Advanced Engineering Mathematics

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Lecture No. # 01

Review Groups, Fields & Matrices

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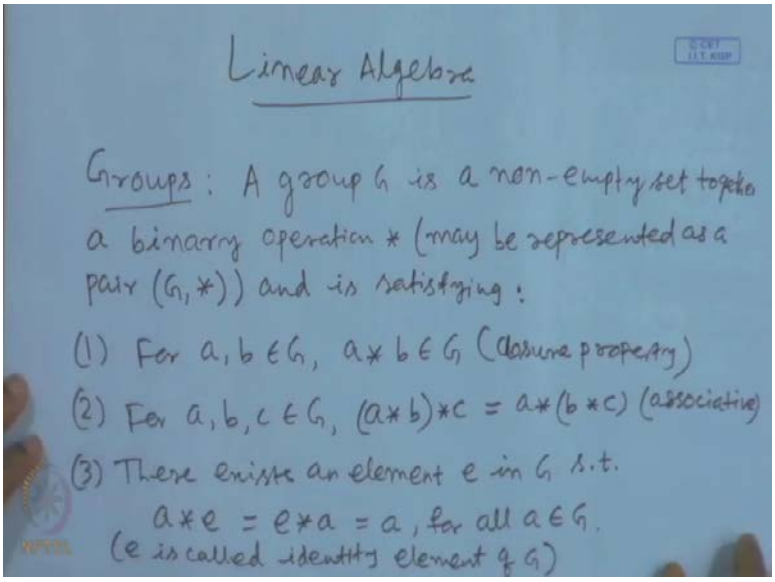

We learn the topic Linear Algebra. Linear Algebra linear algebra is an important, basic topic of our Mathematics. It is also an important tool in Engineering and Sciences, it has lots of application in almost all areas of Engineering and also Science disciplines. The topic linear algebra is based on an important algebraic structure that is called vector space or linear space.

Again in a vector space, we deal with two types of elements that is, vectors and scalars. So, first let us know what is scalars? To know scalars, we have to recall two important basic algebraic structures called groups and fields. So, we will start with this groups; this is very basic topic in Abstract Algebra.

So, a group G is a non-empty set together with, non-empty set together with a binary operation star. This may be represented as, may be represented as a pair (G, star). And it is satisfying the following conditions.

First one is that is called closure property; for any two elements a and b in G, a star b belongs to G. This is called closure property. This is called closure property. For any three elements a, b, c in G, a star b star c is equal to a star b star c. This is called associative. And third property is this; there exists an element e in G such that a star e is equal to e star a is equal to a, for all a in G. This element e is also called identity element; e is called identity element of G.

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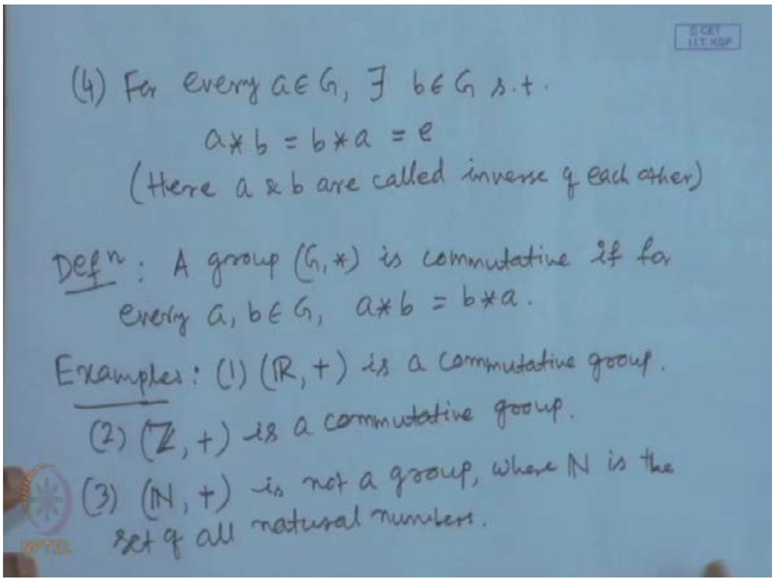

And fourth property is this. For every element a in G, there exist b in G such that, a star b is equal to b star a is equal to e. Here, a and b are called inverse of each other. Again, we say that a group G is commutative, if for every element a and b in G, a star b is equal to b star a.

Next, let us see some examples, some examples of groups. First one is that very natural one that, set of all real numbers with usual addition operation is a commutative group is a commutative group. Similarly, we can also see that set of all integers with addition operation is a commutative group. However, if we see the set of all natural numbers with respect to addition is not a group, where n is the set of all natural numbers, where n is the set of all natural numbers.

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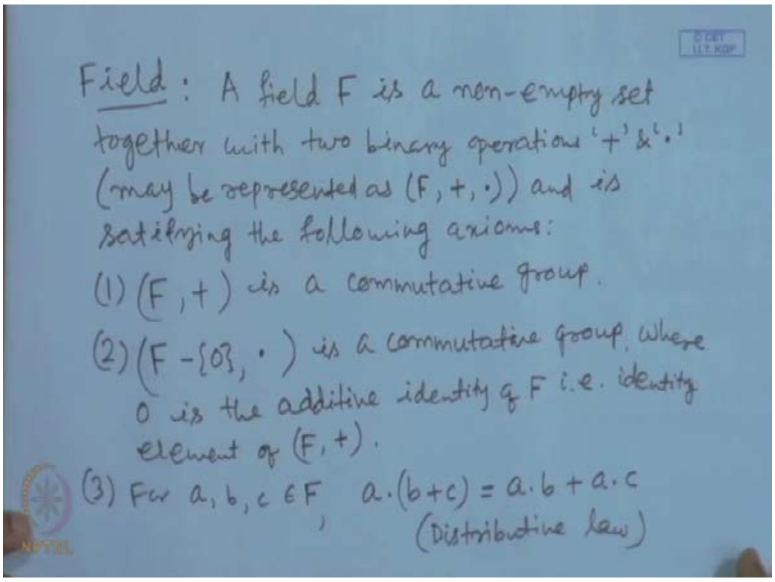

So, next we shall see the definition of a field, which is very much useful in vector space, in defining a vector space. We need this algebraic structure that is called a field. A field F is a non-empty set together with two binary operations two binary operations plus and 'dot'. This may be represented as, may be represented as a triplet; F, plus, dot. And it is satisfying the following conditions or axioms, following axioms.

First one is, F plus is a commutative group. And, second condition is F minus 0 with respect to this dot operation is a commutative group. Therefore, where this 0 is the 0 is the additive identity, additive identity of F, that is, identity element of identity element of this group F, plus. And third condition is also called compatibility condition. In any structure, whenever we are having more than one operation, we should have this compatibility property and this is also called distributive property. For any three elements a, b, c in F, a dot b plus c is equal to a dot b plus a dot c. In other words, this dot operation is distributive over addition. This is called distributive law.

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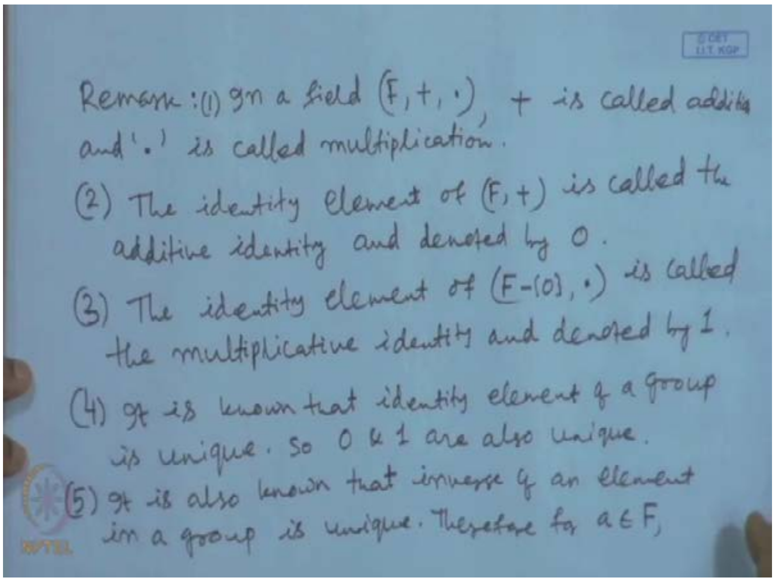

So, here are some terminologies or some definitions that we refer and again and again that will give as a remark set of thing that in a in a field, positive mark is this, in a field F plus dot; this 'plus' is called addition and 'dot' is called multiplication. Of course, they are not necessarily the addition and multiplication operation on the set of real numbers. This 'plus' and 'dot' symbols, they just represent to binary operations. And, their names have been in given as addition and multiplication. There need not be usual additional and multiplication operations on set of real numbers. There may be any binary operation.

So, that we will see example that, here this zero is called or this, is the identity element of F, plus is called the additive identity and denoted by zero. The identity element of F minus 0, dot is called the multiplicative identity and is denoted by one. Then fourth remark is like this. It is known that identity element; identity element of a group is unique. So, zero and one are also unique. So, fifth observation is like this. It is also known that inverse of an element in a group is unique.

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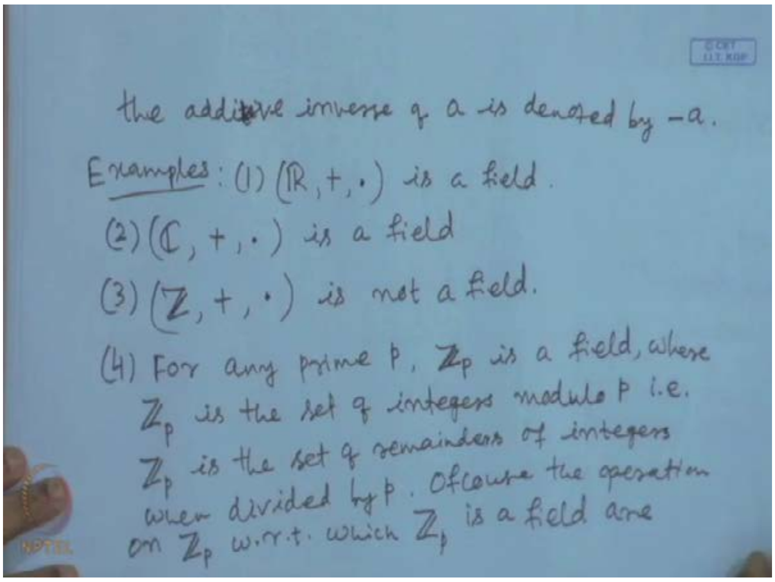

Therefore, for any element a in F, the additive inverse the additive the additive inverse of a is denoted by minus a. Next, we shall see some examples of fields. So, again we are having this. Very natural example is this. Set of all real numbers with usual addition and multiplication, this is a field. And second one also we are having this set of all complex numbers with respect to addition and multiplication. This is also a field. We will see another example that set of all integers with respect to addition and multiplication. This is not a field because we will not get multiplicative inverse of elements in Z. So, fourth example is, we will see an example of a finite field here all the examples we have given are examples of infinite field. So, let us see one example of a finite field.

So, for any prime for any prime p, Z p, Z p is a field, where Z p is the set of integers modulo p; that is, in other words, Z p is the set of remainders set of remainders of integers when divided by p divided by p.

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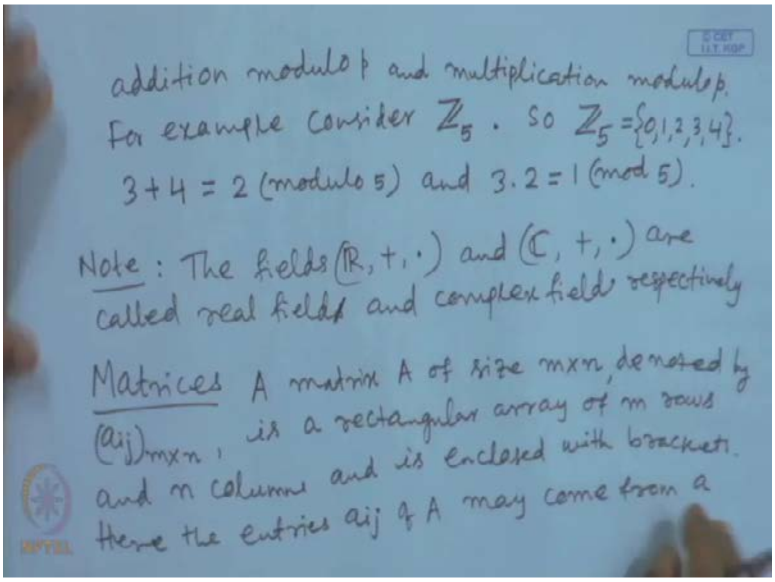

Of course, the operations on Z p with respect to, which Z p is a field are addition modulo p are addition modulo p and multiplication modulo p. For example, consider Z 5. So, Z 5 will be the set of remainders 0, 1, 2, 3, 4. Here on Z 5, we will have this addition modulo 5 operation is like this; 3 plus 4 that is equal to 2 modulo 5 and 3 into 2 that is equal to 1 modulo 5. So, now these fields, here we can have this note that the field R, the fields R, plus, dot and C, plus, dot are called real fields and complex field respectively.

So, now we are ready to give definition of a vector space, but before that we like to recall another important objects that we need in this linear algebra are matrices. Matrices play an important role in linear algebra to represent various concepts. So, let us see matrices. So, as you know this, matrix A, matrix of size A matrix, a matrix A of size m by n denoted by a i j m by n is rectangular array of m rows and n columns and is enclosed with brackets.

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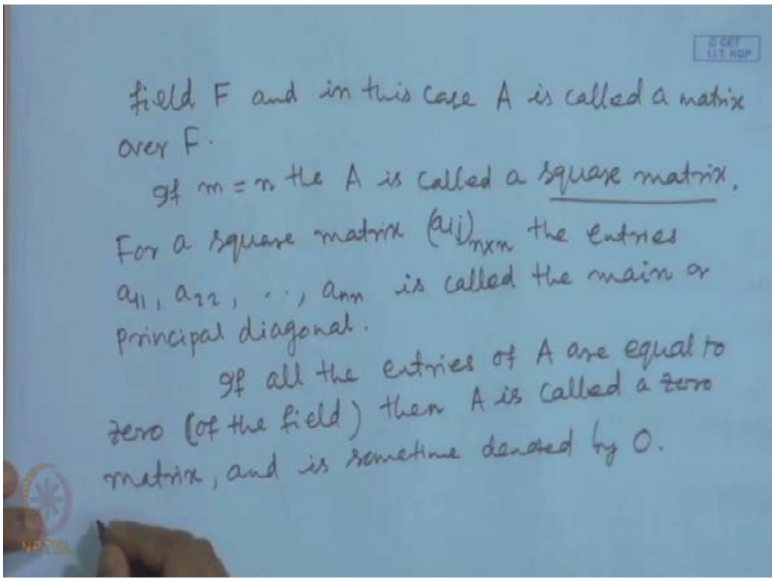

Here this matrices, here these entries, here the entries a i j, entries a i j of A may come from may come from a field, may come from a field F. And, in this case, A is called a matrix over F. If m is equal to n, then A is called a square matrix is called a square matrix. For a square matrix a i j n cross n, the entries a 1 1, a 2 2... a n n is, the entries are called the main diagonal or main or principal diagonal. Again, if all the entries of A if all the entries of A are equal to 0, this 0 is 0 of the field of course, of the field, then A is called a zero matrix. And, sometime we denote by 0 also and is sometime denoted by 0.

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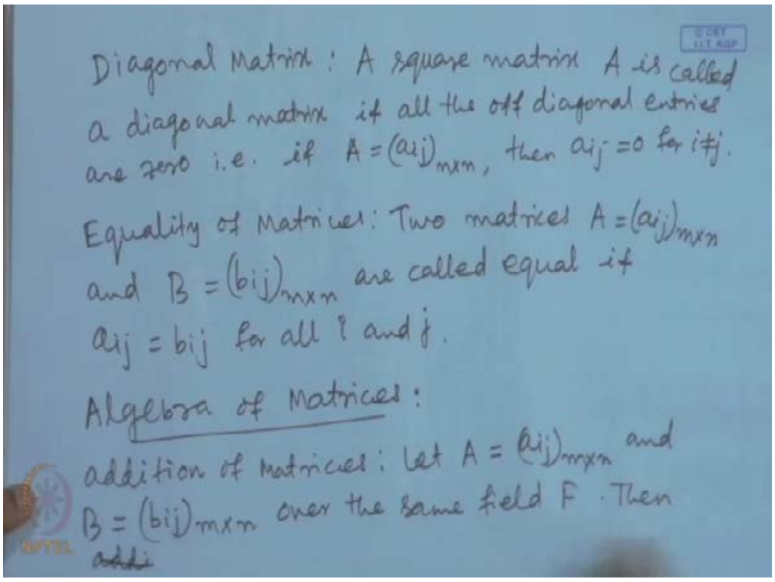

Here, we will have also diagonal matrices. That we will see. So, we can have this diagonal matrix. A square matrix A is called a diagonal matrix if all the off diagonal entries are 0; that is, if the entries of A are a i j, then a i j equal to 0, for i not equal to j. So, now we can also say that two matrices are equal. That equality of matrices, two matrices A and B are said to be equal, two matrices of same size that m by n and B is also a matrix of size m by n are called equal if a i j is equal to b i j, for all i and j.

So, now we have an algebra of matrices that we can add to matrices. We can subtract two matrices. We can have scalar two multiplications of matrices and also we have product of matrices or multiplication of matrices. So, now we can say this algebra of matrices. So, first we shall see addition of matrices, but suppose we are having two matrices A and B of same size, A be a matrix of size m by n and B be a matrix of same size m by n over the same field F.

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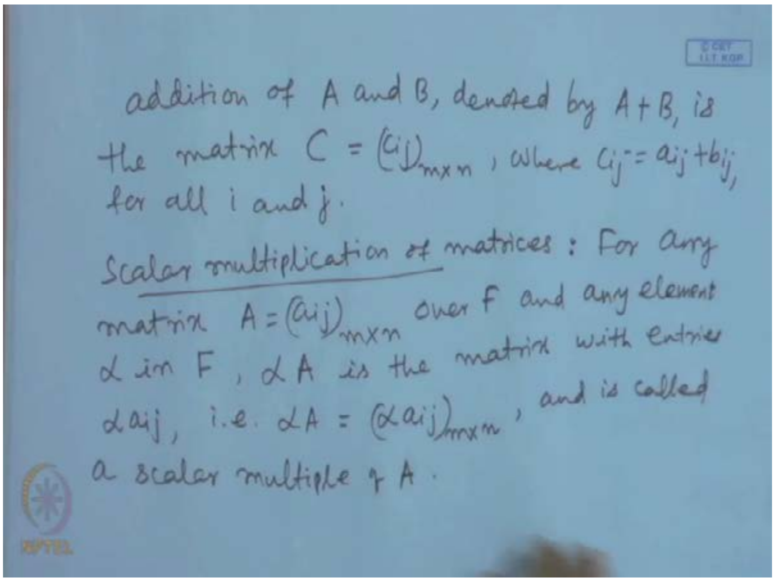

Then addition of A and B then addition of a and b, denoted by A plus B, is the matrix C, whose entries are c i j; where c i j is equal to a i j plus b i j, for all i and j. We can also have scalar multiplication of matrices. But we can have also this scalar multiplication of matrices. That is, for any matrix A with entries a i j of size m by n over a field F and any element alpha in F, alpha A is the matrix is the matrix with entries alpha a i j, that is, alpha A is equal to alpha a i j of size m by n and is called a scalar multiple of A.

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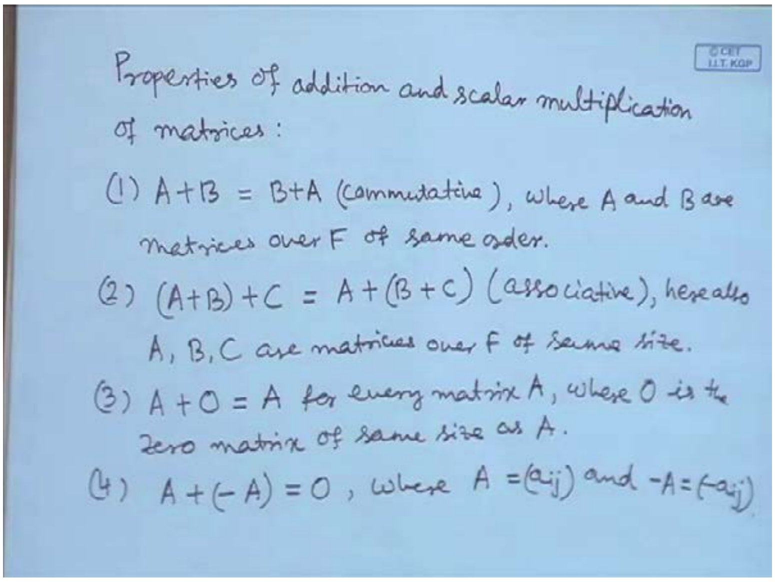

So, next we will have some properties of addition and scalar multiplication, which are very useful. But we will have some properties, some properties of addition and scalar multiplication of matrices scalar multiplication of matrices. That, first property is A plus B is equal to B plus A; that is commutative matrix, addition is commutative. Here A and B are any matrices over same field and same size, are matrices over F and are of same size.

We will also have that, A plus B plus C is equal to A plus B plus C; that is associative property. Here also matrices A, B, C are over the same field F of same size. A, B, C are matrices over F and of same size. So, here if we add the 0 matrix that, A plus 0 is equal to A for every matrix A; of course the size of this matrix A and the size of this matrix 0 are same, where A and 0 are of same size. So, fourth property is that, we can also have this negative of A matrix that A plus minus A. That is equal to 0. So, where A is a i j implies minus A is equal to minus a i j.

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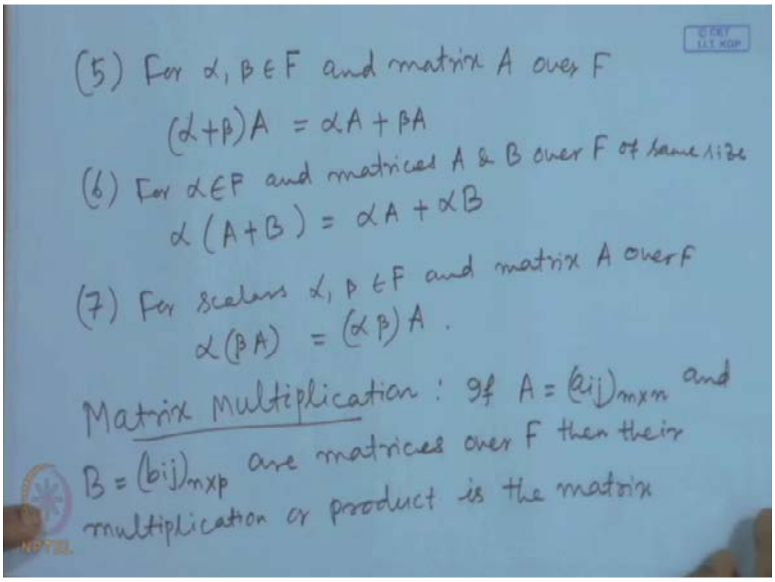

So, next we will have some properties of scalar multiplication. That is fifth property. For scalars, alpha and beta in the field, elements of this field a power four is called scalars for alpha, beta belongs to F and matrix A over F alpha plus beta times A is equal to alpha A plus beta A. And A for alpha in F and matrices A and B over F of same size; alpha into A plus B is equal to alpha A plus alpha B. Or in other words, the scalar multiplication is distributive over matrix addition.

Also, we are having this. For scalars alpha, beta in F and matrix A over F alpha times is beta A; this is equal to alpha beta A. So, next we will see multiplication of two matrices or also called a product of matrices. So, that is called matrix multiplication or product of two matrices. So, it is defined like this. If A be a matrix of size m by n and B is a matrix with entries b i j and size n cross p, n by p are matrices over a field F, then their multiplication or product is the matrix.

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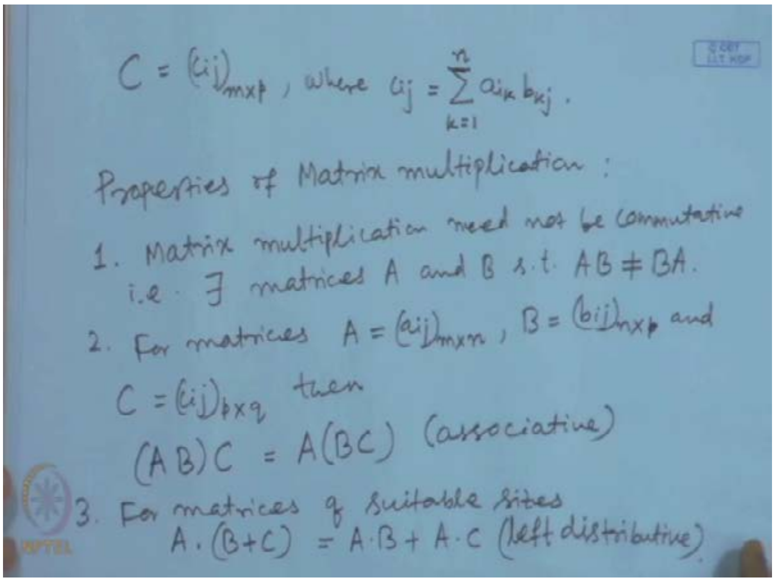

C with entries c i j and this will be of size m by p. And, this entries are given by c i j is equal to sum of a i k b k j and k runs from 1 to n. Next, we will see some properties of matrix multiplication properties of matrix multiplication.

So, one important property of matrix multiplication is that, this need not be commutative. That matrix multiplication need not be commutative; that is, there exist matrices A and B such that, A B is not equal to B A. However, this matrix multiplication is a associative. That, for matrices A with the entries a i j, that is m of size m by n, B is a matrix with size n by p and C is a matrix of size p by q. Then, we have this A into B into C is equal to A into B into C. That is associative.

Also, this matrix multiplication is distributive over addition. For matrices of suitable sizes that we have A dot B plus C that is equal to A dot B plus A dot C. That is left distributive. This property is called left distributive. Similarly, we have that right distributive property that also holds for this matrix multiplication.

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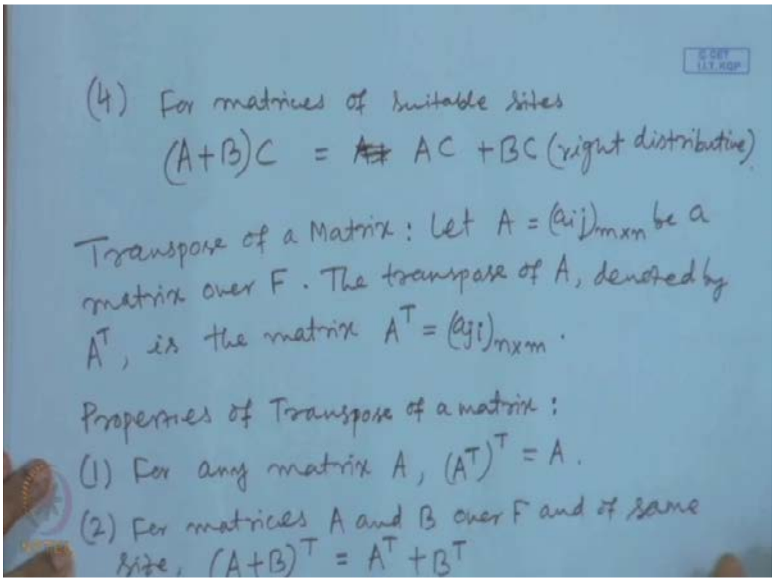

So, that is, for suitable matrices or matrices of suitable sizes, this A plus B dot C that is equal to, I mean A…, that is equal to, sorry, that A C plus B C. This is called right distributive. Then, we are having some special matrices. Before that, we will see this transpose of a matrix, transpose of a matrix. This is very useful throughout this linear algebra. So, if we are having the matrix A with entries a i j and size m by n be a matrix over F, the transpose of A will denote like this; denoted by A transpose is the matrix A transpose is equal to a j i that is of size n by m. Or in other words, we get transpose of A by writing the rows of a h columns of a transpose in order.

So, we can have this properties of transpose of a matrix. So, first property is like this. For any matrix A, if we take its double transpose, transpose the A transpose that will get the same matrix back; that equal to A. Second property is, for matrices A and B over F and of same size, we have this transpose of A plus B; that is equal to transpose of A plus transpose of B.

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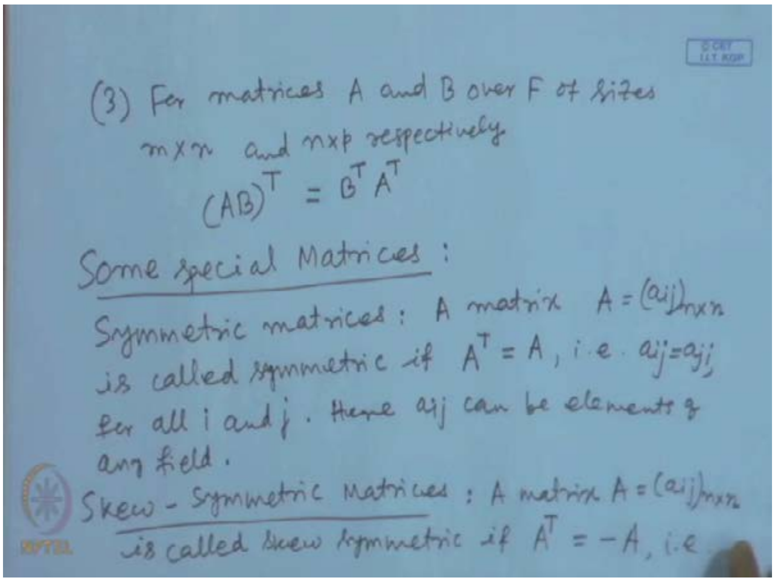

So, another property we will have is this. So, if A and B or for matrices A and B over F of sizes m by n and n by p respectively; that transpose of product of A and B. this is equal to product of transpose of B and transpose of A. So, it will be reverse. The product will be in reverse order.

So, next we will see some special matrices. First one is this symmetric matrices. They are also very important type of matrices. So, a matrix A, basically we will see that square matrix of size n by n is called symmetric is called symmetric, if A transpose is equal to A. Or in other words, that entries of this, that is a i j is equal to a j i, for all i and j. Here the entry is can be element of any field, of course. Here a i j can be elements of any field. Of course some people, they consider elements of a symmetric are real numbers. But, in general we can consider elements be elements of any arbitrary field.

Then, we say that a matrix A is Skew-symmetric, that is, Skew-symmetric matrices. A matrix A with entries a i j, this is also square matrix, and is called Skew-symmetric, if A transpose is equal to minus A. That is, it satisfies this condition that a i j is equal to minus a j i, for all i and j.

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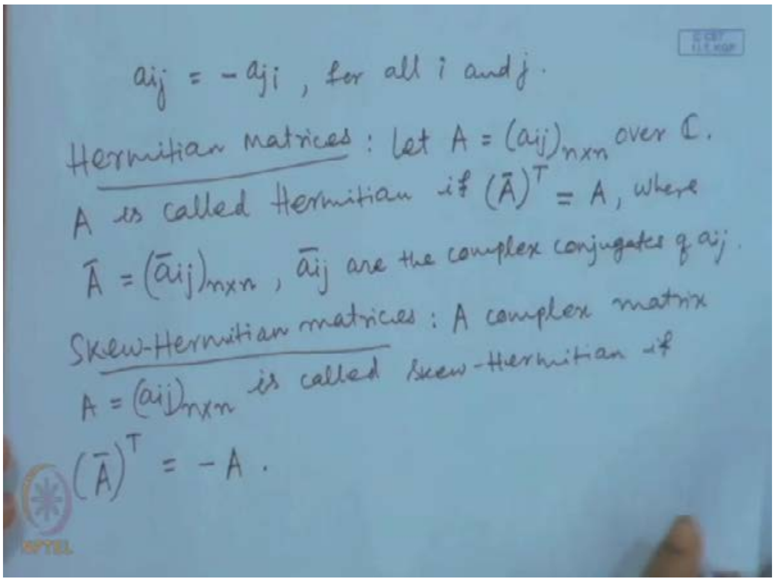

So, another important kind of matrices are Hermitian matrices and they are defined over complex fields only. So, let A be a square matrix over complex field, over C. Then, A is called Hermitian a is called hermitian, if A conjugate transpose is equal to A; that is where, thus A conjugate is equal to a i j bar, that is the complex conjugate, a i j bar are the complex conjugates of a i j.

Here, also we define these Skew-Hermitian matrices. So, we say, again the complex matrix, a complex matrix A with entries a i j, of course this is square matrix. It is called Skew-Hermitian skew hermitian if A conjugate transpose is equal to minus A.

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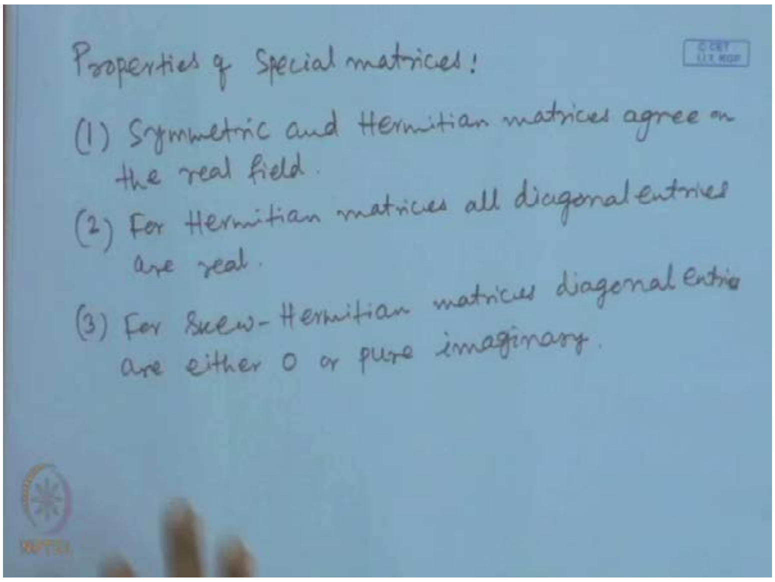

So, quickly we can have some properties of this matrices; that properties of special matrices. That first property is that, symmetric matrices and Hermitian matrices agree on the real field. Second property is that, for Hermitian matrices all diagonal entries are real, all diagonal entries are real. And for skew-Hermitian matrices diagonal entries are either 0 or pure imaginary. So, next we will also discuss.