Measure Theoretic Probability 1

Prof. Suprio Bhar
Department of Mathematics & Statistics
Indian Institute of Technology, Kanpur

Lecture No. 05

Limits of sequences of sets and Monotone Classes

Welcome to this lecture. As before, let us first recall what we have discussed so far and then we shall start with the topics of this lecture. So, this week, we have started talking about events coming from certain random experiments. And when you look at the collection of all events in connection with random experiments, we end up with certain structures on these collections of events. Remember that these events are special subsets of the sample space, where sample space means the set of all outcomes.

Now, once you have that collection of all events, we considered the structures of σ −fields and fields on top of it, we studied general definitions of σ − fields and fields corresponding to any non-empty set. We looked at many of the examples of these we had shown that all σ −fields are fields but not the converse. So, we have shown an example of a field which is not a σ −field.

For examples of fields we have done two different types of approaches, one was a bottom-up approach starting from the trivial σ −field and adding sets to the and obtaining examples. Another was a top-down approach, starting with the power set σ −field and removing certain sets in a certain way, so that I get σ −field. In the second way, we have constructed what we call as σ −field generated by certain collection of sets.

And in this regard, in this way, we have obtained a very important object or very important σ −field on the real line, which we called as the Borel σ −field. This was defined as the σ −field generated by the open sets in the real line. We have shown, after that we have discussed that there are other the classes of sets, which also generate the same σ −field, the Borel σ −field on R, for example, all closed sets will also generate the same things.

We have looked at open intervals, left open right closed intervals and so on. And all we have seen is that, if we look at classes of such sets, then they generate the same σ −field, the Borel σ −field. So, Borel σ −field has a very nice collection of events or very nice collections of subsets of the real line. We have also remarked that there are the sets, there are subsets of the real line, which are not Borel σ −field meaning not in the Borel σ −field, we shall see examples of those later on.

And one other thing we have discussed in the last lecture in fact is that there are notions of Borel σ −field on extended real lines and also for higher dimensional Euclidean spaces and also on Borel subsets of the real line. So, we have discussed all these concepts. Now, with that at hand, we now start with today's lecture, we are now moving on to the slides.

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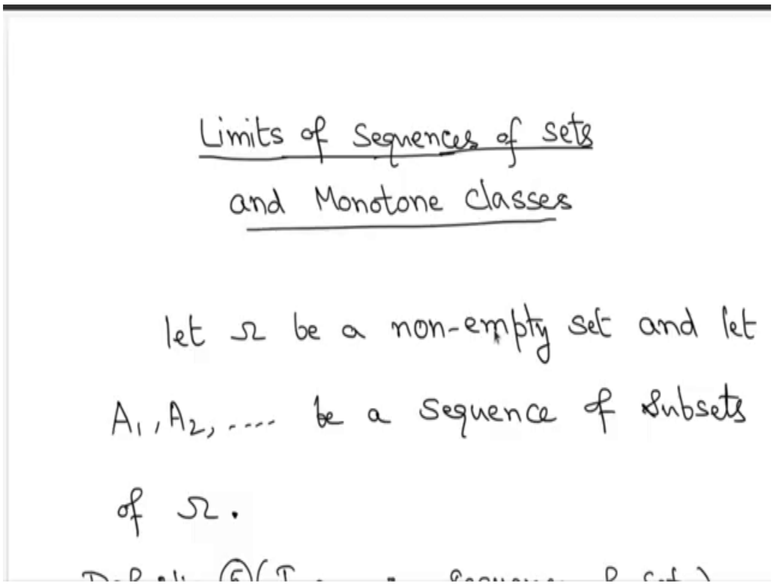

So, this is involving limits of sequences of subsets. So, I am given some non-empty set call it Ω and I am interested in sequences of subsets. So, $A_1, A_2, \ldots, A_n$ and so on. Now, the idea is this involving the σ −fields, we have been able to work with countable unions and countable intersections. And these in some sense are certain kind of limiting operations.

And that is what we want to highlight and do much more closer analysis in this lecture. So, we start off with some ideas of sequences of sets and we introduce some certain notions of monotonicity involving sets. So, that we can call that a certain sequence maybe increasing or a certain sequence may be decreasing. So, let us start with the definitions.

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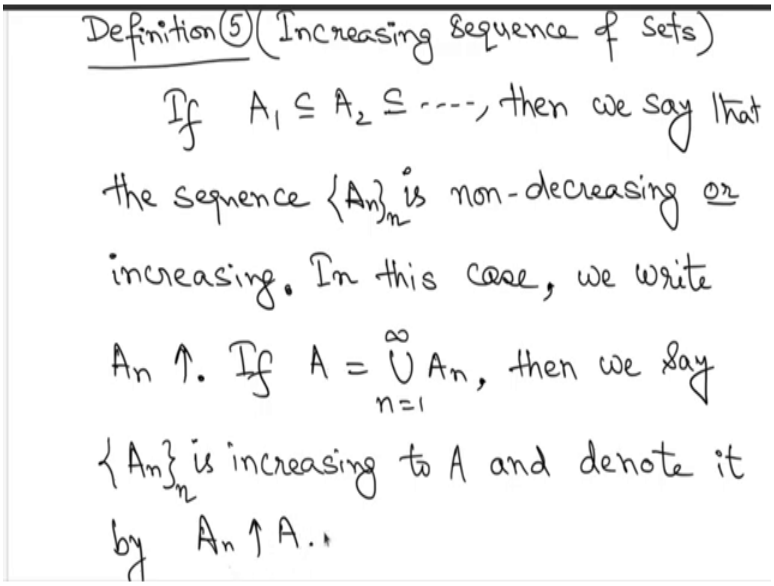

So, I say that, I am given such a sequence of sets A_1, A_2, . . ., A_n. If it so happens that

$A_1 \subseteq A_2 \subseteq A_3$. . . . So, $A_n \subseteq A_{n+1}$. So, then we will say that the sequence $\{A_n\}$ is non decreasing or increasing. In this case, what happens is that if you look at the countable union of these sets, that is what I call as A, let us say then we shall say that $\{A_n\}$ is increasing to A.

So, you have these sets A_1, then it increases to A_2, then it increases to A_3. So, you look at all these unions of all these sets, as they increase you are getting bigger and bigger sets. And at the end, if you denote that set as A, then we say that the sequence $\{A_n\}$ is increasing to A. Now, in this regard, we shall use this notation $A_n \uparrow$ to denote that the sequence is increasing or non-decreasing.

So, here the notion of monotonicity is simply involving set inclusions. So, An should be included in A_{n+1} and this should happen for all n. Great. Now, I am looking at this notation that if A is the complete union of all these sets A_n, I will write A_n increases to A, so this $A_n \uparrow$ A. So, that is what the notation means, we shall see examples of this.

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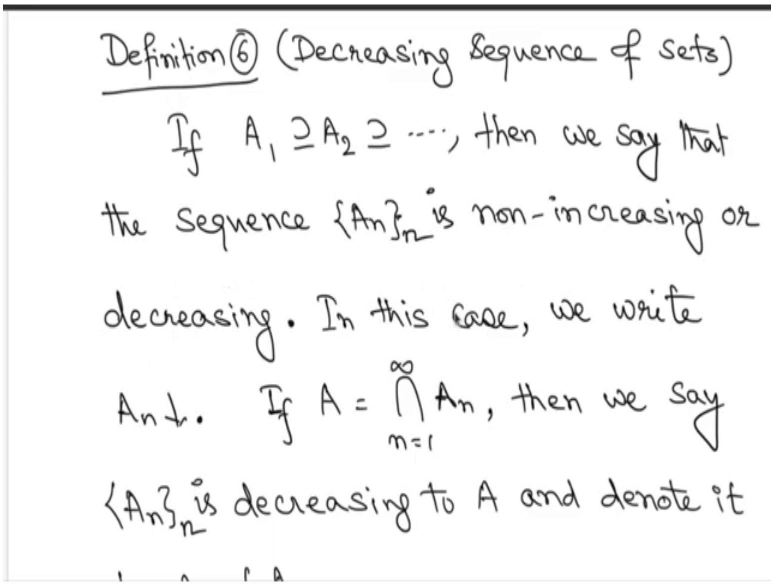

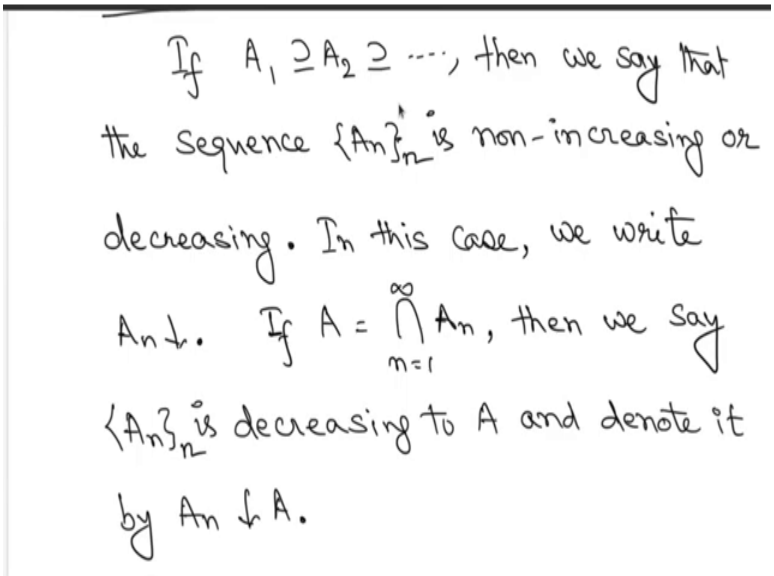

But conversely, you expect to have a notion in the other way. So, if it So, happens that A_1 is the largest set, then you get A_2, so that is slightly smaller, so $A_2 \subseteq A_1$, then $A_3 \subseteq A_2$ and so on. So, $A_n \supseteq A_{n+1}$, if it happens for all such n, then we say that the sequence n is a non-increasing or decreasing. So, in this case, we are using this notation as expected $A_n \downarrow$. So, this will mean that the sequence of sets decreases.

Now, once you look at the sequence of sets, you expect that you first to start with A_1 then you go to a smaller set. So, you can think of A_2 as the intersection between A_1 and A_2, then when you look at A_3, you can think of it as a intersection between A_1, A_2 and A_3. So, in this way, you can expect to get the limit to be this complete intersection of all these sets. And let us say you denote this complete interception by A then we also going to say that this sequence $\{A_n\}$ is decreasing to A.

So, provided the sequence is decreasing, look at this complete intersection and write that A_n decreases to A or $A_n \downarrow A$. So, this simply means that these sets will decrease and are going to decrease to the complete intersection, this complete countable intersection. So, we have these two notions that we have introduced and what you require is that first of all for increasing sequence you require $A_n \subseteq A_{n+1}$ for all n. And for decreasing sequences you require $A_n \supseteq A_{n+1}$. So, you have a decreasing sequence of sets in the second case.

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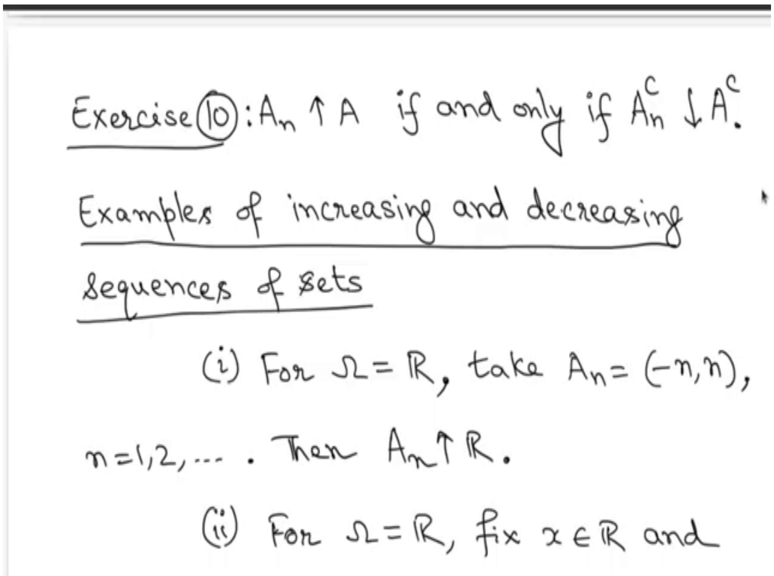

So, now, you first make a connection with these two notions. So, you can immediately try to check that it actually shows that if $\{A_n\}$ is a increasing sequence of sets and increases to a then it will happen if and only if their complements are decreasing and it actually decreases to the set A complement. So, try to check this. So, they should this is an if and only if condition that connects the notion of increasingness to the decreasingness of another sequence. So, just to understand this notion better, let us look at certain examples.

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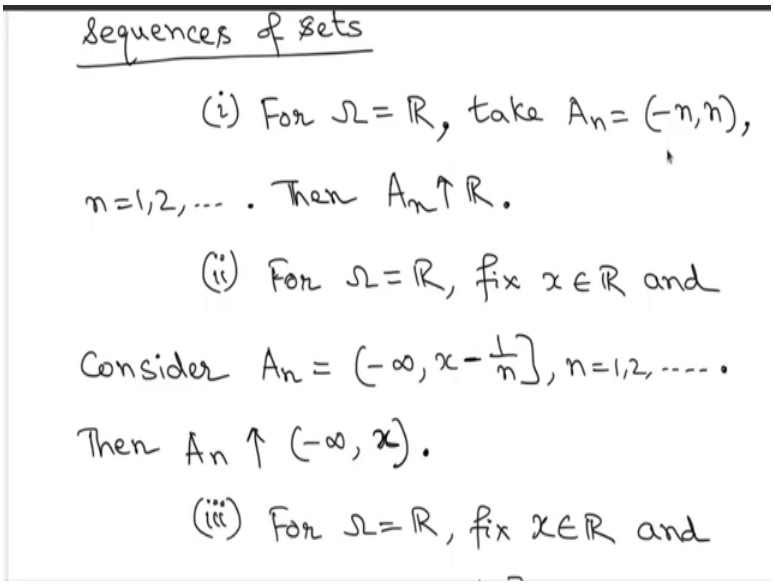

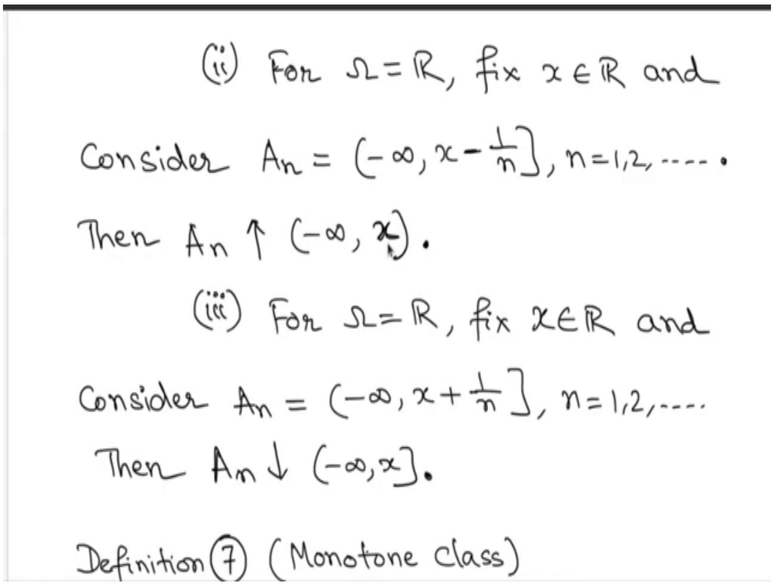

So, take this sequence of intervals $\{A_n\}$ on the real line. So, An is minus n to plus n. So, for n 1, 2, 3 all the non-negative integers, all positive integers. Then what happens? If you look at A_{n+1} that starts from $-(n+1)$ to $(n+1)$. So, therefore, A_{n+1} is a bigger set and it will contain A_n. So, since $n \in A_{n+1}$ and this is true for all possible n then $\{A_n\}$ is increasing.

Now, you are interested in what happens to the limit. So, the sense of limit here is simply for the increasing sequence of sets, all you have discussed is the complete union. So, what is the complete union of all these A_n's is that the whole real line. So, what do we expect is that you look at all these intervals $(-n, n)$, take their union as n goes from 1 to ∞, it will cover the whole real line. So, the complete union is the whole real line. So, that is the first example that you keep in mind.

Now, here is a very interesting example. So again, we are restricting our attention to the real line. So, fix a point x, a real number and then look at these type of sets A_n. So, this is again, minus infinity to the closed, so this is a closed interval $[-\infty, x - \frac{1}{n}]$, so therefore, these sets will look like something like this. So, $[-\infty, x - 1]$, so that corresponds to $n = 1$. Then for $n = 2$, you get the set $[-\infty, x - \frac{1}{2}]$, and so on. Now, we can easily check that these sets A_n are increasing.

And now we are interested in what is the complete union. And it so happens that if you try to verify this complete union, you can rewrite it in this notation, that it actually the complete union is actually this open interval minus infinity to open x, so it will not contain the point x, but it will be $(-\infty, x)$. So, that is a very interesting observation that you have these collections of closed intervals, but their countable union is giving you an open interval. So, please check this.

And similar fashion, there is this interesting example. So, here I am now, instead, I am again fixing x, but I am looking at a different types of closed intervals. So, I am taking $(-\infty, x + \frac{1}{n}]$. So, again, for $n = 1$, $(-\infty, x + 1]$, then $n = 2$, you will get $(-\infty, x + 1/2]$ and so on. Now, you can again check that A_n's are decreasing and what is the limit. So, the limit here, notion of limit is simply the complete intersection.

So, if you now try to check that the what is the complete intersection here, you will end up with a closed interval. So, intersection of all these closed intervals will give you this close interval $[-\infty, x]$. So, again, try to check this. So, this is a very interesting notion of convergence for the sets. So, you have certain limits of sequences here, when sets are increasing in a certain way, or decreasing in a certain way. So, using these notions, we now introduce a interesting type of collections of sets.

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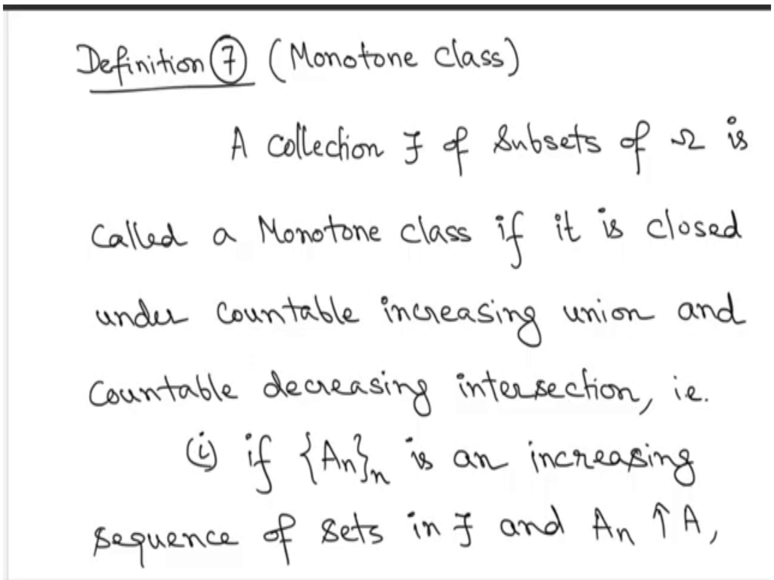

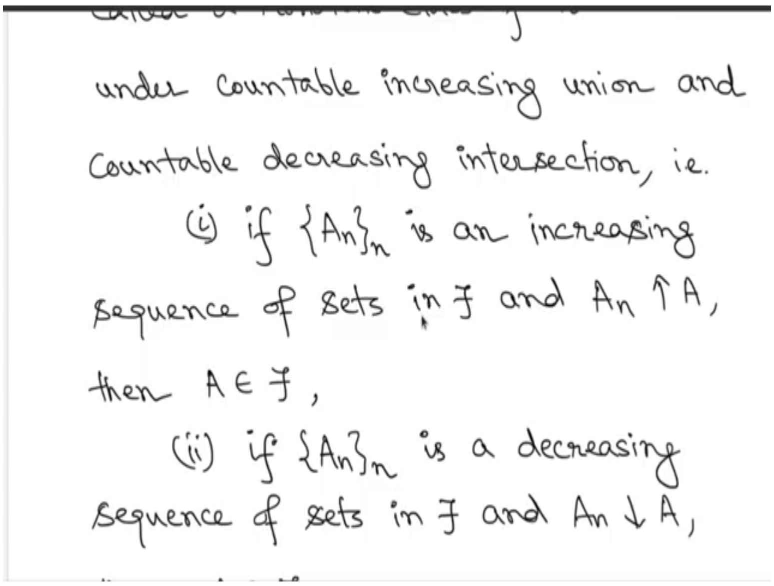

So, I look at a collection of subsets of Ω, some non-empty set, and let us call this collection $\mathcal{F}$. This is some arbitrary collection so far, but I ask that there are certain properties. So, I will call this collection as a monotone class. If it is closed under countable increasing union and countable decreasing intersections. What do I mean by this again? So, let us go to the exact descriptions of the operations, countable increasing unions and countable decreasing intersections.

So, the first property means that you take an increasing sequence of sets and look at their countable union. So, as described earlier, if you are looking at an increasing sequence of sets, you are going to look at the complete union, the complete countable union. And since these sets are increasing, you can call this complete union as a countable increasing union. So, that is what I mean.

So, what do we want is that if you have an increasing sequence of sets in this collection, $\mathcal{F}$, I want their complete union to also belongs to the collection $\mathcal{F}$. So, that is the first property and this is what we will call as closure under countable increasing union. And once you have that property, you again will expect that the similar property for the intersections.

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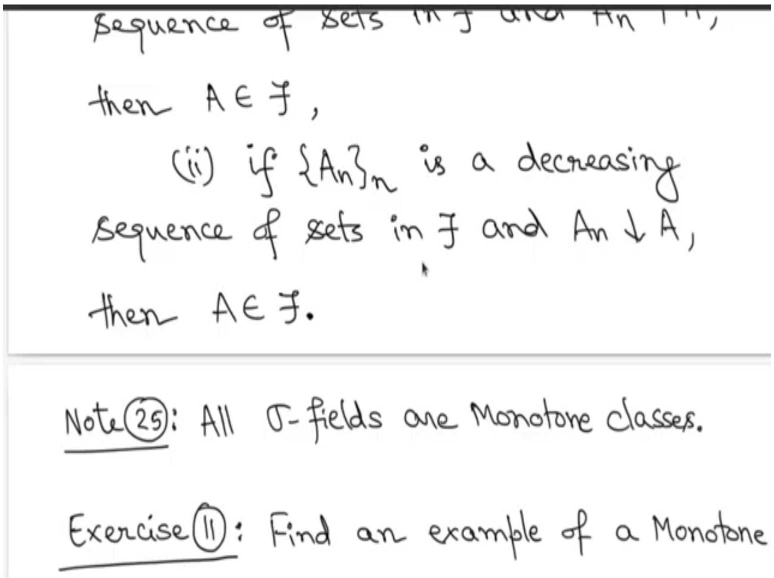

So if you have a decreasing sequence of sets in the collection $\mathcal{F}$, and if it so happens, that A_n's will decrease, and therefore they are going to decrease to the complete intersection. Now, if A_n's are decreasing, then the complete intersection you can now visualize as the countable decreasing intersection. So, that is what I call here. And therefore, that is what your set A is.

And here what you want is that if $\{A_n\}$ is a decreasing sequence of sets and coming from this collection $\mathcal{F}$, I want that complete countable intersection, the countable decreasing intersections to belongs to that collection once more. So, this is why I will call as closure under countable decreasing intersections. So, once you have a collection of subsets $\mathcal{F}$ with these properties, that it is closed under countable increasing unions and countable decreasing intersections, I shall call it a monotone class.

So, now all these operations that you expect, I want to see examples of this. So, the first examples that you will immediately get are σ −fields. So, no matter what kind of sets you take, as long as they are increasing and looking at countable unions of those, so countable unions are allowed in σ −fields. So, therefore, these σ −fields will satisfy closure under countable increasing unions. So, remember σ −fields will support countable union.

So, in particular, it will support countable increasing unions, so, therefore, it is closed under countable increasing unions. Similarly, for a σ −field, you have closure under countable interceptions. So, in particular, it will also support closure under countable decreasing intersections. So, therefore, all σ −fields are monotone classes. We are not going to look at many general classes of monotone classes, many explicit other examples, we will restrict our attention to σ −fields.

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then $A \in \mathcal{F}$.

Note (25): All σ-fields are Monotone classes.

Exercise (11): Find an example of a Monotone Class, which is not a σ-field.

Exercise (12): Let $A_1, A_2, \ldots, A_n$ be sets in a field $\mathcal{F}$. Consider the following sets $B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$ defined by

Exercise (12): Let $A_1, A_2, \ldots, A_n$ be sets in a field $\mathcal{F}$. Consider the following sets $B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$ defined by

$$B_1 := A_1, \quad B_i := A_i \cap \left(\bigcup_{j=1}^{i-1} A_j \right)^c$$

for $i = 2, \ldots, n$.

$$C_i = \bigcup_{j=1}^{i} A_j, \quad i = 1, 2, \ldots, n.$$

for $i = 2, \ldots, n$.

$$C_i = \bigcup_{j=1}^{i} A_j, \quad i = 1, 2, \ldots, n.$$

Verify that (i) $B_1, B_2, \ldots, B_n \in \mathcal{F}$.

(ii) $B_i \cap B_j = \phi$ for $i \neq j$,

i.e. they are pairwise disjoint.

(iii) $C_1, C_2, \ldots, C_n \in \mathcal{F}$

(iv) $C_i = \bigcup_{j=1}^{i} B_j, \quad i = 1, 2, \ldots, n$

But it is a good exercise to find an example of a monotone class, which is not as σ −field, try to work this out, there are many, many examples, you can work them out. So, it is so, we are not going to use too much of monotone classes in this course, we are not going to spend too much time on it. But it is always good to have an understanding of monotone classes and that is why this exercise will help you. So, please try to find examples of monotone classes, which are not σ −fields.

In fact, this monotone classes and σ −fields and fields, they have very interesting connections in between them. So, just to describe that, here are now a set of exercises for you. So, exercise 12 here, this is in relation to a field. And that is what something we are going to use later on, let us take some number of sets a finite number of sets in a field. So A_1, A_2, A_n are sets in a field. Consider these sets let us say $B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$, I define them. So, what are these $B_1, B_2, \ldots, B_n$?

So, I define B_1 to be A_1 itself, but I look at B_i for $i = 2$ onwards for next numbers onwards to be Ai, but I remove all the items, all the elements that I have seen so far. So by that I mean that I look at $\bigcup_{j=1}^{i-1} A_j$, which we have already encountered so far. And then I take them out from Ai. So, I look at A_1 for B_1, then B_2 is simply $A_2 \cap A_1^{\,c}$, then B_3 will be $A_3 \cap (A_1 \cup A_2)^c$, so like that.

So, we are looking at the new terms that are arriving in A_i. But I do not want the terms that I have already seen up to A_{i-1}. So, those are I am removing. So, that is what, that is how I am defining B_i's. And what are C_i 's? C_i 's are nothing but $\bigcup_{j=1}^{i} A_j$. So, C_1 is A_1 itself, but C_2 is $A_1 \cup A_2$. So, what you can now try to show is that if the sets A_1, A_2, A_n are in the field, then the sets B_1, B_2, . . ., B_n are also in the field, but using the structures of the B_i 's as defined here, you can try to show that B_i and B_j are pairwise disjoint by that I mean that if i naught equals to i, then B_i and B_i will not have any common element. So, their intersection is an empty set.

Similarly, for the C_i's, which are defined as some kind of a cumulative union. So, what you are having is that for, if A_1, A_2, A_n are in the field, then C_1, C_2, C_n and also in the field. Interestingly, you can also show some certain connections with Ci's and the sets B_j 's. So, you can again show that C_i is nothing but again kind of a cumulative union of the B_j 's. So try to verify these relations.

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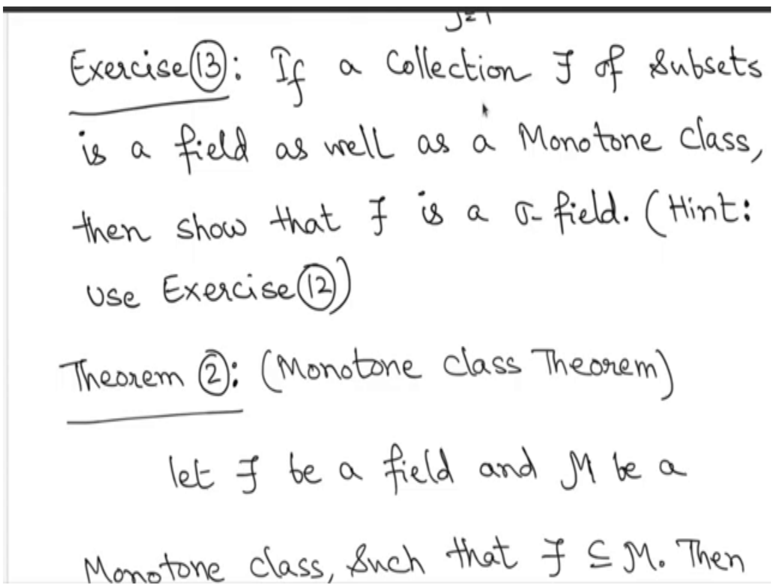

And then here is the interesting connection, that if you can have a collection of subsets, which is a field as well as a monotone class, so remember, fields will allow you complementation finite unions and finite intersections. A monotone class allows you countable increasing unions and countable decreasing intersections. So, suppose a collection of subsets allow all these operations for you, then try to show that $\mathcal{F}$ must be a σ $-$field.

So, field together with a monotone class implies you actually get a σ $-$field and a hint for this is use this previous exercise that we have just discussed. And an important theorem or important result follows these ideas. And this is what is called the monotone class theorem.

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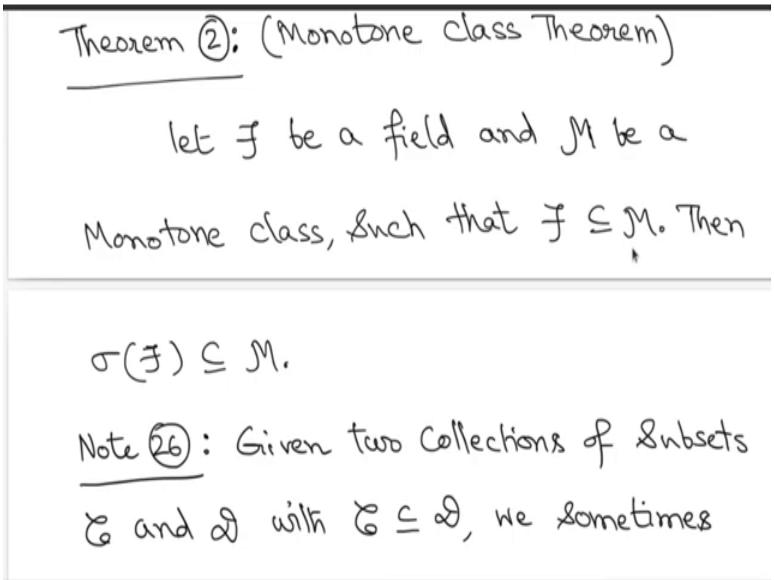

So, let us come to this point. So, let $\mathcal{F}$ be a field and $\mathcal{M}$ be a monotone class with the property that the field is contained in a monotone class. So, I am looking at non-empty set Ω, I am looking at some collection of subsets, which is $\mathcal{F}$ with these complementation finite unions and intersections that is a field and then I am looking at a monotone class $\mathcal{M}$, again on the same non-empty set.

So, on monotone class I have countable increasing unions and countable decreasing intersections with the property that the original list of field, original collection of sets $\mathcal{F}$, that is a field all those sets are already included in the list in the collection $\mathcal{M}$. So, that is what this inclusion means, as we have discussed earlier.

Now, the conclusion of the theorem says that under this situation, the σ −field generated by the collection by the field is contained in the monotone class, this is a very important result that connects fields the generated σ −field and the monotone class. So, again just to repeat, if you have a field that is contained in a monotone class, then the σ −field generated by the field is also contained in a monotone class.

So, all we are saying is that, if you have a field, then you will also expect to have sequences of sets from the field and then their complete unions and things like that also will belong to the monotone class. That is what this generated σ −field contained in monotone class implies. So, if you have a field contained in a monotone class, then the σ −field generated by the field is also content in the same monotone class. So, this is a very important theorem, and we do not prove this we are just stating it as part of the course.

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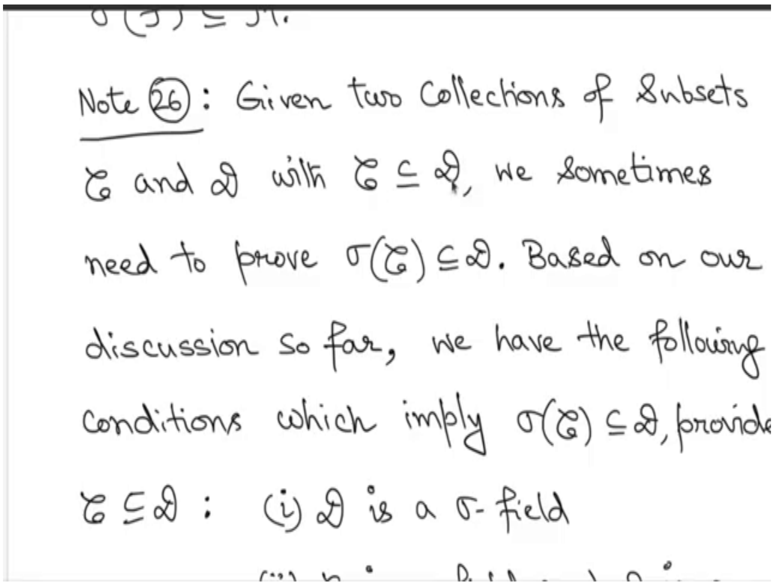

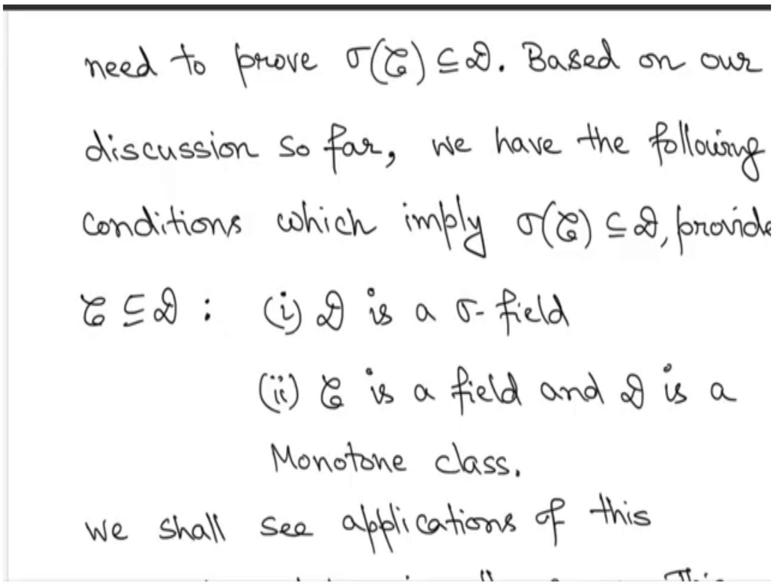

Now, what are the implications of this, so, the way we are going to use this is discussed in this note. So, if you have two collections of subsets, arbitrary collections of subsets, $\mathcal{C}$ and $\mathcal{D}$, so, again, I am fixing the non-empty set and discussing this. So, here, if it so, happens that $\mathcal{C}$ is contained in $\mathcal{D}$. So, this is a larger collection $\mathcal{D}$ is a larger collection, we sometimes will need to prove that the σ $-$field generated by $\mathcal{C}$ is contained in $\mathcal{D}$.

We shall need that we will discuss such situations later on. And then to claim such inclusions provided that we have $\mathcal{C}$ is already there inside $\mathcal{D}$, we shall need certain results. And based on our discussion so far, we have the following conditions which imply this. So, again the setup is that we are already given the information $\mathcal{C}$ is contained in $\mathcal{D}$, but, I want to claim that σ $-$ field generated by $\mathcal{C}$ is also contained in $\mathcal{D}$.

So, what are the situations? So, the first situation is this that if $\boldsymbol{C}$ is contained in $\mathcal{D}$, and $\mathcal{D}$ is a σ $-$field, then since σ $-$field generated by $\boldsymbol{C}$ is the smallest σ $-$field containing $\boldsymbol{C}$, as a definition, immediately it will follow that σ $-$field generated by $\boldsymbol{C}$ is contained inside $\mathcal{D}$. So, this is something we discussed as part of the previous lecture.

But then, there is another condition which we just got introduced to in the monotone class theorem. And that suggests that if you have field and if $\mathcal{D}$ is a monotone class, then the same inclusion holds. So, that means, if $\boldsymbol{C}$ is a field contained in a monotone class $\mathcal{D}$, then the σ $-$field generated by the field $\boldsymbol{C}$ is also contained in $\mathcal{D}$. This is the implication provided in the monotone cluster.

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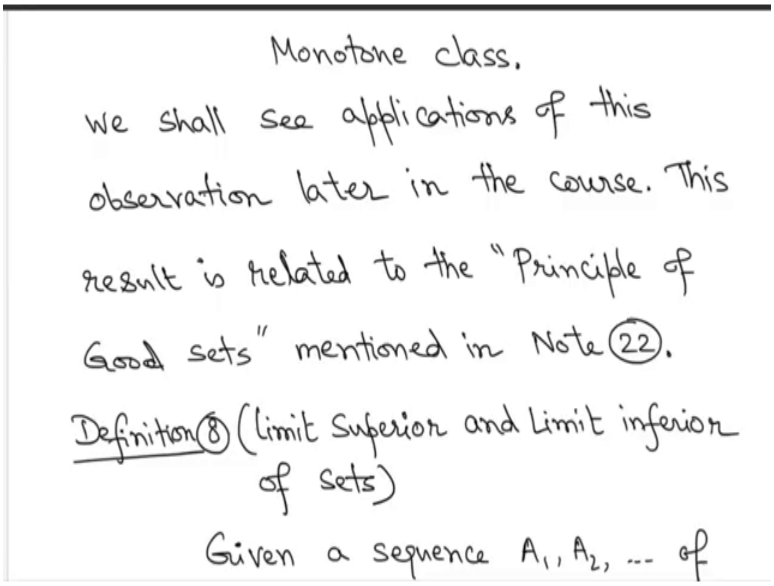

We shall see applications of these observation later on. And this result is again related to the principle of good set. So, we had mentioned this earlier in note 22. But again just to repeat due to time constraints, we are not going to discuss these things in detail later on.

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Good sets" mentioned in Note 22.

<u>Definition 8</u> (limit Superior and Limit inferior of sets)

Given a sequence $A_1, A_2, \ldots$ of Subsets of Ω, Consider the limit Superior and limit inferior of the Sequence defined as follows:

defined as follows:

$$\limsup_{n \to \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

$$\text{and } \liminf_{n \to \infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

<u>Exercise 14</u>: Consider the Sequences of

Now, we come to a very interesting operation called $lim\ sup$ and $lim\ inf$ of sets. So, again just to recall so far, what we have done is that we have looked at increasing sequences subsets and decreasing sequences subsets, and we have discussed certain ideas of limits there. So, in case of increasing sequence of subsets, we had looked at the complete union, the complete countable union as the kind of a limit and for the decreasing sequence of subsets, we had looked at the complete intersection as the limit.

Now, it may so happen that you have a general sequence of sets, which is neither increasing nor decreasing. In that case, you may ask, is there any notion of a limit? Again, we take motivation from real analysis. So, in real analysis, if you are given a general sequence of real numbers, it need not be convergent, it need not relevant. However, you have this notions of $lim\ sup$ and limits inferior and they always exist for a real number sequence.

Of course, $lim\ sup$ and $lim\ inf$ for a sequence of real numbers can be $\pm\ \infty$, but they always exist. So, we take motivation from that and define the $lim\ sup$ and $lim\ inf$ of a sequence of sets. How do you do this, so, I am given this sequence of sets they wanted to a and so on, I define this $lim\ sup$ this way. So, I first look at unions from n onwards. So, that is the inner union.

The outer intersection says that I now look at intersections of such things. And $lim\ inf$ is exactly the opposite thing. I first look at intersections from n onwards, that is the inner operation. And outside I am now looking at unions of all such things. So, get used to these notations $lim\ sup$ of sets A_n and $lim\ inf$ of the sequence $\{A_n\}$'s. So, I have a sequence of sets An and I am defining $lim\ sup$ and $lim\ inf$ this way. Now, you shall ask what happens as examples.

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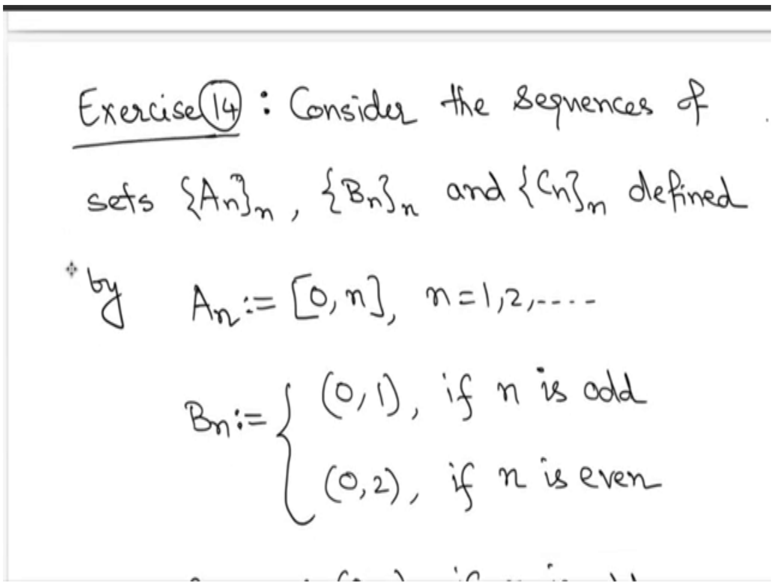

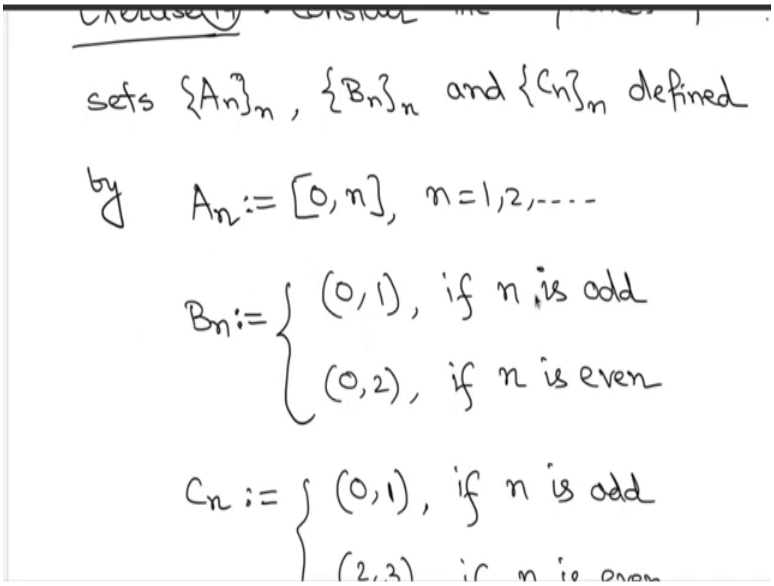

sets $\{A_n\}_n$, $\{B_n\}_n$ and $\{C_n\}_n$ defined by $A_n := [0, n]$, $n = 1, 2, \ldots$

$$B_n := \begin{cases} (0,1), & \text{if } n \text{ is odd} \\ (0,2), & \text{if } n \text{ is even} \end{cases}$$

$$C_n := \begin{cases} (0,1), & \text{if } n \text{ is odd} \\ (2,3) & \ldots \end{cases}$$

Now, just for your hands on examples, I have left you with some exercises. So, consider these three types of sequences in A_n, B_n and C_n. So, A_n is the set $[0, n]$, as you can, as you can immediately see, this sets A_n increase. What about B_n's? They have some kind of oscillating feature you see, if n is odd, like n equal to 1, 3, 5, 7 so on. So, there I am taking the set $(0, 1)$. If n is even like 2,4, 6, so on, so then I am taking the set $(0, 2)$. Then try to see what happens to $lim\ sup$ and $lim\ inf$.

So, again the way of computations just to give you some ideas. So, just going back to computations involving $lim\ sup$, first fix some n, small n. Look at this inner union first compute that write try to write in terms of one set, but it will depend on n and then take the complete intersection as n goes from 1 to infinity. To compute limit individuals, what you have to do? You have to do exactly similar thing, first compute the inner intersection from n onwards. So, whatever you get, you write it in terms of one set and then you take the complete union from as n goes from 1 to infinity, try to work this out.

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by $A_n := [0, n]$, $n = 1, 2, \dots$

$$B_n := \begin{cases} (0,1), & \text{if } n \text{ is odd} \\ (0,2), & \text{if } n \text{ is even} \end{cases}$$

$$C_n := \begin{cases} (0,1), & \text{if } n \text{ is odd} \\ (2,3), & \text{if } n \text{ is even.} \end{cases}$$

$(0,2)$, if n is even

$$C_n := \begin{cases} (0,1), & \text{if } n \text{ is odd} \\ (2,3), & \text{if } n \text{ is even.} \end{cases}$$

Compute the liminf and limsup of these sequences.

Note 27: $x \in \limsup A_n$ if and only if

So, that is for the A_n's and B_n's and C_n for the C_n's. So, in B_n's I have taken sets $(0, 1)$ and $(0, 2)$, they have certain intersections in C_n they have no intersections, but they have this oscillating type of feature. So, try to work this out, I have left them as exercises and this will help you understand what exactly happens in terms of $lim\ sup\ \ lim\ inf$.

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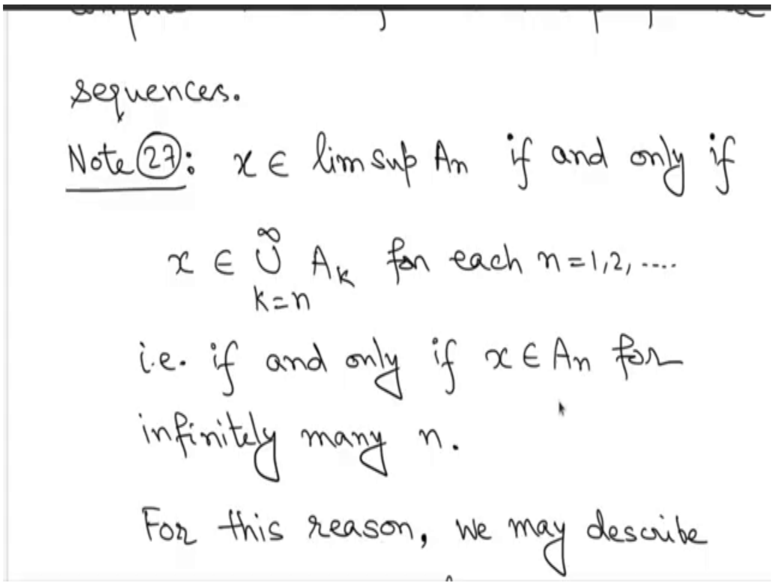

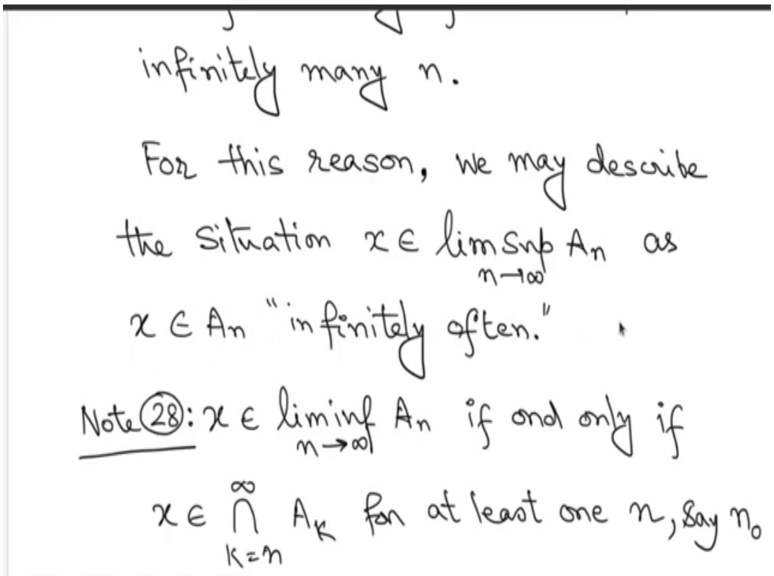

Now, there are certain interesting interpretations for $lim\, sup$ and $lim\, inf$. So, if a point belongs to $lim\, sup$, so this means, if you go to the definition, let us go to the definition once more. So, let us look at $x \in lim\, sup$. So, that means x belongs to this intersection, intersection is outside. So, that means, x belongs to this inner union for all possible n. So, just to repeat if x belongs to this $lim\, sup$ that means, from the definition of intersections, x must belong to the inner unions for each fix set small n.

So, that is what we write. So, if $x \in \lim_{n \to \infty} sup\, A_n$, then x must belong to this inner unions into ∞ for each n, but this will happen if and only if $x \in A_n$ for infinitely many n. You try to now go back to the definition of x belonging to this union. So, if it happens for each n, then x must fall in A_n's for infinitely such things.

I am not saying that it will belongs to each and every n, but I am saying there will be some sequences of n's such that you will get $x \in A_n$, this is simply following the definition. And for this reason, we may describe the situation x belongs to the $lim\ sup$ as $x \in A_n$ infinitely often. So, $x \in A_n$ for infinitely many n and in English language we shall refer to it as $x \in A_n$ infinitely often.

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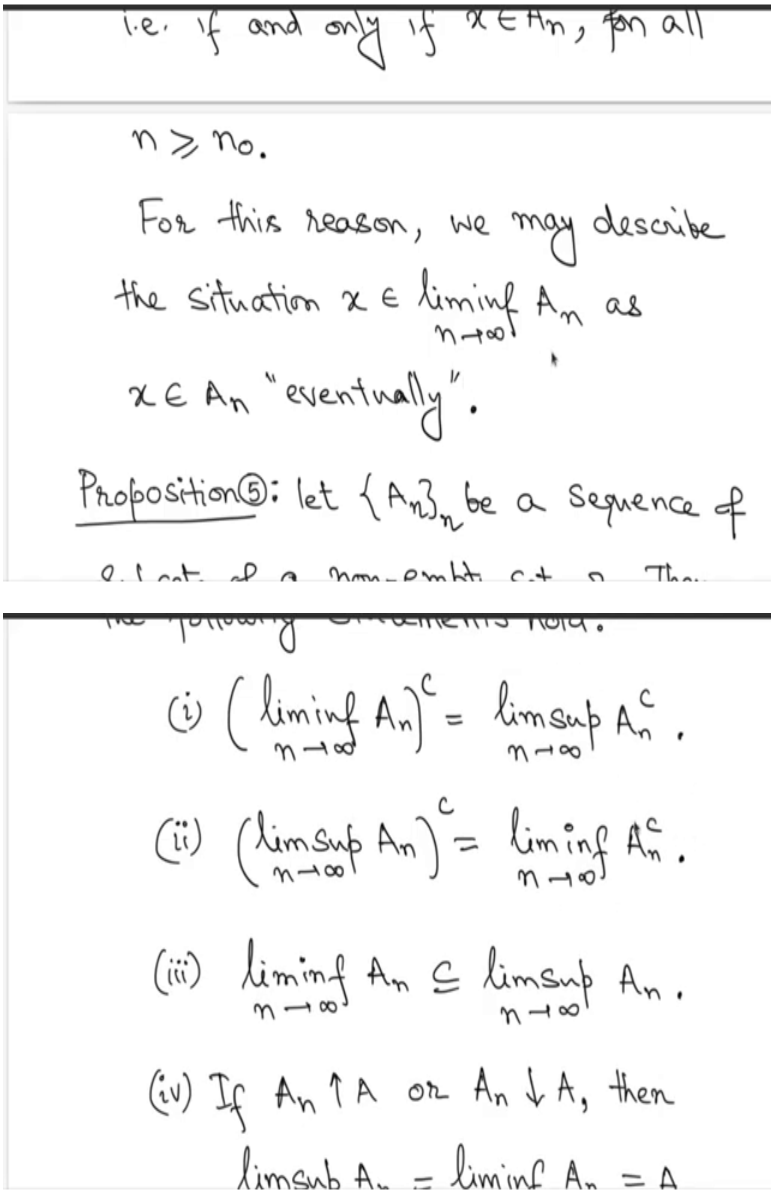

There is a similar interpretation for $lim\, inf$ of A_n's. So, that is goes like this. So, x belongs to $lim\, inf$. So, remember, just going back to the definition, if you look at the definition x belongs to this $lim\, inf$ means $x \in \bigcup_{n=1}^{\infty} A_n$. That means, x belongs to this inner intersection for at least one n. So, let us write that down.

So, x belongs to $lim\, inf$ that means, x belongs to this intersection for at least one n. So, let us say that n is n_0, some positive integer, but then that means that x belongs to this intersection for all n after an including n_0. So, x belongs to this intersection from n_0 onwards. So, that means x is a common element in all this A_k's as k varies from n_0 onwards.

So, for this reason, we may describe this situation that x belongs to $lim\, inf$ of A_n's as $x \in A_n$ eventually, that means, there exists a stage in n_0 such that after that stage $x \in A_n$'s. So, that gives you some very nice interpretations of $lim\, sup$ and $lim\, inf$.

Now, there are some interesting properties that connects $lim\, inf$ and $lim\, sup$, they are given in this proposition. Again, these are some of the operations using the usual set operations and you can easily check them. So, if you look at $lim\, inf$ and look at the whole complement, that can be written as the $lim\, sup$ of the complement of the sets. So, that is the first property.

The second property says, if you look at $lim\, sup$, and look at the complement that can be written as the $lim\, inf$ of the complements of the sets. So, again these properties are simply involved in the complementation, intersections and unions. So, using set standard set operations properties. Now, in the real line, when you define $lim\, inf$ and $lim\, sup$ of a sequence of real numbers, you had this property that $lim\, inf \leq lim\, sup$.

For sets, that notion simply belongs, simply becomes a notion of subset, that means, we are getting this property that $lim\, inf\, A_n$ is a subset of $lim\, sup$. Again from the discussion that we have done for $lim\, sup$ and $lim\, inf$, you can immediately check this. So, check that, that a point is in $lim\, inf$, that means, you can immediately show that it actually belongs to $lim\, sup$ also, therefore, $lim\, inf$ is a subset of $lim\, sup$.

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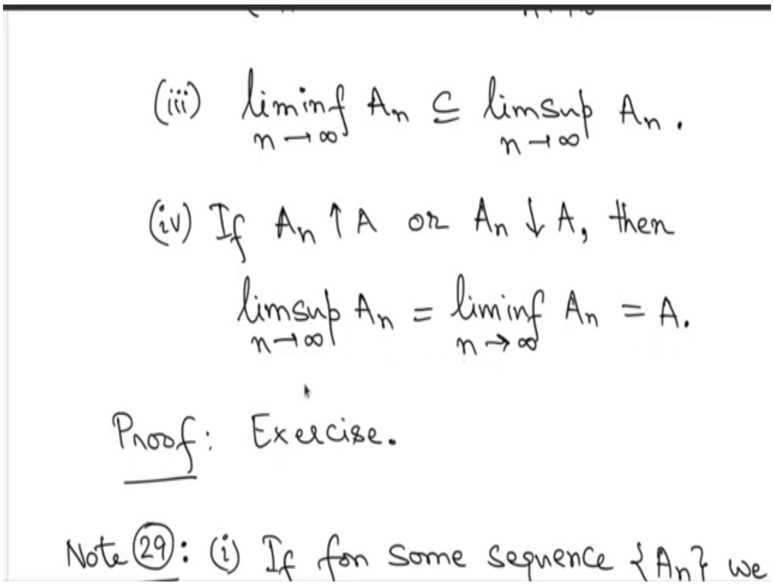

And then it will now connect to the notions of limits that we have already considered for increasing sequences of sets and decreasing sequences of sets. So, if it so happens that A_n's increase to A, so in case of our increasing sequence or A_n's decrease to a that for the case of decreasing sequence subsets, then you can immediately check that both the notions $lim\ sup$ and $lim\ inf$ exist.

Of course, as per the definition here, they will match. So, that is the important point here. So, $lim\ sup$ will equal $lim\ inf$ in the case the sequences increase or decrease and it will actually give you the set A. So, no matter if you are taking increasing sequences subsets, then A will become the complete union. So, then $lim\ sup$ limit and $lim\ inf$ both will be the complete union.

If A_n's are decreasing, then it decreases to the complete intersection and therefore, $lim\ sup$ and $lim\ inf$ will match and will be the complete intersection. So, try to work this out. So, that is left as exercise these are very simple exercises involving set operations.

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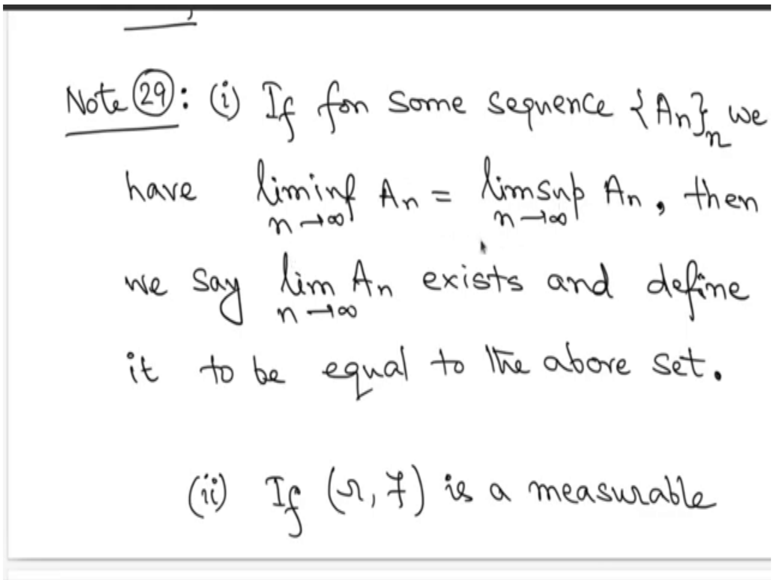

Now, again some further comments, if it happens for some sequence $\{A_n\}$, that $lim\,sup$ and $lim\,inf$ match, then we say that limit of the sets exists. So, this would happen if as we have discussed in the proposition above, if the original sequence A_n is increasing or decreasing this will happen that was a part of the proposition above.

So, in general if it so happens that $lim\,inf$ and $lim\,sup$ match, then we shall write that limit of A_n's exist and we it will be the equal, it will equal that $lim\,inf$ or $lim\,sup$ in this equality case. So, the limit will be equal to the $lim\,inf$ and $lim\,sup$, that equals the common value here.

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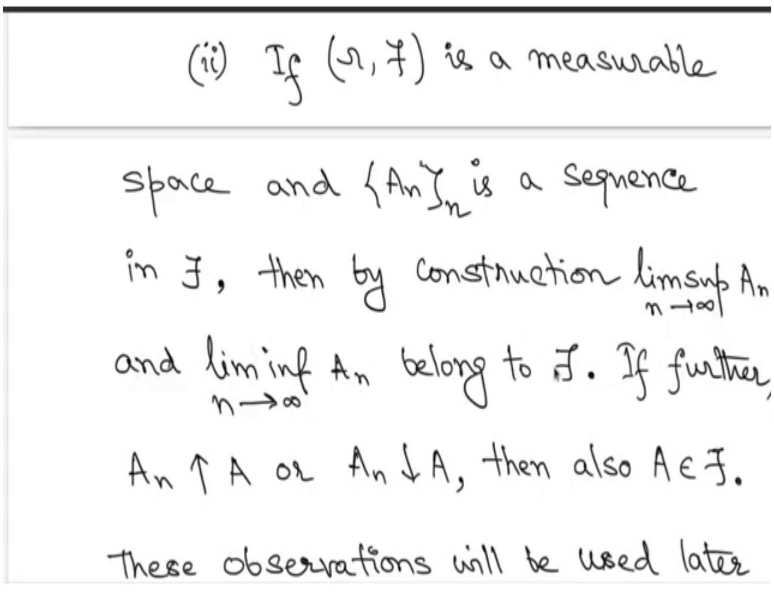

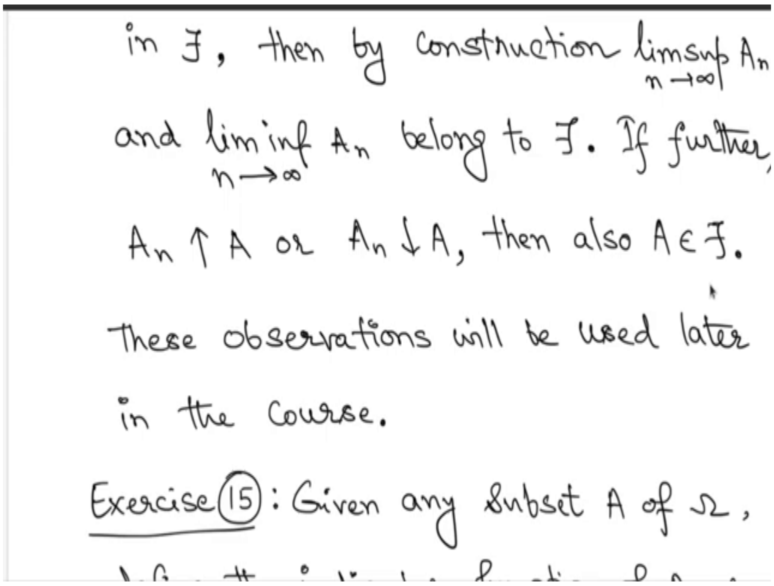

in $\mathcal{F}$, then by construction $\limsup\limits_{n\to\infty} A_n$ and $\liminf\limits_{n\to\infty} A_n$ belong to $\mathcal{F}$. If further, $A_n \uparrow A$ or $A_n \downarrow A$, then also $A \in \mathcal{F}$.

These observations will be used later in the course.

Exercise 15: Given any subset A of Ω,

And another important observation take a measurable space. So, that means $\mathcal{F}$ is a σ −field on this non-empty set Ω. And if it so happens you have a sequence of sets in the σ −field. Then if you look at this construction of $lim\ sup$ and $lim\ inf$, they involve certain countable unions and countable intersections. You will immediately claim that since a σ −field $\mathcal{F}$ is closed under countable unions and countable intersections, the $lim\ sup$ and $lim\ inf$ must belong to $\mathcal{F}$.

So again, this is immediately following from the definition of a σ −field. Further, if you have additional information, like A_n's increase or A_n's decrease, then of course, this limit this common value of the $lim\ inf$ or $lim\ sup$. So, that is the common value. So, let us call that A again.

So, that will also belongs to the σ −field. So, we will again use this later on, but this is an important observation that take any sequence of sets coming from a σ −field, then their $lim\ inf$ or $lim\ sup$ are certain sets involving countable unions and countable intersections, that is how they are defined and they will belong to the σ −field.

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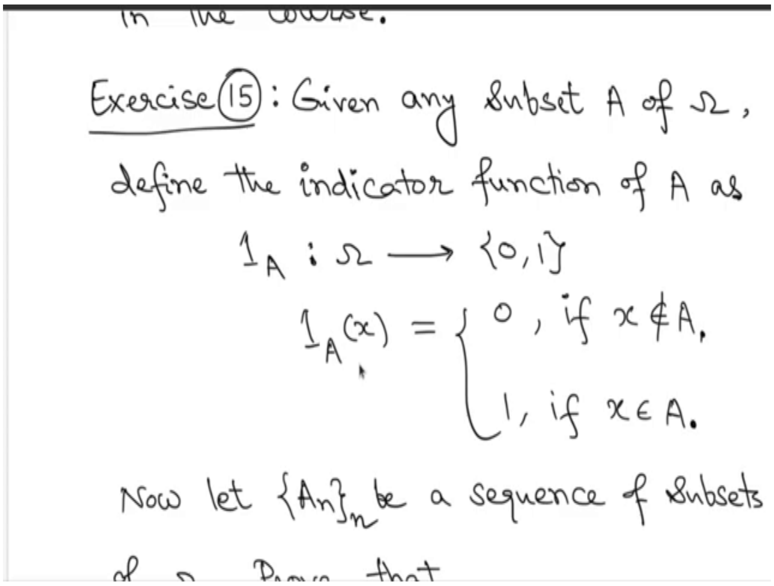

So, in this regard, there is this interesting connection with the usual notions of $lim\ sup$ and $lim\ inf$ that we have seen for sequences of real numbers. So, that connects involving certain indicator functions of sets. So, what are these, so take any subset $A \subset \Omega$ a non-empty set, then you say that the indicator function of the set A is this function, I will write 1_A, it will it is to has a domain Ω and it takes values 0 or 1.

So, if you take a point in Ω, $x \in \Omega$, then the indicator function will give you 0 if the point is not in the set A, otherwise it will put 1 if the set if the point is in A, then it will put 1, so that is the indicator function. So, it has to two values you just check if an arbitrary element x belongs to the set or not. If it belongs there, you put 1 if it does not belong you put 0, that is the indicator function of the set.

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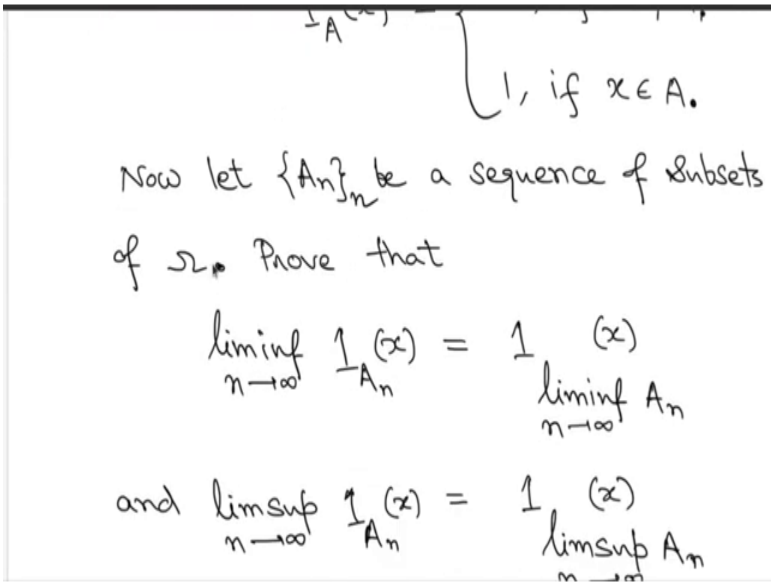

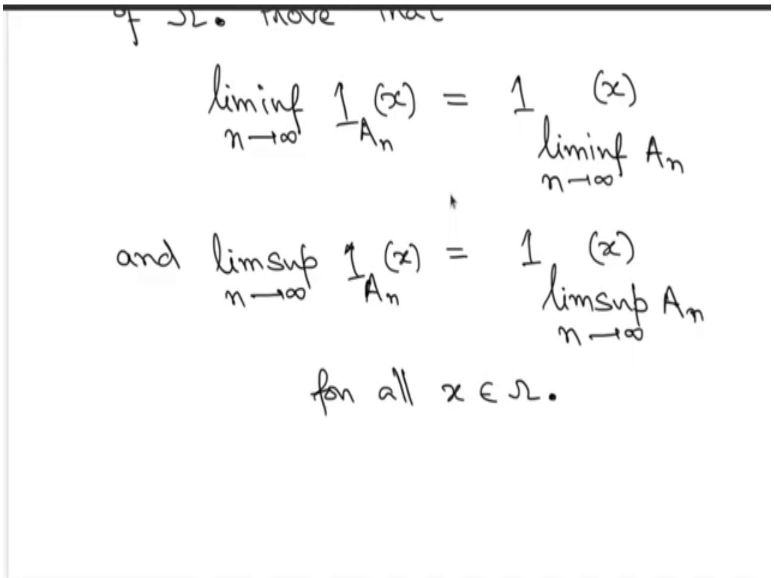

Now, look at a sequence of subsets, what you can now try to show that for all $x \in \Omega$, if you look at $lim\,inf$ of this indicator $A_n(x)$, so the left hand side, if you fix your x, then indicator $A_n(x)$, this is a sequence of 0's and 1's. So, this is a sequence of real numbers, I am looking at $lim\,inf$ of that. And on the right hand side, I am looking at indicator of the $lim\,inf$ of the sets.

Now this is a set $lim\,inf$ of A_n's, this is one set that we have discussed. So, I am looking at an indicator of that and I am evaluating at the point x. So, on the left hand side, I have a sequence of 0s and 1s I am looking at $lim\,inf$ of that. On the right hand side, I also have a 0, 1 value sequence, but the $lim\,inf$ is taken in terms of sets.

A similar statement also holds for $lim\,sup$. So again, on the left hand side, I have a sequence of 0s and 1s. So, I am just putting this I am fixing that x, I am just checking whether the x is belongs to the A_n's or not. So, that is how I am getting this sequence of 0's and 1's and I am computing the $lim\,sup$ and on the right hand side, I am computing the indicator of the $lim\,sup$ that we have defined.

So, on the right hand side $lim\,sup$ is in terms of the sets and then I am evaluating that function at the point x. So, these will connect the notions of $lim\,inf$ usual sense of real numbers, sequence of real numbers with notions of $lim\,sup$ and $lim\,inf$ of sets. So, we stop here.