Measure Theoretic Probability 1
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Lecture No. 09
Properties of Measures – II

Welcome to this lecture. So, as before, let us quickly recall what we have done so far. So, in this week, we have been looking at the measures of sets, which allows us to look at how likely certain events are in some random experiment or how large or important is one subset, some certain subsets in your measurable space.

So, far we have studied many examples of such measures and in particular probability measures. And we have also looked at many interesting properties of measures, which we called as algebraic properties involving many inequalities or equalities and involving usual algebraic operations like addition and subtraction. Now, in this lecture, we start looking at certain continuity properties of these measures.

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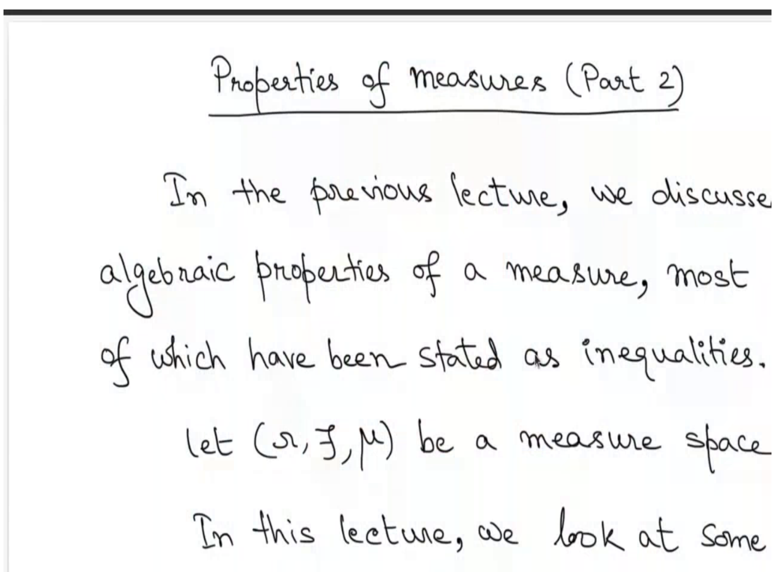

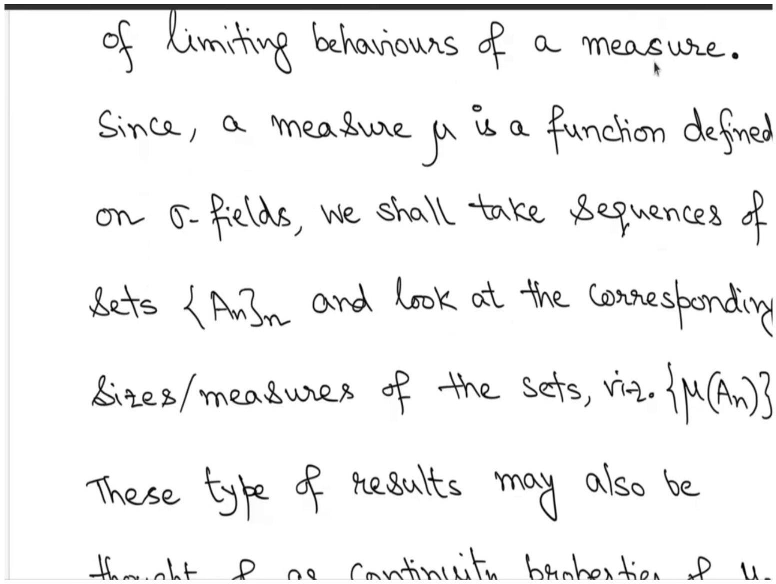

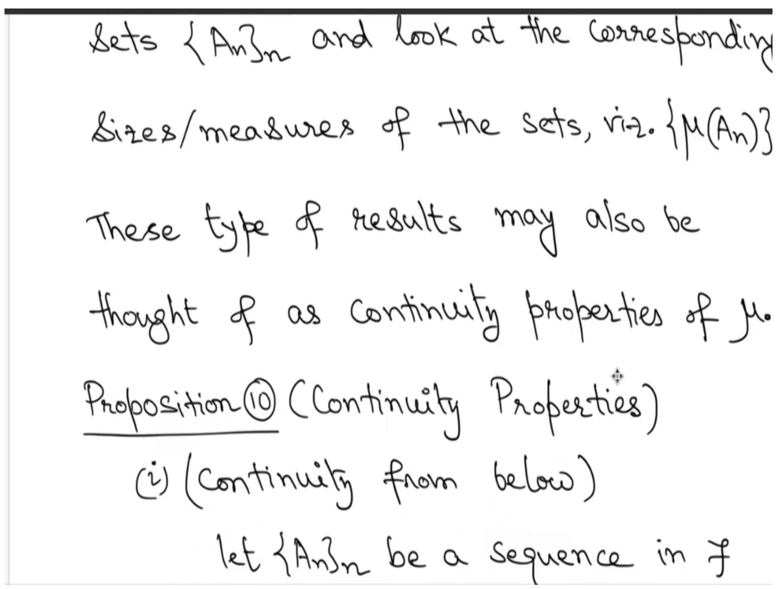

So, let us start by moving on to the slides. So, in the previous lecture, we have discussed these algebraic properties. Now, we start by looking at a slightly different type of properties. So, as before let us fix the notation first $(\Omega, \mathcal{F}, \mu)$ is a measure space. So, here μ is a measure on the measurable space $(\Omega, \mathcal{F})$. Now, what we are interested in limiting behaviors of this measure. What do I mean by that?

We look at this quantity. Since a measure μ is a function defined on this σ −fields, some collection of subsets, then we can take sequences of sets $\{A_n\}$ and look at the corresponding sizes or measures of these sets, which is $\mu(A_n)$. So, original collection, original sequence that was there that was from the σ − field. And now, correspondingly we are looking at the sizes of the sets and that allows us to construct a sequence of non-negative numbers, which of course could be ∞, which of course could include ∞.

Now, we are interested in the limit values of these measures of An. Now, these type of results may also be thought of as continuity properties of μ. Since, we have already discussed certain limiting notions for the sets An. So, therefore, we would like to connect the notions of limits of the sets A_n with the notions of $lim\,\mu(A_n)$ and that is why we are going to think of all these properties as continuity properties of μ or limiting behaviors of μ.

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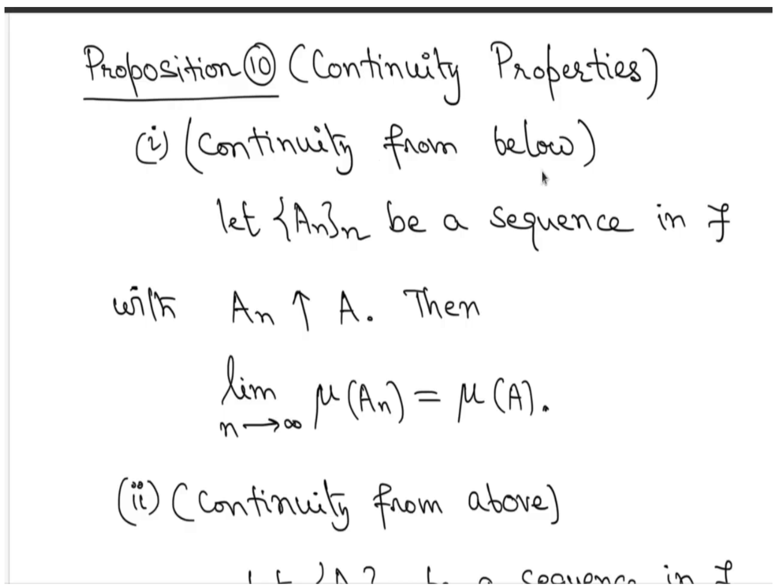

So, in this regard, so this is the result we are going to study continuity properties. The first part says that a measure μ is continuous from below. What is this? It is like this. So, take a increasing sequence of sets A_n and call the complete union, call the countable union of Ans as capital A. This is the usual notation used earlier. Then we claim that $\mu(A_n)$ increase, an increase to this limit value which is nothing but the $\mu(A)$. So, this is continuity from below. So, if A_n increase to the set A, then $\mu(A_n)$ also increase to the $\mu(A)$. So, that is the continuity property.

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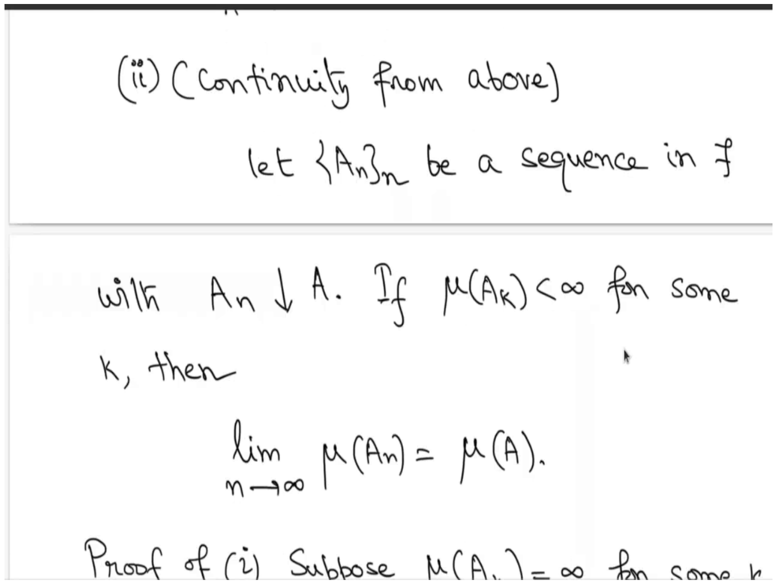

The second property says it is also continuous from above, but with certain restrictions. What is this? We start with a decreasing sequence of sets going down to the complete intersection, going down to the countable intersection, which we have used, this notion we have used earlier. So, A denotes the countable intersections of A_n. Now, we have this additional condition that at least one of the sets Ak should have finite mass.

Now, remember we are dealing with a general measure. So, $\mu(A_n)$ could be ∞. The sequence that you have taken may have infinite sizes. The sequence of sets that you have taken may have infinite sizes. What we want is that at least one of the sets should have finite mass. Then we claim that this limit also holds that $\mu(A_n)$ decrease to the $\mu(A)$. So, if A_ns decreased to A, then $\mu(A_n)$ decrease to $\mu(A)$.

So, now, just to quickly clarify, here in the statement it is written in terms of a limit. So, that is the second part again. And in the first part, it is also in terms of the limit. But I have mentioned that this is an increasing limit. These measures increase here and in the second part the measures decrease here.

So, why is that? This is because of the fact that when you are looking at increasing sequence of sets, then by the monotonicity property that was proved in the previous, that was discussed in the previous lecture, we have $\mu(A_n)$ must increase. Similarly, if A_n's are decreasing, then $\mu(A_n)$ must decrease. This is again by the monotonicity properties that was discussed in the previous lecture. Using that, we are now going to continue and talk about the actual limit value.

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Proof of (i) Suppose $\mu(A_k) = \infty$ for some k.
Since $A_k \subseteq A_{k+1}$, by the Corollary to Proposition (7) part (ii) of the previous lecture, we have, $\mu(A_k) \leq \mu(A_{k+1})$ and hence $\mu(A_{k+1}) = \infty$. Likewise we get $\mu(A_n) = \infty, n \geq$
Since $A_n \subseteq A \ \forall n$, we have $\mu(A) = \infty$.

Thus the required equality is true in this case.

Suppose $\mu(A_n) < \infty$ for all n. In this case, note that the set $A = \bigcup_n A_n$ can be written as a countable union of pairwise disjoint sets as follows:

$$A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \cdots .$$

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By the corollary to Proposition 7 part (i), we have,

$$\mu(A_{n+1} \setminus A_n) = \mu(A_{n+1}) - \mu(A_n), \ \forall n.$$

Here we have used $\mu(A_n) < \infty \ \forall n$.

Hence,

$$\mu(A) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(A_{n+1} \setminus A_n)$$

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$$= \mu(A_1) + \sum_{n=1}^{\infty} \left[\mu(A_{n+1}) - \mu(A_n)\right].$$

On the right hand side, we have the convergent series $\left\{\mu(A_1) + \sum_{n=1}^{\infty}\left[\mu(A_{n+1}) - \mu(A_n)\right]\right\}$ whose m-th partial sum is $\mu(A_{m+1})$.

So, for the first part, there was no restriction. We are going to consider increasing sequence of sets A_n and we are going to show that $\mu(A_n)$s increase to μ(A). But here we, here for simplicity, we split it into two parts. First, assume that $\mu(A_k)$ is ∞ for some k. So, if it so happens that $\mu(A_k) = \infty$ for some k, then since A_{k+1} already contains A_k, then by the monotonicity property that was proved in proposition 7, you immediately claim that $\mu(A_{k+1})$ is also ∞, because $\mu(A_{k+1})$ must be at least $\mu(A_k)$ and $\mu(A_k)$ is taken to be ∞ as per the assumption here.

Likewise, you can repeat this argument and claim that measures of $A_n = \infty$ for all $n \geq k$, because the sets are increasing. So, if one of the sets has infinite mass, then all the next sets will also have infinite mass. But then what happens to the limit value. The limit value is nothing but ∞. So, $\lim (\mu(A_n))$ is ∞.

But on the other hand, if you are looking at $\cup A_n \subset A$, the complete union, now $\mu(A)$ must also be ∞, because it contains the set A_k. Therefore, $\mu(A)$ is also ∞. Thus, the required equality holds, because the limit value is ∞ and $\mu(A)$ is also ∞.

Now, in the second part, now we look at the other case. So, in the first case we assume that one of the sets has infinite mass. But now you assume the contrary, the opposite condition that all the sets has finite mass. So, this is an interesting case. So, if all the sets has finite mass, now look at the complete union A once more, but then you apply your familiar disjointification. So, you construct these sets $B_1, B_2, \ldots, B_n$, but since these sets Ans are increasing, you can actually simplify it and write it using this pairwise disjoint union.

So, this construction was done earlier. But for the case, for the special case of increasing sequences, you get this decomposition. What is this? So, the complete union, unions of A is nothing but first A_1 then $A_2 \backslash A_1$ then $A_3 \backslash A_2$ and so on. So, earlier recall it was like A_1 then $A_2 \backslash A_1$, but the third set was $A_3 \backslash (A_1 \cup A_2)$. But since the sets are increasing, $A_1 \cup A_2$ is nothing but A_2 itself. That is why you get this simplified relation.

Therefore, so by this earlier construction, these sets A_1, $A_2 \backslash A_1$ and so on, these are pairwise disjoint. So, you apply now, you would like to apply now, the countable additivity of the measure. But then note that by the corollary of the proposition 7 that we have discussed again by the monotonicity properties, you also have this relation that measures of this thing $A_{n+1} \backslash A_n$ is nothing but the subtraction.

This is also true, this is true, because, this is true, because of these two reasons; first A_{n+1} is the union, is the disjoint union of A_n and $A_{n+1} \backslash A_n$. Repeat, just to repeat A_{n+1} is the disjoint union of

A_n and $A_{n+1} \backslash A_n$. So, you have these two sets pairwise disjoint, apply finite additivity, you get that $\mu(A_{n+1})$ is nothing but the addition of the individual sizes of A_n and $A_{n+1} \backslash A_n$.

But then, since you are dealing with sets with finite mass, we have made that assumption, you are allowed to subtract and get this relation that μ of the $\mu(A_{n+1} \backslash A_n)$ that is on the left hand side is equal to $\mu(A_{n+1}) - \mu(A_n)$. So, now, all you have to do of the countable additivity of the measure μ on this complete union A. Since you have written it as a countable disjoint union, what you get is that you get this relation that $\mu(A)$ is nothing but the addition of all these measures, additions of these sizes. But then, you have proved that $\mu(A_{n+1} \backslash A_n)$ is obtained by subtracting $\mu(A_n)$ from $\mu(A_{n+1})$.

Now, look at this series. This series has the partial sum given by a $\mu(A_n) + \sum_{n=1}^{m} \left(\mu(A_{n+1}) - \mu(A_n)\right)$ of this telescoping sums. If you are looking at the m-th partial sum of the series, you will simply get $\mu(A_1) + \sum_{n=1}^{m} \left(\mu(A_{n+1}) - \mu(A_n)\right)$ of this thing. But then, for the finite sum you can easily compute that this is becomes a telescoping sum many of the terms cancel off and the remaining term is $\mu(A_{n+1})$.

But since this countable additivity holds, you are basically saying that $\mu(A)$ is nothing but the limit of this series or the summation of the series, but nothing but, that is nothing but the limit of the partial sums. But the partial sums themselves are nothing but measures of A_{n+1} and therefore, you immediately get that $\mu(A)$ is nothing but the limit of partial sums which is nothing but this quantity.

So, here we have used the fact that a series is equal to the limit, the series value is equal to the limit of the partial sums and that allows us to claim this equality. So, that proves the case when the sets are increasing.

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$$n \to \infty$$

Proof of (ii): Fix k such that $\mu(A_k) < \infty$.

Then $A_k \setminus A_n \uparrow A_k \setminus A$ as $n \to \infty$.

Hence, by part (i)

$$\mu(A_k \setminus A) = \lim_{n\to\infty} \mu(A_k \setminus A_n).$$

This relation is now re-written as

$$\mu(A_k) - \mu(A) = \lim_{n\to\infty} \left[\mu(A_k) - \mu(A_n)\right]$$

But when the sets are decreasing, you also have that assumption that there is at least one set Ak for which the measure is finite. So, you start with that. Then observe this that the sets $A_k \setminus A_n$, so here you fix k and vary n. You look at these sets $A_k \setminus A_n$. Since A_n's are decreasing, you will get that $A_k \setminus A_n$ will increase and increases to $A_k \setminus A$. But then, here you can apply the first part because this sequence now is an increasing sequence. So, if you apply that then $\mu(A_k \setminus A)$ that is the limit value of the $\mu(A_k - A_n)$ as $n \to \infty$.

But then, since you are dealing with sets with finite mass, since A_k you will have finite mass then $A_k - A_n$ also has finite mass. Then you can rewrite this equation equality as $\mu(A_k) - \mu(A)$, because A was a subset of A_k, so $\mu(A)$ is also finite. So, you can write the left hand side as the difference of $\mu(A_k) - \mu(A)$.

On the right hand side, again you will apply the same fact. Since $\mu(A_k)$ is finite, $\mu(A_k - A_n)$ is also finite, and you can write it as a subtraction of these quantities. Therefore, you get this relation. But then $\mu(A_k)$ is finite. So, you can cancel off $\mu(A_k)$ from both sides. And that required result follows. So, you get a limit of $\mu(A_n)$ is nothing but $\mu(A\ )$. But this is using the fact that at least one of the sets as finite mass.

But now we are going to ask that what happens if you are considering a decreasing sequence and all of the sets has infinite mass. What happens there? So, for this case, we have to, for this case, there are counter examples.

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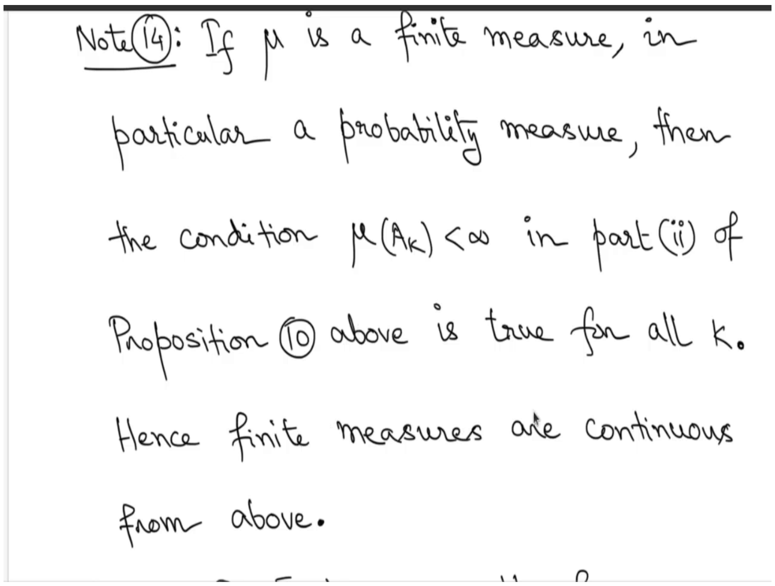

So, now, if you are dealing with finite measures, then this condition is automatically satisfied. So, you do not have to worry about it. So, in particular for probability measures this condition is automatically satisfied. So, therefore, what you actually have proved from that proposition is the fact that finite measures including probability measures are both continuous from above and continuous from below.

So, take any increasing sequences or decreasing sequences, then the appropriate limit holds. So, that is what I mean by continuous from above and continuous from below. So, for finite measures and in particular for quality measures, you have this fact.

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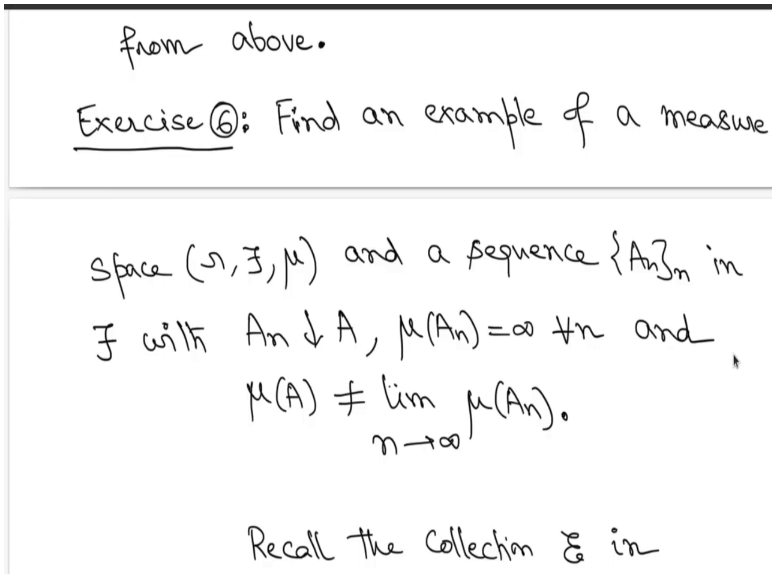

But then, for an infinite measure you can find examples where the all the measures of the sets that you are considering have infinite mass the sets decrease to that countable intersection and then you can try to show that $\mu(A) \neq \lim_{n\to\infty} \mu(A_n)$. So, you have to nstruct such an example explicitly. Please work this out. So, this will give you a counter-example to the fact that when there are infinite measures, there can be some sequences which are decreasing, but the equality does not hold. So, the measure need not be continuous from above in that case. So, please work out, please find such an example.

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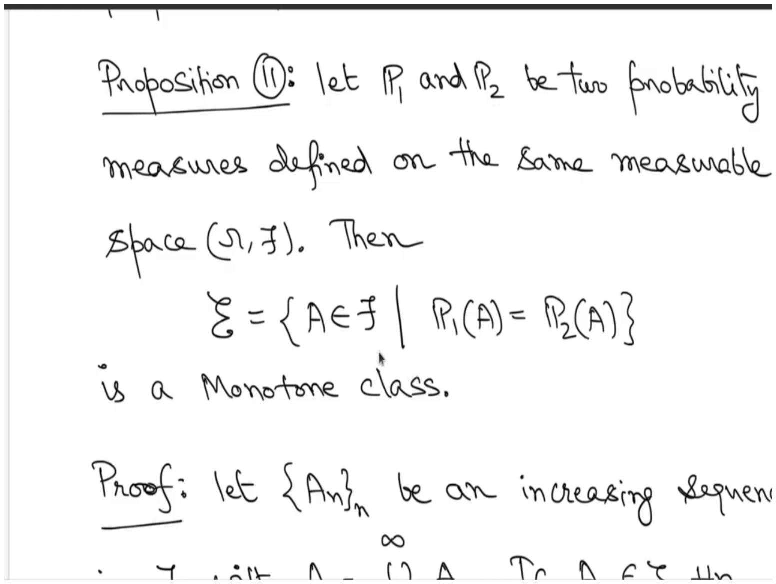

Now, we restrict our attention to familiar collection that was considered earlier in the previous lecture. What is this? So, consider two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ and look back at the collection of subsets where the probabilities match. So, probability of A under $\mathbb{P}_1$ and $\mathbb{P}_2$ must be the same. So, look at those special subsets and call that collection as $\mathcal{E}$. Earlier, we had proved that this collection is non-empty and closed under complementation, finite disjoint unions and countable disjoint unions.

Now, we are going to show that this collection also has some nice structures like that this is a monotone class. How do you show this? So, to show this, all you have to verify that this collection $\mathcal{E}$ is closed under countable increasing unions and countable decreasing intersections. How do you show this?

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Proof: let $\{A_n\}_n$ be an increasing sequen

in $\mathcal{F}$ with $A = \bigcup_{n=1}^{\infty} A_n$. If $A_n \in \mathcal{E}$ $\forall n$,

then $\mathbb{P}_1(A_n) = \mathbb{P}_2(A_n)$ $\forall n$.

we want to show $A \in \mathcal{E}$, i.e.

$$\mathbb{P}_1(A) = \mathbb{P}_2(A).$$

Now,

$$\mathbb{P}_1(A) = \lim_{n\to\infty} \mathbb{P}_1(A_n) = \lim_{n\to\infty} \mathbb{P}_2(A_n) = \mathbb{P}_2(A).$$

$$\mathbb{P}_1(A) = \lim_{n\to\infty} \mathbb{P}_1(A_n) = \lim_{n\to\infty} \mathbb{P}_2(A_n) = \mathbb{P}_2(A).$$

Hence, $\mathcal{E}$ is closed under countable increasing unions.

A similar argument shows $\mathcal{E}$ is closed under countable decreasing intersections

Hence $\mathcal{E}$ is a Monotone class.

So, let us take the first conditions. So, take our increasing sequence A_n with A being the union. Then, if A_n's are already in this collection, you would like to show that the union is in the collection $\mathcal{E}$. Now, you observe that if A_n's are already in this collection, then the probability of A_n's according to $\mathbb{P}_1$ and $\mathbb{P}_2$ must match. We want to show that the probability of A, the union must match under $\mathbb{P}_1$ and $\mathbb{P}_2$. To share this you apply continuity from below. Why?

If A_1, if the sets A_n's increased to the set A, then by the continuity from below of the probability measure $\mathbb{P}_1$, you get the probability of $\mathbb{P}_1(A)$ is equal to $\lim \mathbb{P}_1(A_n)$ that was just proved as continuity from below. But then if $\mathbb{P}_1(A_n)$ is nothing but $\mathbb{P}_2(A_n)$, you rewrite it in terms of $\mathbb{P}_2$. And here you apply the continuity from below of the probability measure $\mathbb{P}_2$ and you end up having the probability of A in terms of $\mathbb{P}_2$.

So, therefore, the union has the same probability under $\mathbb{P}_1$ and $\mathbb{P}_2$. So, hence $\mathcal{A}$ is closed under countable increasing unions. And a similar argument involving the continuity from above for this probability measure $\mathbb{P}_1$ and $\mathbb{P}_2$ will show you that it is also closed under countable decreasing intersections. And hence, $\mathcal{E}$ is a monotone class.

But here for the increasing, countable decreasing intersections what you have to use is the continuity from above and just note that any finite measures in particular for probability measures this property is true. So, for any probability measure, it is continuous from above and continuous from below. So, this is what implies that this collection $\mathcal{E}$ is closed under countable increasing unions and countable decreasing intersections and hence it becomes a monotone class.

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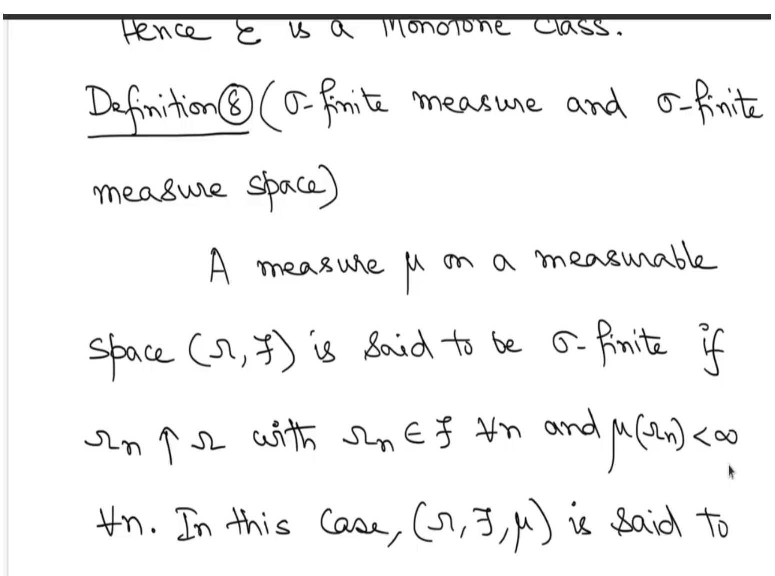

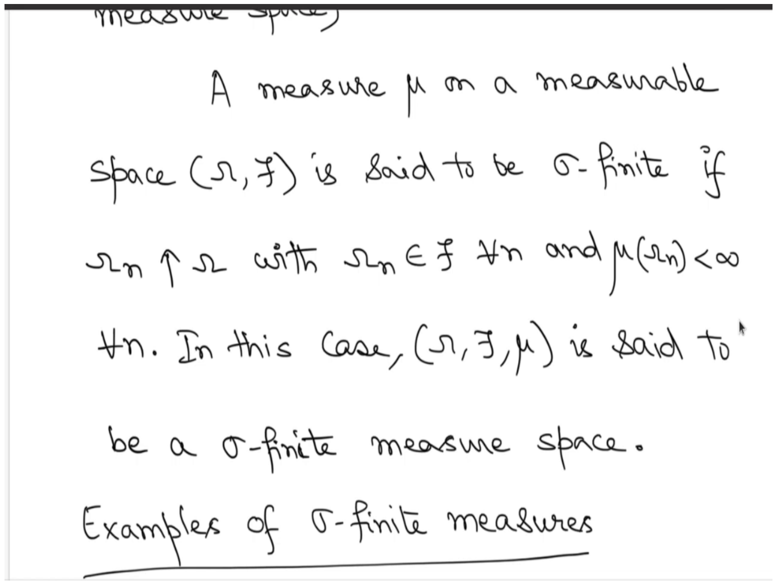

We now consider a concept related to increasingness and decreasingness of sequences of sets. So, a measure μ on a measurable space $(\Omega, \mathcal{F})$ is said to be $\sigma -$ finite, if you have a sequence of sets Ω_n increasing to Ω and this Ω_n first of all must be from your collection from your $\sigma -$ field and with the additional condition that measures of Ω_n are finite.

So, again we need to, we are considering measures on some measurable space, we would like to figure out increasing sequence of subsets Ω_n such that Ω_n' s have finite mass, but Ω_ns increase to Ω. So, in this case, we are going to say that $(\Omega, \mathcal{F}, \mu)$ is a $\sigma -$ finite measure space.

Now, a quick clarification, here we are not saying anything about the $\mu(\Omega)$. We are only restricting that measures of Ω_n's, those special subsets that you consider must have finite mass. So, if there exists one sequence Ω_n like that, if there exists increasing sequence Ω_n like that, then you get that the measure, then you say that the measure is $\sigma -$finite and call the corresponding measures is as $\sigma -$ finite measure space. But again you are not saying anything about the $\mu(\Omega)$. It could be finite. It could be infinite.

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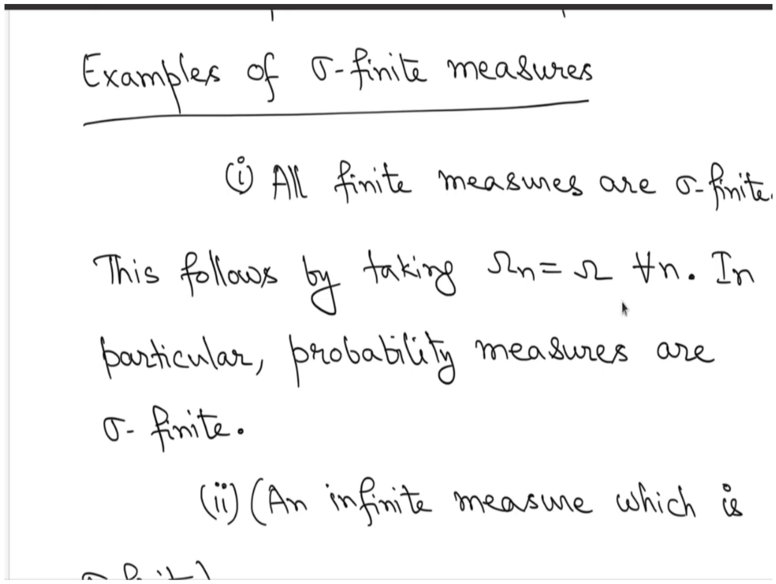

So, let us first understand what are some examples. So, if you look at a finite measure, then of course, you can take Ω_ns to be Ω itself. So, it is a constant sequence of sets. So, all the sets are Ω. These are, of course, subsets as per our definition. And each of which has finite mass because we have taken a finite measure. So, therefore, you get an increasing sequence of sets and you get that this finite measures are σ −finite. In particular, all property measures are σ −finite. But then you would like to get examples for the case of infinite measures.

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(ii) (An infinite measure which is σ-finite)

Consider the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers. Let μ be the counting measure. Since $\mu(\mathbb{N}) = \infty$, μ is an infinite measure. Take

measure. Since $\mu(\mathbb{N}) = \infty$, μ is an infinite measure. Take

$$\Omega_n = \{1, 2, \ldots, n\}, \quad n = 1, 2, \ldots$$

Then $\Omega_n \uparrow \mathbb{N}$ and $\mu(\Omega_n) = n < \infty \ \forall n$. Hence μ is σ-finite.

Note (15): Not all infinite measures are σ-finite. In this course, we do not

Note (15): Not all infinite measures are σ-finite. In this course, we do not consider any measures which are not σ-finite. In later lectures, we shall see examples of σ-finite measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

So, an infinite measure which is σ −finite is discussed here. So, consider the measurable space of integers of natural numbers $\mathbb{N}$ and the corresponding power set. So, look at the measurable space given by the, consider the measurable space with the set of natural numbers with the power set. Here consider the counting measure. So, here the set of natural numbers $\mathbb{N}$ is countable. So, therefore, a set of natural numbers is countably infinite. So, the number of elements there is infinite. So, $\mu(\mathbb{N})$, the size of the set $\mathbb{N}$ is infinite. So, that is already given to us. So, this is an infinite measure.

But then consider the sets Ω_n which is like this $1,\ 2,\ ...,n$. So, it has exactly n elements Ω_n for each n. Then look at the union. The union is nothing but the set of natural numbers. But then $\Omega_n \subset \Omega_{n+1}$, because Ω_{n+1} is the connection $1,\ 2,\ ...,n$ and $n + 1$.

So, therefore, $\Omega_n \subset \Omega_{n+1}$ and their union is nothing but the set of natural numbers. But here observed that number of elements in Ω_n is nothing but n itself, which is finite. So, it is finite for each n, but $\mu(\Omega)$ is infinite. And here, you end up having an example of an infinite measure which is σ −finite.

So, you get these pieces Ω_n where the sizes are finite, but the sets increase to the whole set and gives you the examples of a σ −finite measure. But it is a word of caution that not all infinite measures are σ −finite. But in this course, we are not going to consider such measures. We are not going to consider any infinite measure which is not σ $-$finite. We are not going to consider such measures.

But we are going to consider σ $-$finite measures on the real line. And these will be some very interesting examples, which of course, includes probability measures on the real line. So, this we are going to see in later lectures. We stop here and we are going to continue the discussions about properties of measures in the next lecture.