this is v. So, this can be written as from the lemma that inner product of u a conjugate transpose times v and this equal to inner product of u and inner product of this a v because a is Hermitian.

This we get because A is Hermitian and then this inner product is equal to u and a v that is equal to beta v. So, if e take this scalar out then we get its complex conjugate beta bar times inner product of u and v. So, this whole thing implies that we get alpha minus beta bar times inner product of u and v. That is equal to 0. So, let us say this equation one. So, if we check alpha equal to beta then, u equal to v and one will be this alpha minus alpha bar multiplied with inner product of u with itself that is equal to 0.

But u is a non zero vector. So, since u is not equal to 0; we get this alpha minus alpha bar is equal to 0 and this implies that alpha is equal to alpha bar. So, this says that alpha is a real number. So, from here we get that, all Eigen values of a Hermitian matrix are real numbers. So, next if alpha and beta are distinct they are different Eigen values then, this alpha minus beta bar is not equal to 0 and from one will get this inner product of u v we get to zero that is u and v are orthogonal. u and v are orthogonal So, it says that the Eigen vectors corresponding to the distinct Eigen values are orthogonal. so And this how we prove this result. Next, since we are discussing about the nature of Eigen values of Hermitian matrices here, we can also see the nature of Eigen values of skew Hermitian matrices.

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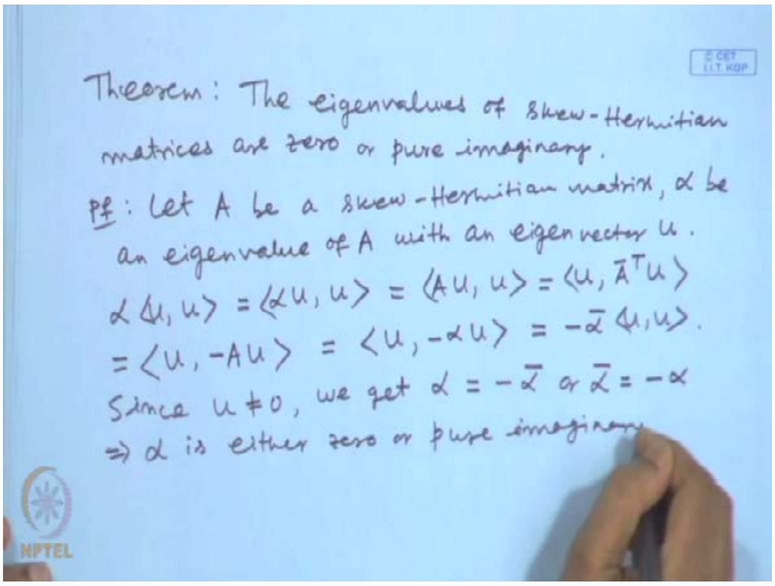

So, the theorem says the nature of Eigen values of skew hermitian matrices so the eigen values of skew Hermitian matrices are 0 or pure imaginary skew hermitian matrices are zero or pure imaginary purely imaginary numbers. So, this can be checked in the following way. This is analogous to the proof of the previous theorem. So, let us consider A be a skew Hermitian matrix skew hermitian matrices and this alpha be an Eigen value of a alpha be an Eigen value of a with an Eigen vector u with an Eigen vector u.

Then we get in this similar way that alpha u, u is equal to this alpha u, u and that is equal to u, u is inner product and from that lemma we get inner product of u A conjugate transpose times u and since a is skew Hermitian. That we get minus a times u and this we get u and this A u is equal to alpha u u minus alpha times u and that is equal to, we get this minus alpha its conjugate u, u. So, from here we get since this u is not equal to 0, we get this alpha is equal to minus alpha bar or complex conjugate of alpha that is alpha bar is equal to minus alpha. That means another we are taking complex conjugate, we are getting the negative of that complex number that is minus alpha.

So, this says that this says that alpha is either zero or pure imaginary. So, this how we prove that the Eigen values of skew Hermitian matrices are 0 or pure imaginary numbers. So, next we shall discuss about the nature of Eigen values of unitary matrices.

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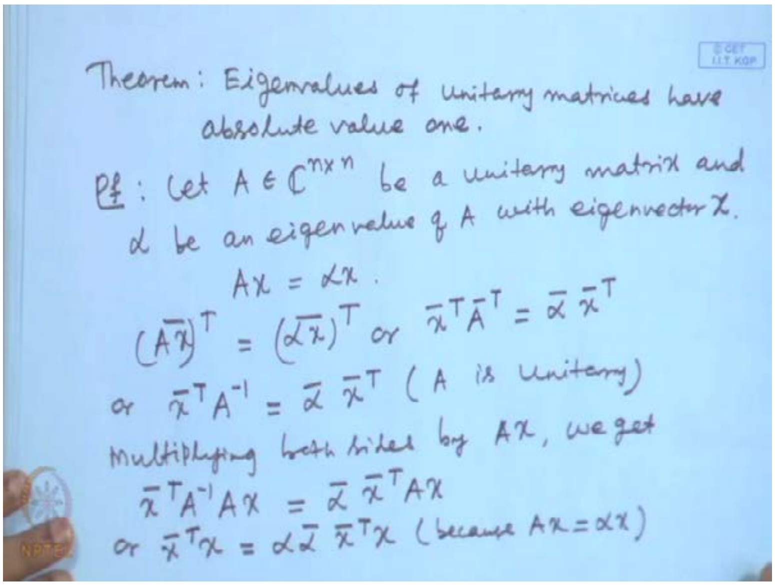

So, that that again we write as a result that it says that Eigen values of Eigen values of unitary matrices unitary matrices have unitary matrix matrices have absolute value one absolute value one. That is if alpha is the Eigen value of a unitary matrices then absolute value of alpha is equal to 1. So, let us consider let A belong to c n cross n be unitary, be a unitary matrix and alpha be an Eigen value of a with Eigen vector x say. So now, we have this a x is equal to alpha x taking conjugate transpose on both sides that A x its conjugate and then transpose similarly, on the right hand side conjugate transpose so we get that x conjugate transpose times x its conjugate transpose of A. This is equal to this alpha bar and conjugate transpose of x.

Since alpha is scalar, it transposes itself. So no point of taking transpose of this scalar quantity or we can write this as x conjugate transpose of x and conjugate transpose of a that is equal to A inverse because a is a unitary matrix. So, conjugate of alpha, conjugate transpose of t. So, this is true because A is a unitary. So, the next we shall multiply both sides by A x. So multiply by both sides by A x we get this a conjugate transpose of x a inverse multiplied by A x is equal to alpha bar conjugate transpose of x times A x or this a inverse a that is equal to identity. Therefore, we get this conjugate transpose of x, conjugate transpose of x times x is equal to this A x, the value of equal to alpha x. So, we get this alpha, alpha bar conjugate transpose of x times alpha x. This is because a x is equal to alpha x.

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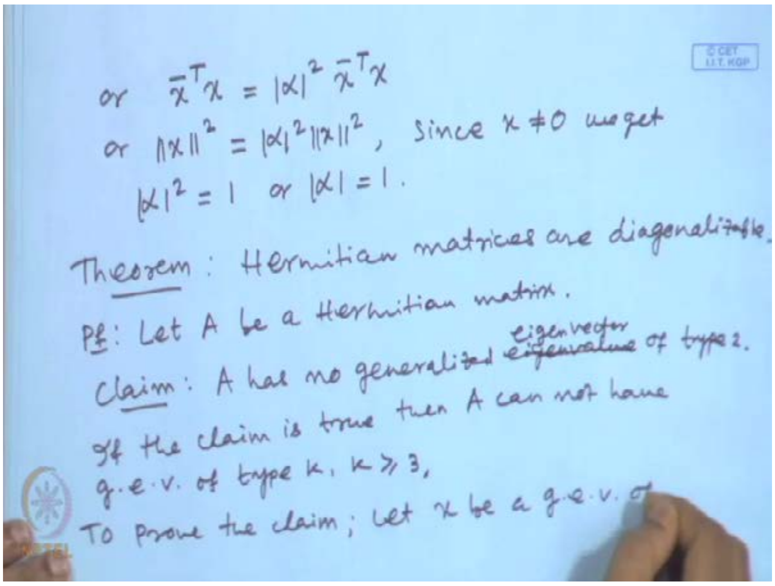

Now, this alpha, alpha bar is equal to mod alpha square or we get this x conjugate transpose of x times x is equal to mod alpha square times conjugate transpose of x times x of this is equal to norm of x square or we get this norm of x square is equal to mod alpha square norm of x square. So, since this x is a non zero vector we get mod alpha square is equal to 1 or mode alpha is equal to 1, and this proves that result that the Eigen values of unitary matrices have absolute value equal to 1.

So, another important result that we can prove for Hermitian matrices is that the Hermitian matrices are diagonalizable. So, this is an important property of Hermitian matrices that Hermitian matrices are diagonalizable diagonalizable. So in this case, we saw that for any Hermitian matrices no generalized Eigen value of type two or more will exist. So let us consider so let A be a Hermitian matrix. So our claim is that a has no generalized Eigen value of type two a has no generalized eigen value of type two. So, if this claim is true, then we cannot have generalized Eigen values of type k, k greater than or equal to three. So, if the claim is true then A cannot have a generalized Eigen value generalized eigen value of type k, k greater than or equal to 3, because that if this matrices a has a generalized Eigen value of k, k greater than or equal to three then the chain.

Generated by the corresponding Eigen vector will contain an Eigen vector. This this will be Eigen vector, this we can write this Eigen vector. So, cannot have generalized Eigen vector of type k for k greater than or equal to 3. So if this has a generalized Eigen vector of type k k greater than or equal to 3, then the chain generated by this generalized Eigen vector will contain an Eigen vector generalized Eigen vector of type two and that will can contradict to claim can have cannot have generalized Eigen vector of type k.

Therefore, so let us prove this claim. Once, so we prove this claim is true, then this matrix a will have only a generalized Eigen vector of type one. So, they are those generalized Eigen vectors of type one are nothing but ordinary Eigen vectors. So, this matrix a will have therefore, n number of Eigen vectors, their distinct Eigen vectors and they are all linearly independent and hence this will be diagonalizable. So, let us prove this claim.

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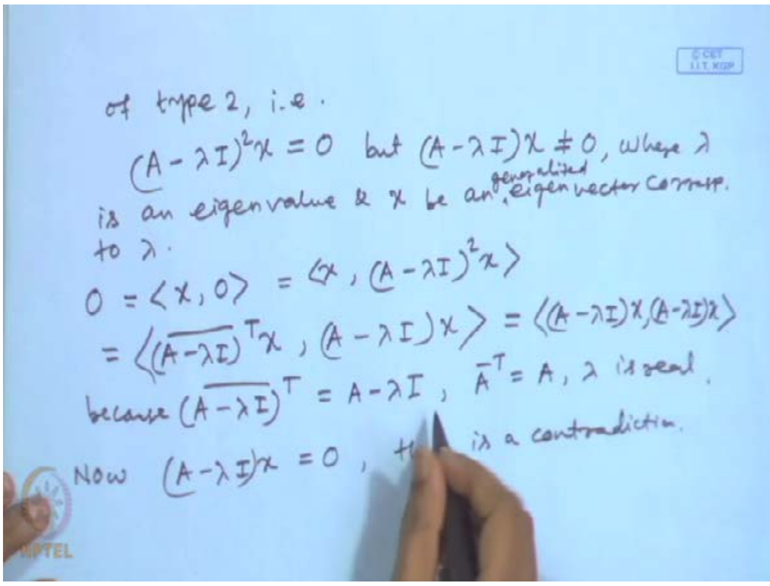

So to prove the claim consider that, to prove the claim let us consider that let x be generalized Eigen vector of A of type 2. So, that is this A minus lambda i whole square x is equal to 0 but A minus lambda I x is not equal to 0, where lambda is an Eigen value and x be an Eigen vector corresponding to an Eigen vector, corresponding to this generalized Eigen vector generalized Eigen vector corresponding to this lambda. So, here we get that this zero is equal to inner product of x and 0 and in place of this 0, we can write a minus lambda I whole square x or from that lemma, we can write this a minus lambda I its conjugate transpose times x A minus lambda i times this inner product and this is same as that A minus lambda I x and this inner product a minus lambda I x, because the conjugate transpose of A minus lambda i is equal to A minus lambda I.

This is true because this A is a Hermitian matrix its conjugate transpose will be a again and lambda is real. So, this is true, because lambda this is scalar, this is real that a conjugate transpose is equal to A lambda is a real and its identity matrix is obviously satisfy that is conjugate transpose, which is again a itself. In other words we get so this minus lambda I x is equal to 0. So, this is a contraction this is a contraction because our assumption was that this A minus lambda I x this is not equal to 0. So, this shows that A cannot have any generalized Eigen value of type two or more; so all generalized Eigen value are type one, which are exactly ordinary Eigen vectors. Therefore, A will have n number of linearly independent Eigen vectors, and hence this A is diagonalizable.

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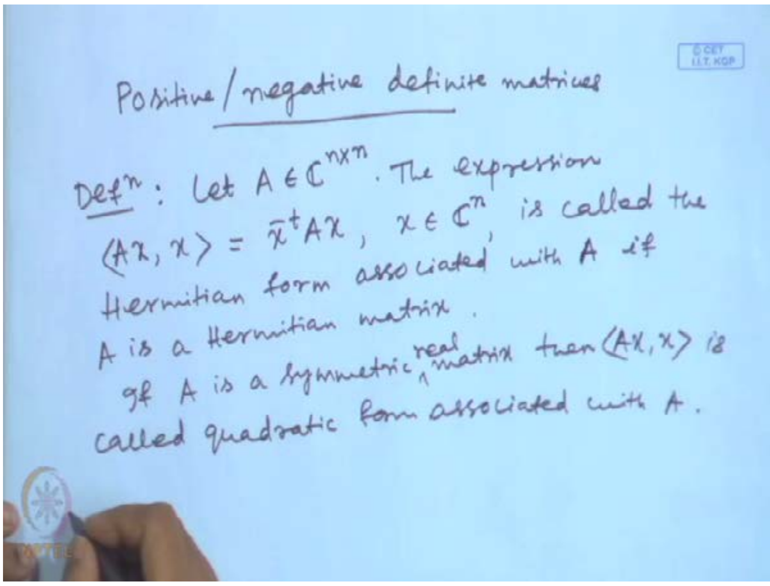

So, next we shall see this positive and negative positive or negative definite matrices; so for this, we shall define one terminal (( )) that is called Hermitian form. So let A belongs to A c n cross n or the matrix of size n with complex entries. Then the expression the expression a x x that is equal to a conjugate transpose a times x for x belong to this complex or this c n is called the Hermitian for associated with the matrix A called the hermitian form associated with a this associated with a this matrix A with hermitian form A, I said that with a if A is a Hermitian matrix and this is called quadratic form; if A is a symmetric matrix if a is is a symmetric matrix then this A x x is called quadratic form. So, this is a symmetric matrix, is a symmetric real matrix in fact, then this is called the quadratic form associated with A.

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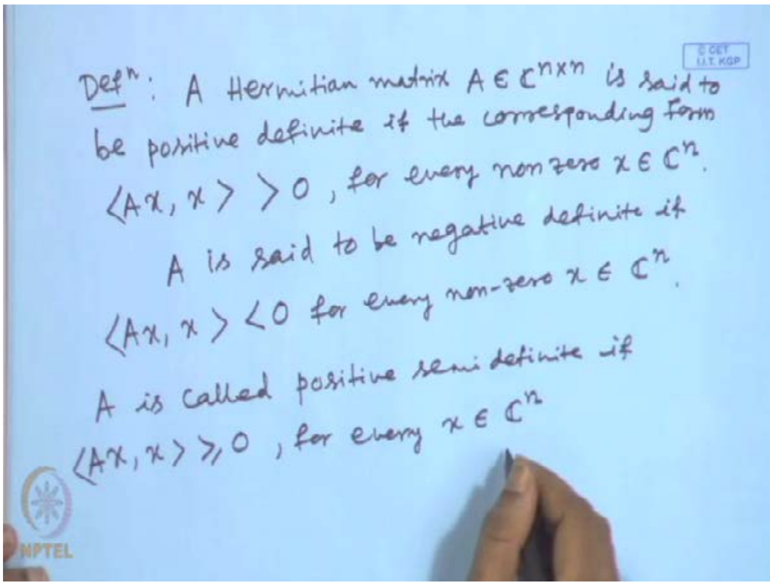

So, next we shall define these positive and negative matrices. So this will define in terms of these forms. So, let us consider a Hermitian matrix. So, a Hermitian matrix hermitian matrix a belongs to this c n cross n is the said to be said to way positive definite def if the corresponding form corresponding form A x x this inner product of A x x is strictly greater than 0 for every non zero x in c n and this is A is said to be negative definite.

A is sais o be negative definite if the corresponding form a x x a x x this is strictly less than 0 for every non zero x n c n. We also define that positive semi definite matrices; so this A is called positive semi definite semi definite if this corresponding form A x x is greater than or equal to 0 for every x in c n. So here, this x may be taken as 0 vector. So similarly, one also define the negative semi definite matrices.

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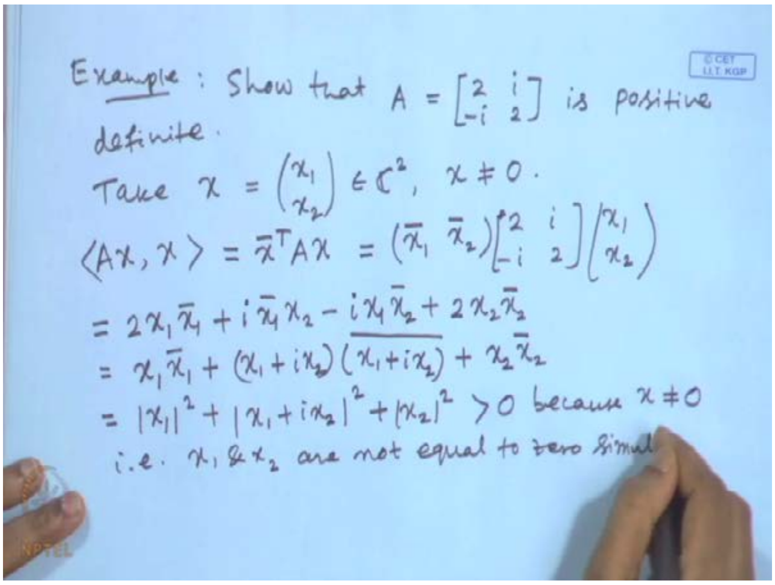

So, let us see one example of this positive definite matrix. This example shows that here we consider a matrix, show that this matrix A with entries 2 i minus i 2 is positive definite. For this we have to verify that if the corresponding form A x x is greater than zero or non zero x. So, let us consider this vector x is x 1 x 2 with of course this is in c 2 and this x is not equal to 0. Now, consider the form A x x and this can be written in, from the definition of the inner product that x conjugate transpose times A x or this is equal to conjugate of x 1 conjugate of x 2 and this matrix A that is 2 i minus i 2 multiplied with x 1 x 2.

So, this on simplification or on multiplication multiplying this we get that this is equal to two times x 1 x 1 bar plus i times x 1 bar x 2 minus i times x 1 x 2 bar plus 2 times x 2 x 2 bar or rearranging this term we can write this x 1 x 1 bar plus x 1 plus i x 2 multiplied with x 1 plus i x 2 its conjugate, one can adjust this term and get like this x 2 x 2 bar and this is exactly the x one norm square plus x 1 plus i x 2 mod square, because they are complex numbers plus x 2 mod square. And this is strictly greater than 0, because x is not equal to 0 this is strictly greater than zero because this x is not equal to zero x that is x 1 and x 2 are not equal to 0, simultaneously not equal to 0 simultaneously so here whatever may be the values of x 1 and x 2 or whatever the x that is of course, non zero we can show that this quantity that x bar transpose A x that is always greater than 0. Therefore, this matrix A is next positive definite matrix. So, similarly, one proves for that negative definite matrices or positive semi definite matrices.

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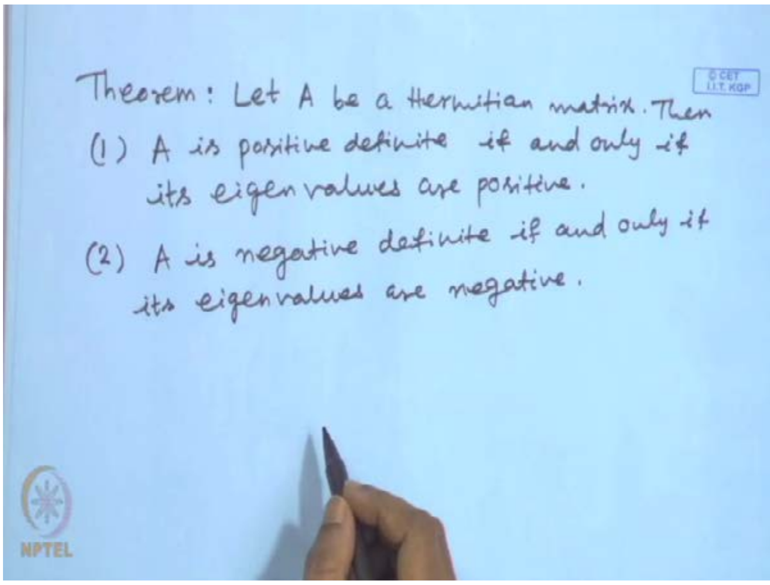

So next, let us state that the nature of next let us state that the nature of Eigen values of positive definite matrices and negative definite matrices. And of course, we shall know this result states in a nature of Eigen values of positive definite matrices and negative definite matrices. So, let A be a Hermitian matrix and matrix then we will have the following; A is positive definite if and only if if and only if its Eigen values are positive its Eigen values are positive. It says that for a positive definite matrix all its Eigen values are positive and for any Hermitian matrix if its Eigen values are positive a positive definite matrix. Similarly, A is this second one is like this; A is negative definite if and only if its Eigen values are negative Eigen values are negative.

So for negative definite matrices all its Eigen values are negative and on the other way that if for a Hermitian matrix all its Eigen values are negative then the matrix will be negative definite matrix. So, one can prove this results are easily and here we skip this proof and that is all for the lecture and we stop here.