Measure Theoretic Probability 1

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Lecture No. 06

Measures and Measure Spaces

Welcome to this lecture. This is the first lecture of week 2. And we are going to start the discussion of a new topic involving Measures and Measure Spaces. Before we go forward, we first recall what we have already done in week 1. So, in week 1, we looked at events that are arise from certain random experiments and then we considered this collection of events. And we looked at three specific type of structures on top of these collections, $\sigma -$ fields, fields and Monotone classes. These involved certain set operations that the collections are closed under.

We had also looked at specific examples of the subfield, $\sigma -$ fields and model classes. In particular, we have spent a lot of time in understanding many of these ways of generating $\sigma -$ fields. We restricted our attention to the Borel $\sigma -$ fields on the real line and we understood the different generating classes that generate this Borel $\sigma -$ field on the real line. We have also extend this notion of Borel $\sigma -$ fields on higher dimensions and also to the external real number system.

At the end, we have also looked at limits of sequence of sets. We did it in two ways. One, using set inclusions, we had looked at two different set theoretic inclusions, one was the interestingness and other was a decreasingness and by means of that we tried to define the notions of limits. And another notion of these limits were taken for general sequences of sets, where we considered this notions of $lim\ sup$ and .

And using these notions of limits, we had also considered the notions of Monotone classes. And towards the end, we saw the connections between the fields, $\sigma -$ fields and Monotone classes through the Monotone class theorem. So, that is what we had discussed in week 1. Now, we will start on with this notion of measures and measure spaces.

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Measures and Measure spaces

In the previous lectures, we have seen examples of events in sample spaces or more generally, sets in measurable spaces. To understand how likely it is for a specific event to occur or to under-

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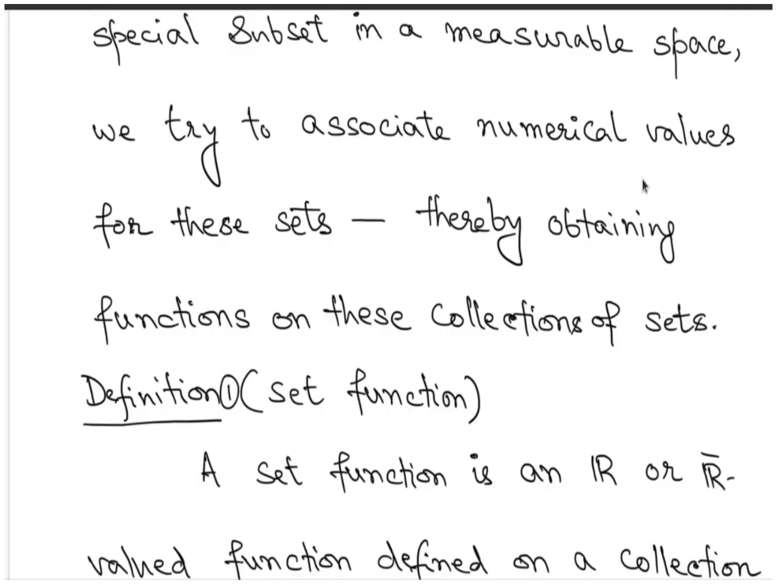
stand how important/large is some special subset in a measurable space, we try to associate numerical values for these sets — thereby obtaining



So, as before I switch over to the slides. So, in the previous lectures, we have seen examples of events in sample spaces or more generally, certain collections of subsets in measurable spaces. Now, we would like to understand this specific question that given certain collections of events, with respect to our random experiment, we would like to understand which of these specific events are more likely to occur than the others.

Or in the general setting for a non-empty set, if you are given a collection of sets, we will have to understand if we can attach certain importance to these subsets and which are more important than the other. You can simply think of it in terms of the size of the sets. We would like to understand if certain subsets are larger than the other.

To understand these concepts, what we are going to do is that we are going to associate certain numerical values to these sets. So, we will start off with certain collection of sets and then we will associate values to each of these sets that is in your collection. And thereby we are going to obtain certain specific type of functions. And this is what we define now.

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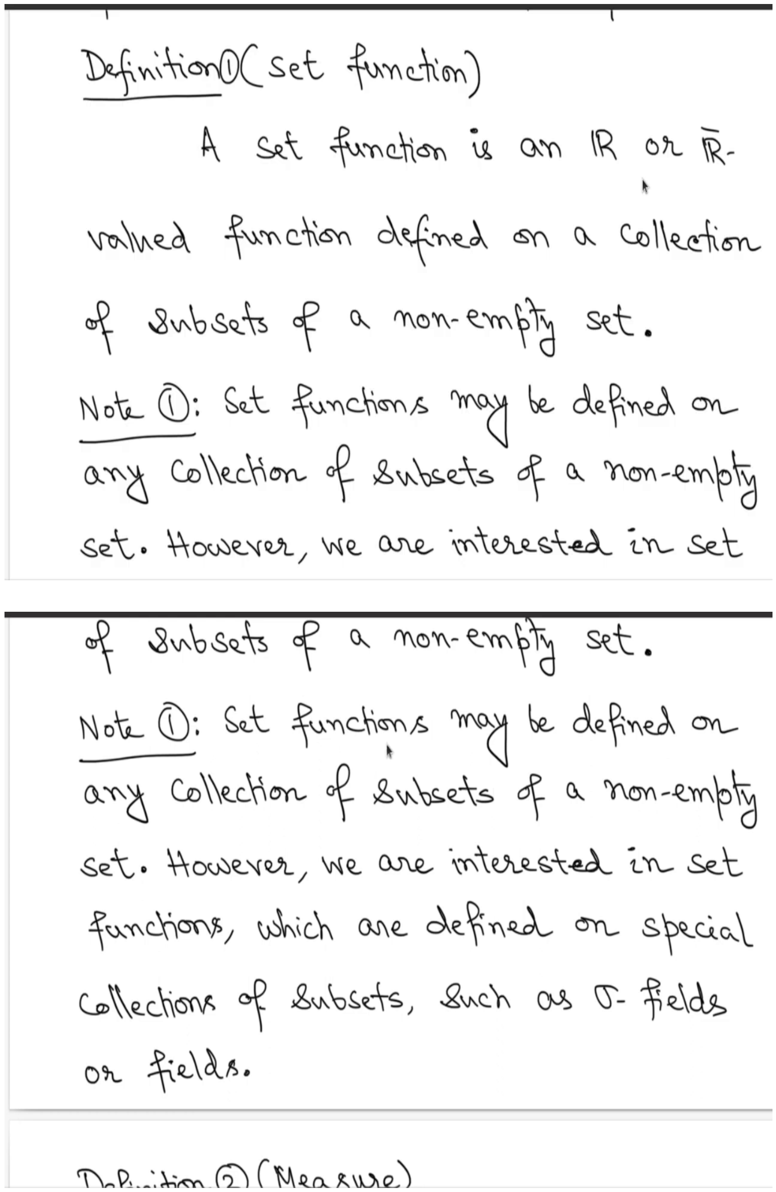

So, this is what this notion of set function is all about. A set function is a real valued or extended real valued function defined on a collection of subsets of a non-empty set. So, the definition is pretty simple. You take some collection of subsets and for each set in your collection you associate a value. This value is allowed to be real number or excellent real number and then you get a set function. But there are certain clarifications that are required.

So, in general, what we have defined so far is that set functions are defined on any collections of subsets. However, in practice, what we need is certain structures on this type of set functions and the corresponding collections. And what we are going to consider are set functions on special collections subsets, which will typically be σ −fields or fields. Now, the notion that we immediately follow from this discussion is these notions of measures of sets.

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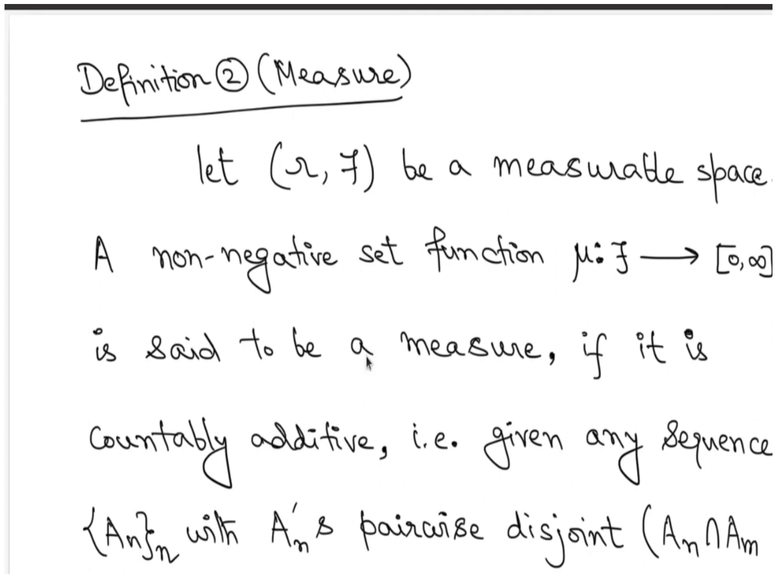

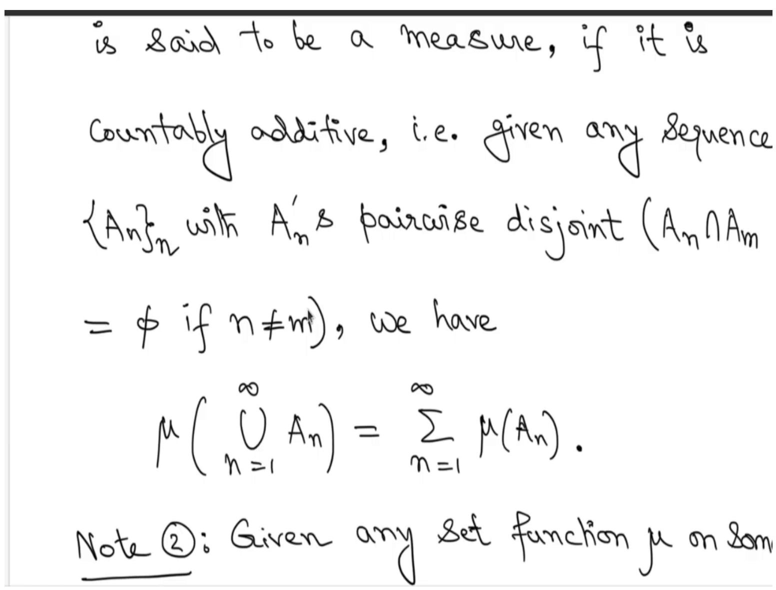

So, what we are interested in again is to understand how likely for an event to occur or how large a certain subset. Now, to do that, we are now associating certain values to this, which we will think of it as size or a measure of these sets. Now, the idea is, if you have two distinct subsets, if they are pairwise disjoint, then if you are adding them up, then their sizes should be added up. So, that is the basic idea.

So, if you have two pieces of things, if they have individual sizes, when you mix them up, they should have the total size computed as the addition of the two individual sizes. So, with this kind of notions at hand, we are going to consider these notions of measures and this is our next definition. So, we are going to consider non-negative set functions on $\sigma -$ fields. So, we will start off with a measurable space and for each set in the $\sigma -$field, we are going to associate some numbers which is between 0 and ∞. ∞ is allowed. And we are going to say that this is a measure if it satisfies this property of countable additivity.

So, what is countable additivity? We are going to look at certain sequences from the $\sigma -$field with these properties that An’s should be pairwise disjoint. So, what is this? This means that $A_n \cap A_m$ is empty if $n \neq m$. So, they do not have any element in common if you are looking at two distinct sets.

Now, for this type of sequences, this union is of course there in the $\sigma -$field. So, you can talk about the size of that or the function value of that, because the function is defined on the whole collection. So, you can talk about this value on the left hand side. But we want it to be equal to the right hand side, which is made up of the individual sizes and adding them up. So, that is how you get the series on the right hand side. So, this is what countable additivity is all about.

So, as long as these sets are pairwise disjoint, they do not have any element in common. You look at their individual sizes and add them up. If it so happens for that function that you have taken then this function is countably additive. And if the function is, in addition, is non-negative, then we are going to call it a measure.

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Note ②: Given any set function μ on some collection of subsets $\mathcal{F}$ of a non-empty set, we may consider the countable additivity of μ on $\mathcal{F}$ as follows. We say μ is countably additive on $\mathcal{F}$, if for any sequence $\{A_n\}_n$ in $\mathcal{F}$ with A_n's pairwise

μ is countably additive on $\mathcal{F}$, if for any sequence $\{A_n\}_n$ in $\mathcal{F}$ with A_n's pairwise
disjoint, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

provided $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Now, an arbitrary collection $\mathcal{F}$ need not be closed under countable unions, i.e, it is not necessary that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Thus to verify countable additivity of μ, we only consider sequences $\{A_n\}_n$ in $\mathcal{F}$ with A_n's pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Note ③: Many authors put the additional

But there is a clarification here that if you consider any arbitrary set function, this need not be non negative, you can still consider the notion of countable additivity. And more interestingly, you can, in fact, consider general collections of subsets $\mathcal{F}$ which need not be significant and you can still consider this countable additivity. So, how do you define this?

So, again, we have to go back to the basic principle that if you have a sequence of sets in that collection, which need not be σ −fields now, if you have such a sequence, which are pairwise rejoined, once more, then you should be able to add up the sizes. So, that is the basic principle here. But then, the main concern is whether this union, this countable union of these sets is in your collection or not. If you are considering a general collection, which is not closed under countable unions, then this is difficult to ensure.

So, what you do is that, you take those sequences, so you take only those sequences with these two properties that the sets are pairwise disjoint and union is in the collections $\mathcal{F}$. So, just to clarify, earlier, when you had the σ −fields, then you did not have to restrict the collections of sequences that you are going to reconsider. But now, you are going to consider those specific sequences.

So, now, the restrictions is on the choice of the sequences is that you have to take these sets in your sequence to be pairwise disjoint and the union should be in the collection, only then you can talk about the size of this because the domain of μ is exactly $\mathcal{F}$. So, you can talk about that. So, that gives your left hand side in the countable addivitiy property. So, that is the size of the whole union. And on the right hand side, it is the sum of the individual sizes. So that is basically the idea. So, that is the general notion of countable additivity.

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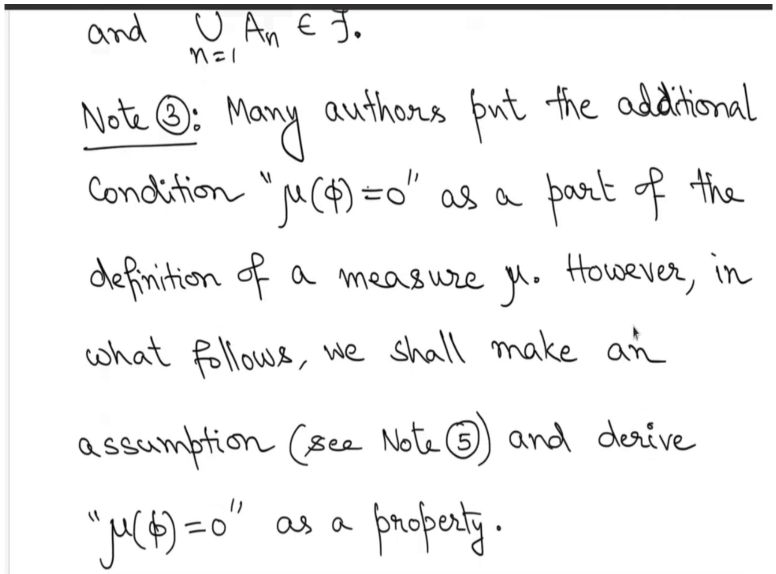

But let us now concentrate our attention to measures. So, again, measures were non-negative, countably additive set functions on σ −fields. Now, many authors put this additional condition that the measure of empty set is 0 as a part of the definition of a measure μ. So, we did not assume any such thing, but many authors would prefer to do that. What we are going to do, we are going to make an additional assumption later on which we will justify, of course, and we will motivate that.

And after that assumption, we are going to show that this fact that measure of the empty set is 0 followed as a property. So, we do not follow the other authors who would prefer to put it in the definition. We are going to make an assumption with some appropriate motivation and then we will derive it from the assumption. So, we will see that in note 5.

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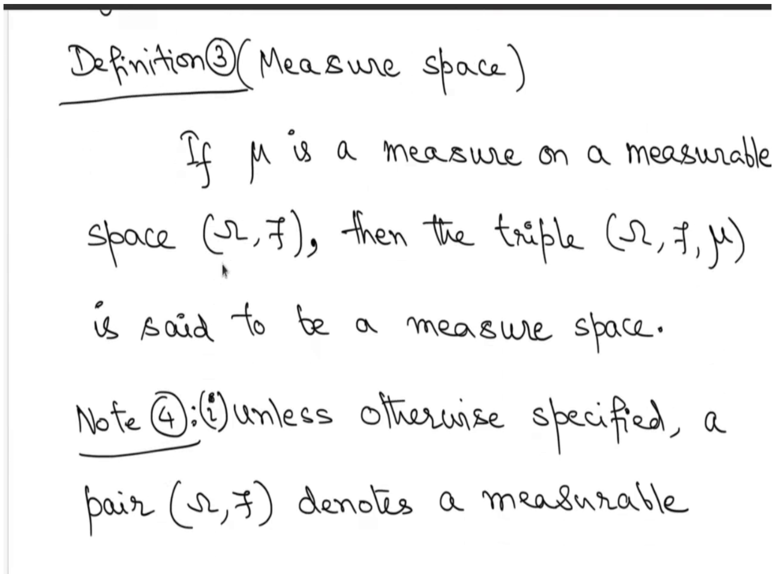

But before that, once you have a notion of a measure on a measurable space Ω f, then you can look at this triple. So, first is an empty set, then you have a significant on top of that and there is a measure defined on this collection of sets on this σ −field. So, if you have this triple like that, you are going to call it as a measure of space. So, first you had non-empty sets. You added a σ −field on top of that. You had these notions of measurable spaces.

Now, on top of this collection of sets, the σ −fields, once you have a measure with this non-negative countably addivitive set function then you get a measure space. So, on top of measurable spaces, if you have a way of measuring the sizes of sets, you will now call it measure spaces.

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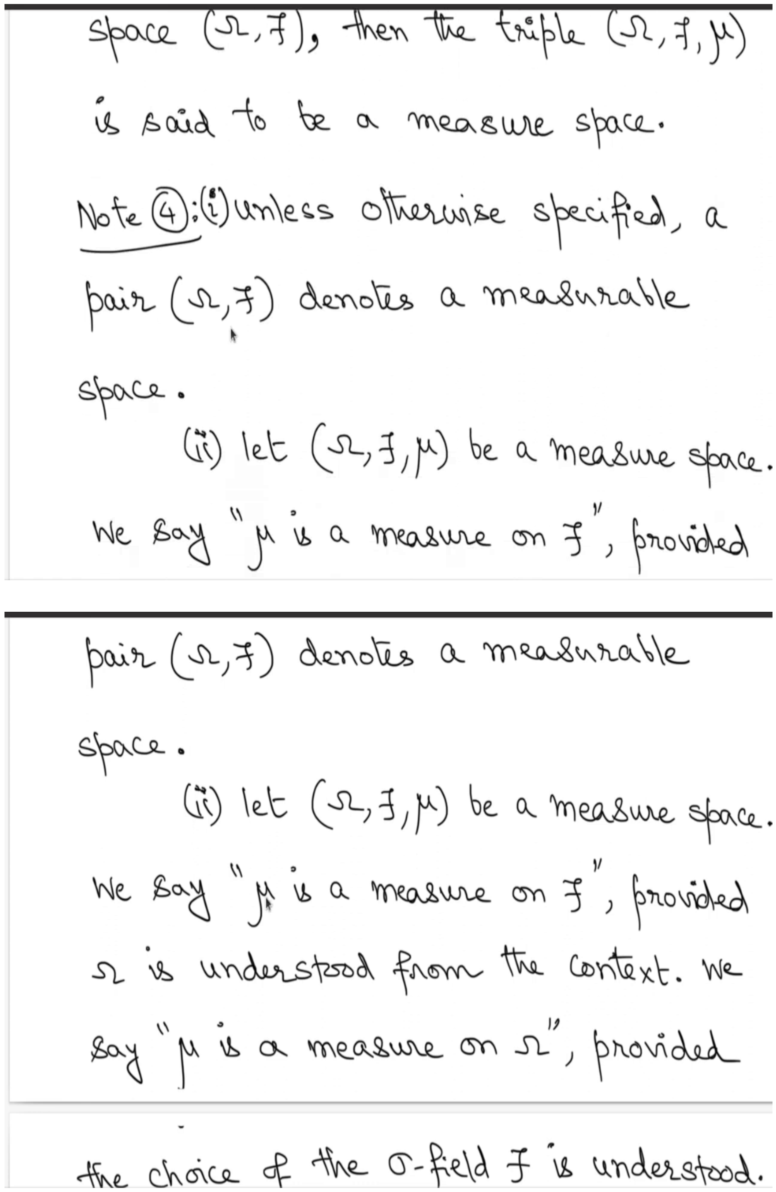

So, now, unless otherwise specified, if you are looking at this pair, if you see this notation anywhere, this will denote a measurable space. And now start off with this explicit clarification that if this is a measurable measure space, then we are going to say that μ is a measure on this σ −field $\mathcal{F}$, provided Ω is understood from the context. So, we are going to not write down all these things. We are going to simply say that μ is a measure on the σ −field $\mathcal{F}$.

On the other hand, there is also the equivalent statement that μ is a measure on the domain Ω. So, again, as long as the σ −field is understood from the context, you can simply write this. This is our way of simplifying the notation. As long as the signal fields and the domain is understood from the context, you can simply say that the measure is defined on the σ −field itself, instead of saying that it is defined on the measurable space or you can simply say that it is a measure on the domain set Ω instead of saying it is defined on a measurable space $(\Omega, \mathcal{F})$.

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the choice of the σ-field $\mathcal{F}$ is understood.

Examples of measures

First we list simple examples. More useful examples will be discussed later.

Check that the following non-negative set functions are measures on any measurable space $(\Omega, \mathcal{F})$ (Exercise)

(i) $\mu(A) := 0 \quad \forall A \in \mathcal{F}$.

(ii) $\mu(A) := \infty \quad \forall A \in \mathcal{F}$.

(iii) $\mu(A) := \begin{cases} 0, & \text{if } A = \phi \\ \infty, & \text{if } A \neq \phi. \end{cases}$

Note ⑤: The example (ii) above does not

So, with this clarification at hand, we are now going to start off with examples of measures. So, there are certain simple examples we are going to see, but more useful examples will be discussed later on, when we consider certain properties of these measures and then which will allow us to construct more examples. So, we are going to look at three specific examples, which are measures on any measurable space. So, this is very interesting.

You take any arbitrary measurable space and look at these three types of functions. And then we can show that these are measures. So, otherwise, there will be restrictions, depending on the choice of the Ω and $\mathcal{F}$, you will get different, different types of examples. But in general, these three types of examples are always there for you. So, what are these? Let us take them one by one.

So, the first example is that which associates to all sets with the value 0. So, you say that the all the sets are of size 0. So, that could, that is the first example. And how does this work? So, you take up pairwise disjoint sequences of sets, so you have a sequence of sets in your σ. And if the sets are pairwise disjoint then you look at their countable union. This is also in the σ $-$field.

Now, what happens? So, the countable union is in the σ. So, therefore, μ of the countable union as per the definition, this is 0, because all the sets gets associated the size 0. So, in your countable additivity property, one side is already 0. Now, we are going to consider the summation of the individual sizes, but then individual sizes are also 0. So, if you sum them up, you will again get 0 on the right hand side. So, countable additivity is verified for any sequences of sets, where the sets are pairwise disjoint. So, therefore, this first example is a genuine example of a measure on any arbitrary general measurable space.

What is the second example? Second example says that you associate the value ∞ to all the sets. So, you say that all the sets are of size ∞. Again, how does this work? So, you take again sequences of sets, these sequences, if it so happens that the sets that you have they are pairwise disjoint, look at their union that is again in the σ $-$field. Then what is the size of the union? Since all the sets are getting infinite size, so that union also has infinite size.

But on the right hand side, what happens you are adding up the individual sizes, which all of which are ∞. So, if you add up ∞ countably many times you still end up with ∞. So, therefore, this is also another example of a measure on a general measurable space $(\Omega, \mathcal{F})$.

Try to write down these arguments and also consider this third type of example which says that you will look at whether your subset is empty or not. If it is empty, you associate the value 0, otherwise you associate the value ∞. So, if your set has at least one element, if your set capital A has at least one element, you associate the size ∞. If the set is empty, you associate the value of

0. So, this is another example and try to see that this also satisfies the countable additivity, which has to be checked.

Now, one important thing that you need to keep in mind that these three set functions that we have taken are non-negative and taking values in between 0 and ∞. So that is something we will have to keep in mind. So, we are looking at non-negative set functions, possibly taking values ∞, but they should also satisfy countable additivity.

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Note 5: The example (ii) above does not have any physical significance and hence, we exclude this in further analysis. In what follows, we make the assumption that there exists $A \in \mathcal{F}$ with $\mu(A) < \infty$. Consequently, we get

with $\mu(A) < \infty$. Consequently, we get
the property discussed earlier in Note 3

Proposition 1: Let μ be a measure on $(\Omega, \mathcal{F})$. Then, $\mu(\phi) = 0$.

Proof: Let $A \in \mathcal{F}$ with $\mu(A) < \infty$.

Consider the following sequence of pairwise

But then we are going to look at the second example in a more detail now. So, what does this say? So, the second example says that we are going to associate infinite size to all the sets. However, this does not really make sense in physical sense. So, if everything has infinite size, then it is not really interesting. So, what we are going to say is that we are going to exclude this specific example from your further analysis. So, we do not want to say that all the sets have infinite size.

So, what follows, we make this assumption important, this is an assumption. So, there should be one set which has finite size or finite measure. So, measure of A, $\mu(A)$ is finite, it is not infinite. So, there should be such a set. So, this is the assumption that we are going to make. Consequently, we are going to get this interesting property that we have discussed earlier in earlier note. So, what is this? This is follows in the proposition.

So, what is this? So, suppose μ is a measure and then what will happen is that the measure of the empty set is 0. So, remember, we said that some of the authors put this in the definition, but now we are saying that we do not want to look at that specific example which associates infinite size to all assets.

If you want to exclude that example, then there will be at least one set which has finite size. And put that in your assumption implicitly that if μ is a measure, then you will expect to get one set where your size is finite. So, assuming that you can show that measure of the empty set is 0.

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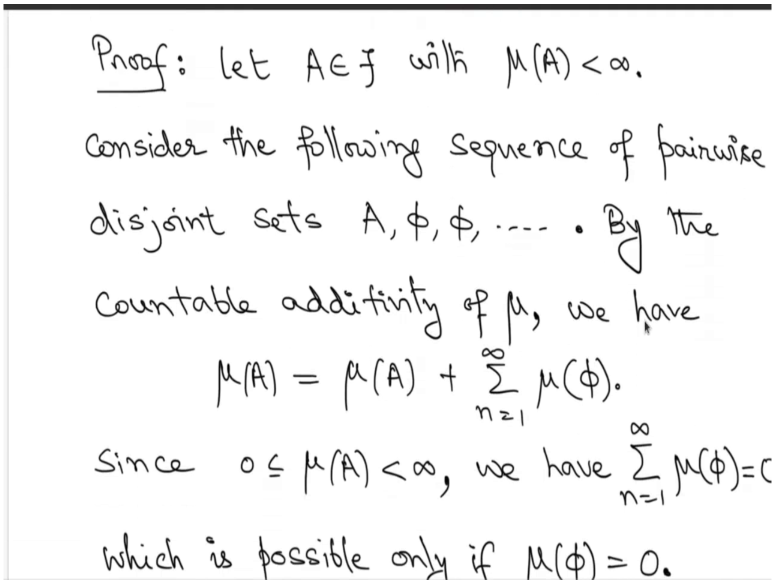

So, how do you show this? So, start with such a set which has finite size and consider this sequence of pairwise disjoint sets. So, take the first set as A itself and then from the second spot onwards, put the empty set. So, this is a sequence of sets in your σ −field. And this is also pairwise disjoint, so now you apply the countable additivity. What happens to the complete union? The countable union, it is just nothing but the original set A itself. So, therefore, on your left hand side in your countable additivity property is that the measure or μ of the whole union, which is nothing but the A itself so that happens to that happens to be on your left hand side.

What happens on the right hand side, you have to add up the individual sizes. So, first size that we need to consider is the size or the $\mu(A)$ itself and then you are adding up all the individual sizes of the empty sets. So, that gives you this additional series on your right hand side. But then, as long as you assume that measure of A, $\mu(A)$ is finite, you can cancel off from the right hand side and the left hand side.

And therefore, what you end up with is that the series summation measure or $\mu(\phi)$ is 0. Now, this is possible only if $\mu(\phi)$ is 0. Remember, $\mu(\phi)$ as per definition itself, it takes some value between 0 and ∞, ∞ included. But then if you have anything positive, anything non-zero, then immediately that summation will diverge because you are just adding up this constant value over and over again throughout the series.

So, therefore, for this to be 0, you must have μ(ϕ) to be 0. So, therefore, following this assumption that there exists a set with finite mass or finite size, you immediately get that μ(ϕ) is 0. So, this is something we are going to use from now onwards.

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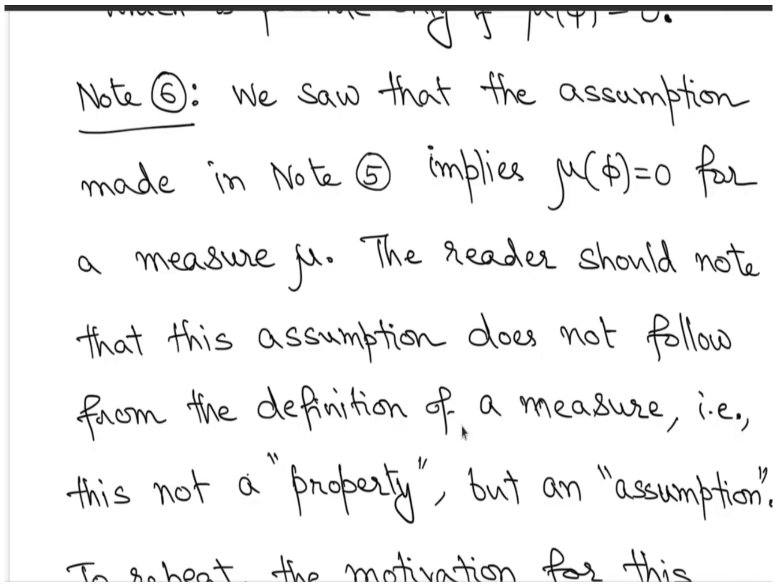

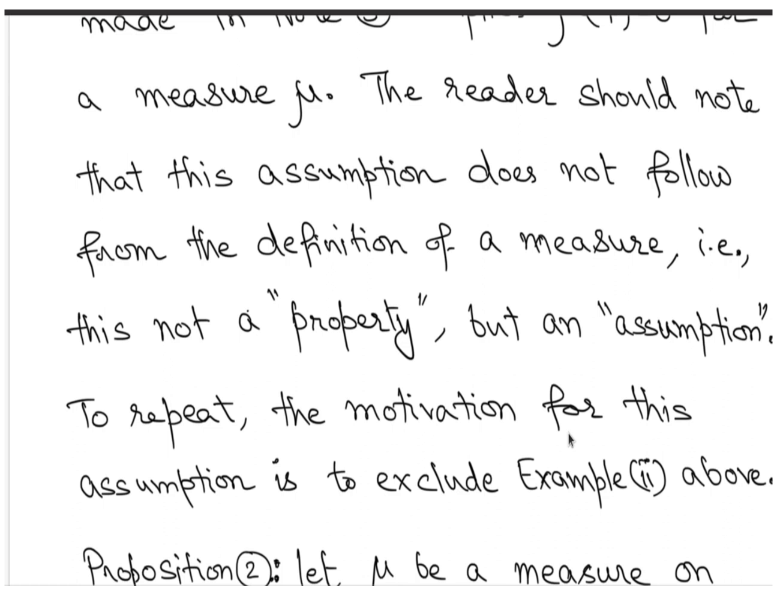

So, now, this is an important clarification. We saw that this assumption that we made implies that μ(ϕ) is 0. This is true for any measure μ. But now this is an important clarification that the reader should note that this assumption does not follow from the definition of a measure. We are putting it as an assumption. This is not a property of a measure. We are putting it as a assumption for our convenience. So, to repeat the motivation for this assumption was to exclude that example which associated infinite mass to all the sets.

So, therefore, from now on we are going to make this assumption, but you should be clear that this is an assumption and not a property. So, this does not follow from just taking a measure. So, if you just have a measure, it does not follow that there exists a set with finite mass.

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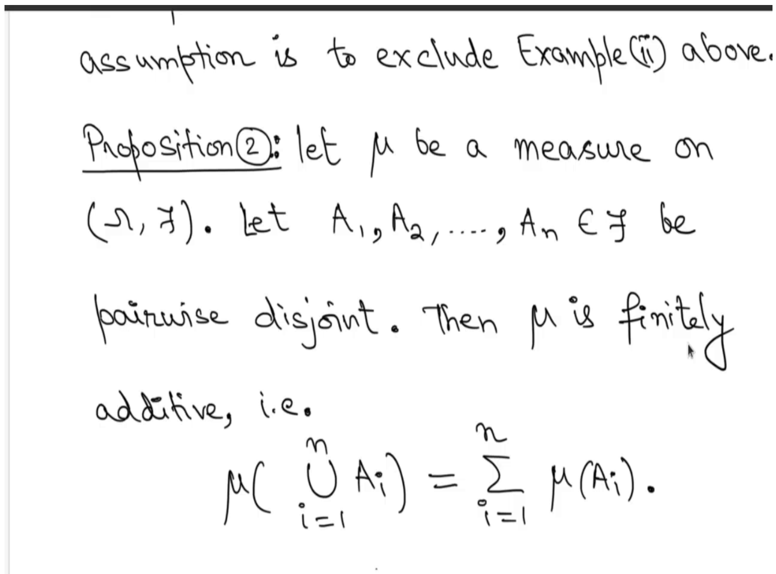

So, now, since the measure of the empty set is 0, this now allows us to prove certain nice properties. So, what are these? So, if μ is a measure and take finitely many sets A_1, A_2, . . .,A_n in your σ −field, if they turn out to be pairwise disjoint, so you have a finite collection of sets A_1, A_2, . . ., A_n and if they are pairwise disjoint, then you get what is known as finite additivity of the set function μ or the measure μ. What is this?

So, you look at this finite union. So, you have to discuss that this finite unions are also in the σ −fields. So, μ of this finite union is equal to the this finite summation of this individual sizes. So, $\mu(A_i)$ you added them up, you get back the μ of the whole union.

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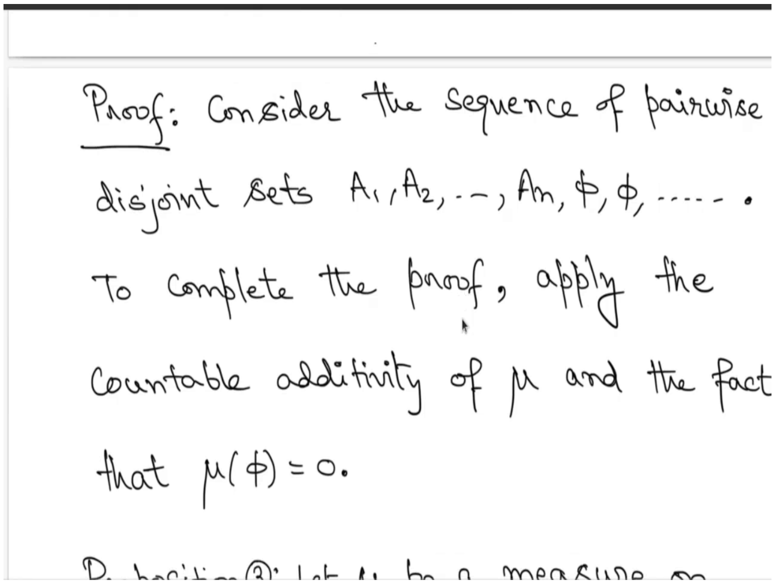

So, how do you prove this? So, this is simple to prove from countable additivity. What you do? You take the original sets as given A_1, A_2, . . ., A_n, they are given to be pairwise disjoint, but from n plus oneth place onwards you put the empty set. So, therefore, you get sequence, but this sequence turns out to be pairwise disjoint. Now, apply countable additivity, what will happen is that, on your left hand side, you are looking at the union of this sequence of sets. So, that union is nothing but this finite union. So, that turns out to be on the left hand side of this countable additivity property.

On the right hand side, you will first add up these sizes of these sets, individual sizes, so you will get measure of A_i, measure of A to measure of A_n and then measures of the empty sets. But since you have proved that measures of empty set is 0, there will be no contribution in that series and therefore, it just reduces to the $\sum_{i=1}^{n} \mu(A_i)$. So, that allows us to prove countable additivity implies finite additivity provided $\mu(\phi)$ is 0. And that followed from that assumption that we have made.

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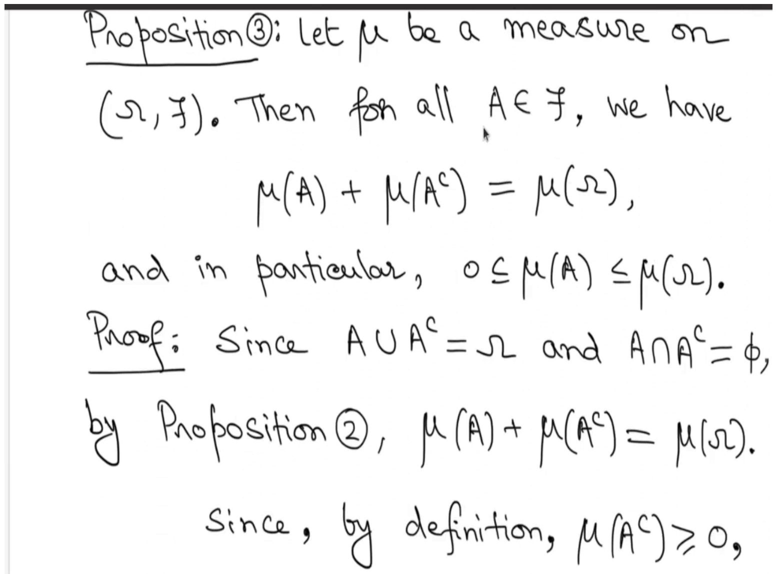

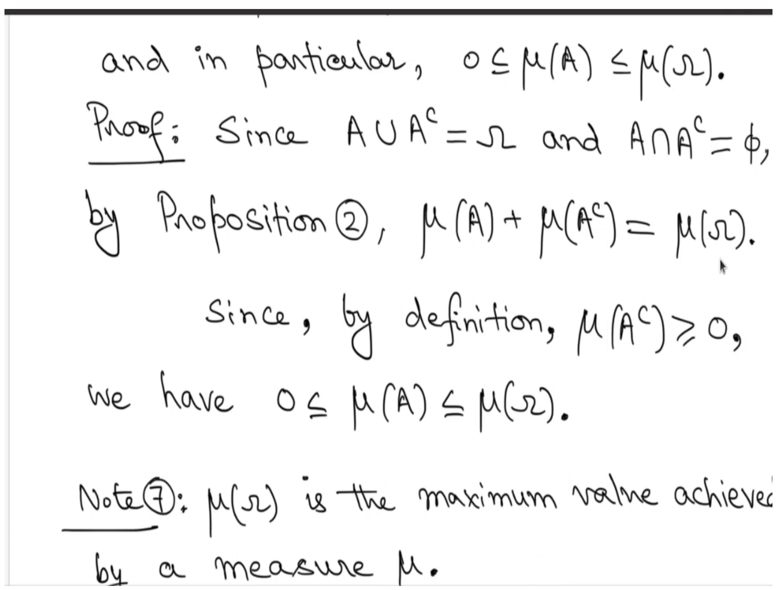

But then this finite additivity has these interesting applications. So, what are these? So, if you have a measure, then for all sets in your collection, in your σ −field, you have $\mu(A) + \mu(A^c)$. If you add them up, they should match with $\mu(\Omega)$. And in particular, you also have this interesting inequality that says that for all subsets $\mu(A)$ is dominant from above by μ of the whole set.

So, how do you prove this? So, first start with this proof of this statement $\mu(A) + \mu(A^c)$ becoming $\mu(\Omega)$. So, how do you prove this? So, take a look at this disjoint union $A \cup A^c$. That union is nothing but the whole set Ω and since this is disjoint their intersection is of course empty. So, now, if you apply countable additivity or the application of countable additivity to finite additivity, if you apply finite relativity to here, then you get $\mu(A) + \mu(A^c)$, if you add up these sizes, you should get the μ of the whole union which is nothing but the Ω.

So, therefore, from finite additivity, you get this property. But then you observe that for any set A, A complement is also in your σ −field and since μ is non-negative, so it takes non-negative values, so $\mu(A^c)$ is non-negative. So, from the equality that $\mu(A) + \mu(A^c) = \mu(\Omega)$, you immediately say that since $\mu(A^c)$ is certain non-negative quantity, you must have that $\mu(A)$ dominated from above by the $\mu(\Omega)$.

So, therefore, what you are basically saying is that $\mu(\Omega)$ is that maximum possible value that is achieved by this measure μ. So, remember, μ is a function defined on this collection of sets and you are looking at individual sizes of the sets and the maximum possible value achieved by this function which is given by $\mu(\Omega)$ itself. The maximum value is achieved at the whole set.

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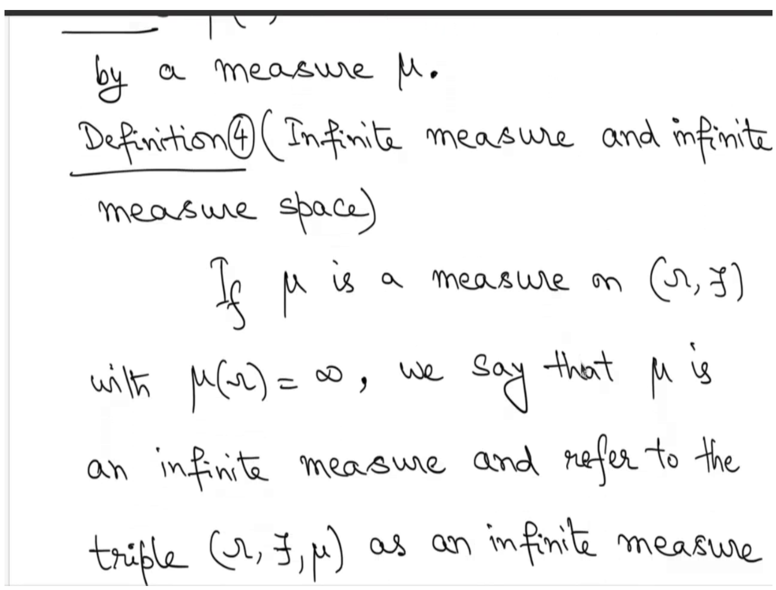

If μ is a measure on $(\Omega, \mathcal{F})$ with $\mu(\Omega) = \infty$, we say that μ is an infinite measure and refer to the triple $(\Omega, \mathcal{F}, \mu)$ as an infinite measure space.

<u>Definition</u> 5 (Finite measure and finite measure space)

If μ is a measure on $(\Omega, \mathcal{F})$ with $\mu(\Omega) < \infty$, we say that μ is a finite measure and refer to the triple $(\Omega, \mathcal{F}, \mu)$ as a finite measure space.

So, with this at hand, we are now going to define these interesting concepts called infinite measures and infinite measure spaces and correspondingly finite measures and finite measure spaces. So, if μ is a measure with the property that the measure of the whole set is infinite, then we say that μ is an infinite measure and refer to this triple as an infinite measure space. So, it is pretty simple. You already said that μ(Ω) is the maximum possible value. Look at that value. If it is infinite, then you will say that μ is an infinite measure and the corresponding measure space will be said to be an infinite measure space.

And you can immediately guess that for finite measures case, all you need is to restrict your attention to the measure of the whole set being finite. So, again μ(Ω) is the maximum possible value achieved by this function μ by the measure μ. And if this is finite, then we are going to say that μ is a finite measure and the corresponding triple, corresponding measure space will be referred to as a finite measure of space. Now, you are going to ask what are examples of this finite measure spaces or infinite measure spaces.

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Example (Counting measure)

Let Ω be a non-empty set. Consider the measurable space $(\Omega, 2^{\Omega})$. Let μ be the non-negative set function on 2^{Ω} defined by

$$\mu(A) := \begin{cases} \#A, & \text{if } A \text{ is finite} \end{cases}$$

the non-negative set function on 2^{Ω} defined by

$$\mu(A) := \begin{cases} \#A, & \text{if } A \text{ is finite} \\ \infty, & \text{if } A \text{ is infinite} \end{cases} \quad \forall A \in 2^{\Omega}.$$

Here, $\#A$ denotes the number of elements in A. Check that μ is a measure on

$\forall A \in 2^{\Omega}$.

Here, $\#A$ denotes the number of elements in A. Check that μ is a measure on $(\Omega, 2^{\Omega})$. Since the values of $\mu(A)$ are exactly the number of elements, μ is called the counting measure.

To go to that we now see this interesting example called the counting measure. So, what does counting measure do? You take a non-empty set Ω and consider this measurable space. So, Ω together with the power set σ which is nothing but the collection of all subsets. So, on this measurable space look at this non-negative set function μ. So, what you do? So, look at any arbitrary subset that is in your power set, look at any arbitrary subset, if it so happens that this arbitrary subset is finite, if it has finite number of elements, then you just add up the number of elements and put that in. So, that is the value or the size of the set.

If the set is infinite, meaning if it has infinitely many elements, then you associate the value ∞. So, just to clarify, if your set has finitely many elements, you will just put that number as the size of the set. Otherwise, if the set is countably infinite or uncountably infinite you associate the value ∞. Again, this $\#A$ denotes the number of elements in A. So, this is for the case when the set A has finite number of elements.

Now, you can easily check that this definition of a non-empty set function is actually countably additivie and therefore, this becomes a measure on this measurable space. Now, please check this. Since the values of these sets or the sizes of these sets are exactly the number of elements, μ is called the counting measure. So, this set function μ what it does, it just tells you the number of elements in your set and that is why you call it a counting measure.

Now, if you have a finite set Ω, if your original set or the whole set is itself is finite, then μ(Ω) is finite just by definition, and therefore, it becomes a finite measure. So, counting measures becomes finite if Ω itself has finite number of elements. If Ω has infinite number of elements, then $\mu(\Omega)$ is infinite. Therefore, μ becomes an infinite measure. So, the corresponding measures spaces become finite measures or infinite measures accordingly.

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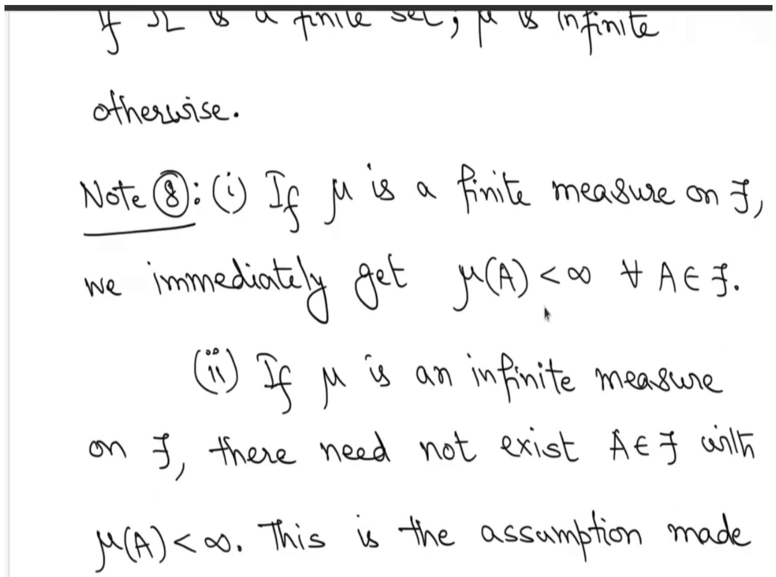

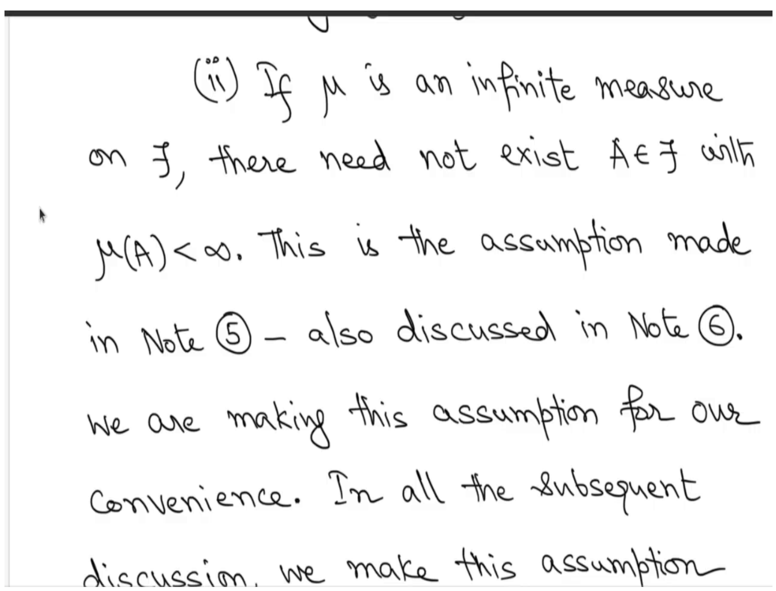

in Note (5) — also discussed in Note (5).
we are making this assumption for our convenience. In all the subsequent discussion, we make this assumption implicitly.

Now, if you have a finite measure, we are immediately getting this fact that $\mu(A)$ is finite. This is because $\mu(\Omega)$ is the maximum possible value achieved by the set function μ, by the measure μ and this maximum value if it is assumed to be finite, then of course, all the individual subsets will also have finite sides. So, therefore, if μ is a finite measure, then you immediately get this property. But just to clarify that assumption that we have made earlier, this is important for the case of an infinite measure. If you have an infinite measure, there need not exist any set A in your σ −field such that $\mu(A)$ is finite.

First of all the measure of the whole set is ∞. And remember that example where the measure was associating value ∞ to all the sets. So, if you just have given infinite measure on some σ −field or on some measurable space, this immediately does not follow that there is a set with finite mass, finite size or $\mu(A)$ being finite. It did not follow. This does not follow from the definition. But we had made it an assumption, and therefore, we are getting all these interesting properties.

So, this is the assumption that we are making here and this is what I am repeatedly stressing on. And this assumption is being made for our convenience only, because you do not want to consider that specific example that is associating infinite mass to all possible sets. That is not really interesting. So, in all subsequent discussions, we make this assumption implicitly. We are not going to state it anymore. We are going to use this and use the fact that the measure of the empty set is 0 and all the subsequent properties.

So, the idea is this, that if you are just given an infinite measure, then it is not necessary that there should be a set with finite mass, but for our convenience, we made this as an assumption and therefore, we are getting these nice properties that follow afterwards. So, in all subsequent discussions, whenever we are discussing other properties and so on, we have, we are going to make this assumption implicitly. So, we are going to stop this lecture. We will stop here and we will continue the discussion in the next lecture.