Measure Theoretic Probability 1
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Lecture No. 04
Borel sigma-field on $\mathbb{R}$ and other sets

Welcome to this lecture. In this lecture, we will discuss some major properties and major facts, and major results involving the Borel sigma field on the real line. But, before we go forward, let us quickly recall what we have already seen in the previous lectures. So, we started with the basic notions of random experiments and saw that the collection of events generated from the collection of random experiments would satisfy certain structures. We looked at a certain collection of structures of subsets inside non-empty sets, and therefore, we obtained these structures of sigma fields and fields.

These involve these operations, the set-theoretic operations, complementation, unions, and intersections. For sigma fields, we allow countable unions and countable intersections, and for fields, we allow finite unions and finite intersections. We have seen many examples of sigma fields as well as fields. We have seen examples that which is a field but not a sigma field. And then afterward, we have seen constructions of sigma fields. One of the approaches was the bottom-up approach, starting with the trivial sigma field and adding sets to that we obtained the sigma fields.

Another approach was the approach of the generating set, where we took a top-down approach. So, what we do is that we start with the power set sigma field and judiciously remove certain sets and look at certain kinds of intersections. And that is how we obtained the sigma fields generated by certain collections that we start with. So, this is what we have already discussed.

And one of the special examples of these sigma fields generated by certain collections was the Borel sigma field on the real line. So, it is the sigma field generated by the collection of all open sets in the real line. So, we start this lecture and discuss results involving the Borel sigma field on the real line, and we will also see some extensions of these to higher dimensions and other sets. So, let us move on to the slides.

So, let us look at the minimal sigma field again generated by the collection of open sets on the real line. For simplicity of notation, we will write by $\mathcal{C}_0$, which to denote the collection of open sets in the real line. So, we will not repeatedly say the collection of all open sets in the real line; we will simply say $\mathcal{C}_0$.

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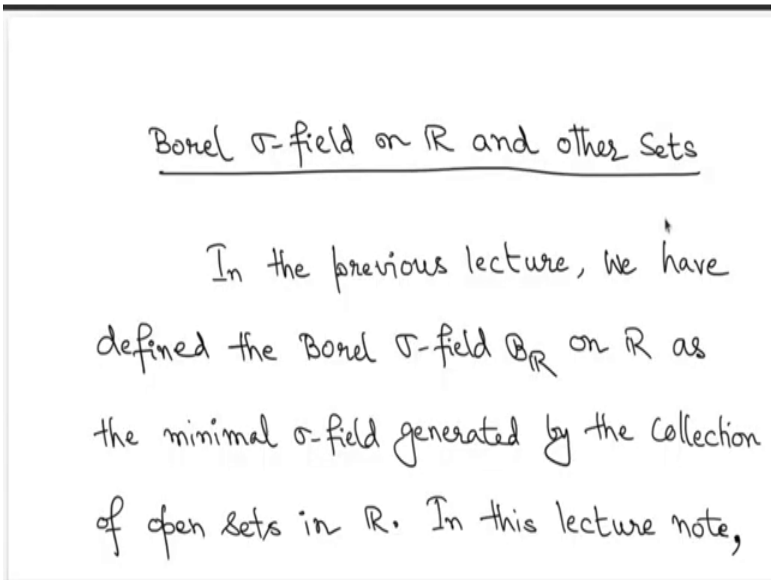
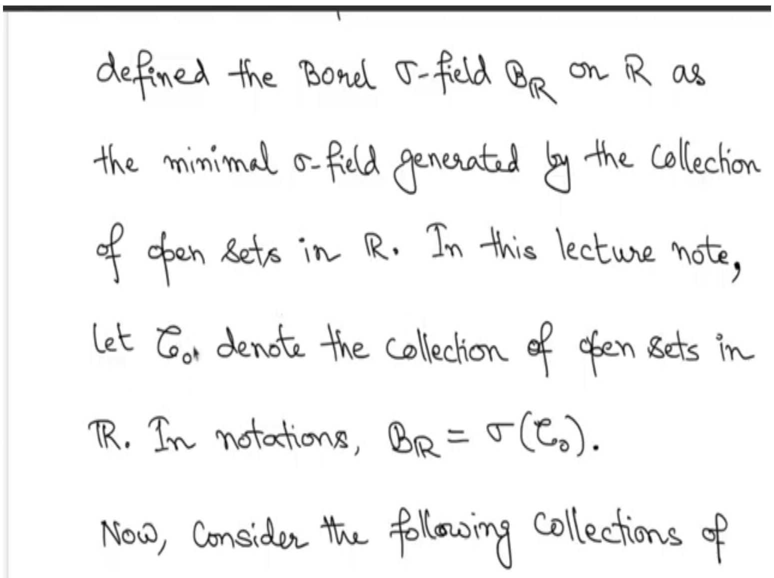


And what we have already defined or whatever the notations we have used so far says that the Borel sigma field on the real line is the same as the sigma field generated by this $\mathcal{C}_0$. So, this is our definition. Now, we consider different collections of subsets of the real line.



So, the first collection first new collection that we see is denoted by $\mathcal{C}_1$. So, this is all the closed sets that are in the real line. Then $\mathcal{C}_2$ and so on, we shall see a certain special type of intervals. So, what are these intervals? So, again we will look at in $\mathcal{C}_2$ we are looking at open intervals in $\mathcal{C}_3$ it is closed intervals, $\mathcal{C}_4$ left open right closed, $\mathcal{C}_5$ left closed right open.

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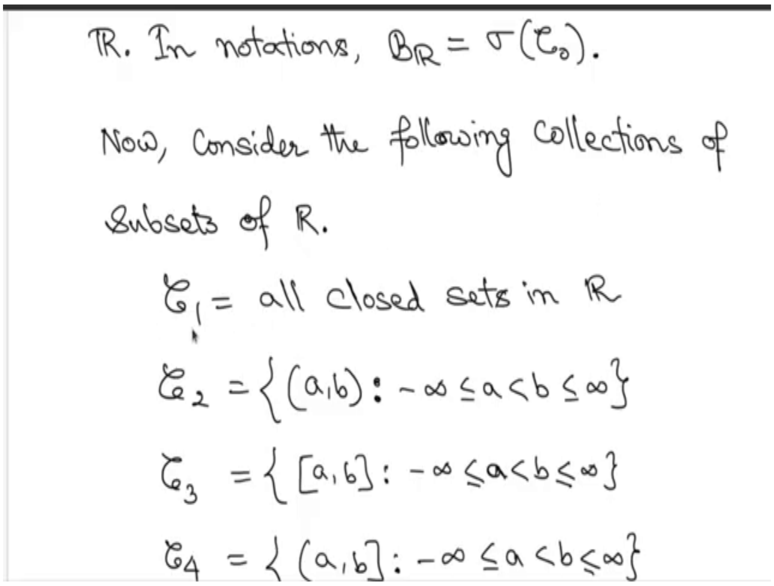
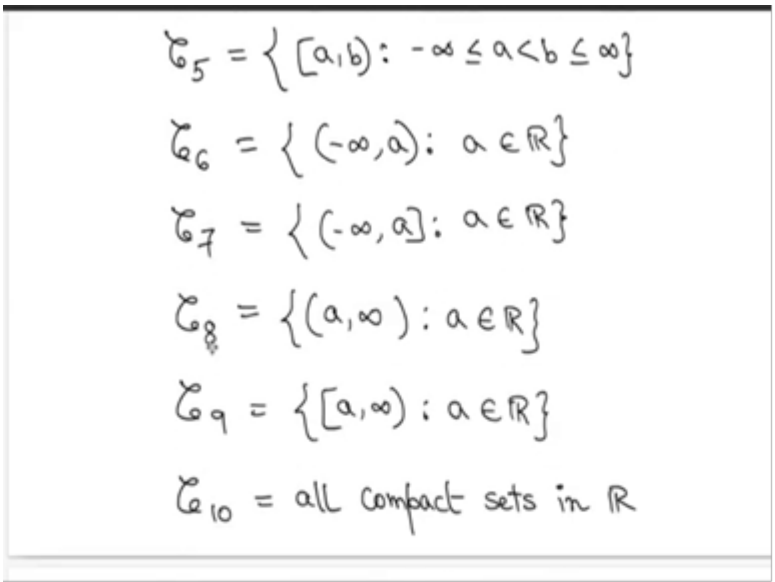


So, in all of these, you will see that the left endpoint and the endpoint a and b could be $-\infty$ or $+\infty$, but this is following the notations that at the end of the day, these sets or these intervals that you consider must be within the real line. So, therefore, for example, if in the $[a, b]$, if a is $-\infty$, you think of it as $(-\infty, b]$. That is how you should think of it. So, this notion has already been explained in the previous lecture.



So, now, in $\mathcal{C}_6$ onwards, we will see unbounded intervals, so one bound of the interval is either ∞ or $-\infty$. So, this gives you 6, 4 more types of intervals.

And then there are two more types, which is $\mathcal{C}_{10}$ which is all compact sets in the real line and then $\mathcal{C}_{11}$ which is our well-known field, which was shown as an example of not a sigma field. So, this is a field but not a sigma field. How was this thing defined? So this was the finite disjoint union of left open right closed intervals.

So, you look at intervals of the form $(a, b]$ and if you have n many such intervals, if they are pairwise disjoint, take their union, that is a typical set in this field, and for the discussion in this note, in this lecture, we will write it as $\mathcal{C}_{11}$. So, we have this list of special sets or a special type of set in the real line. So, this is what we are going to look at the first one was denoted by $\mathcal{C}_0$, other ones are with indexing 1, 2, 3 onwards up to 11.

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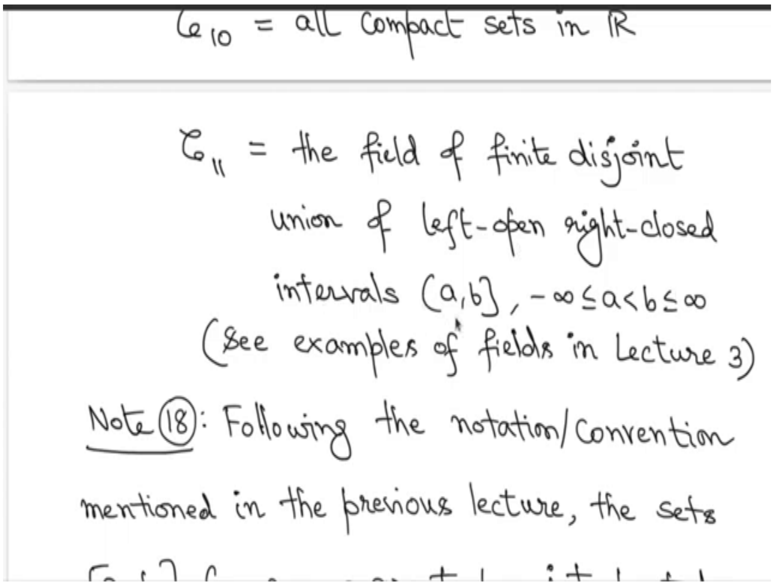
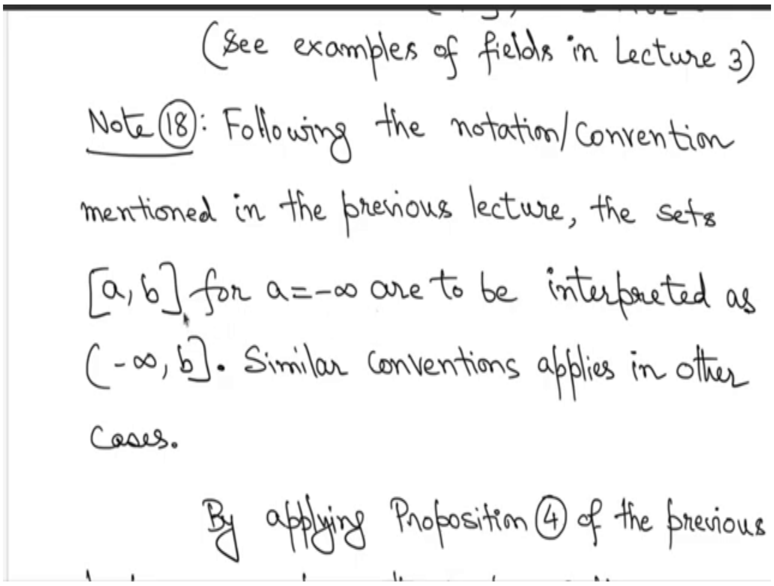


Note 18: So, before we go forward in the definitions and other results, one of the things that I will repeat once more that following this notation or the convention, for example, in closed sets with $-\infty$ or $+\infty$ as the limits or the bounds, then you have to interpret it accordingly. So, this is just to repeat, so there is no cause for confusion, these sets, whatever intervals or sets we are concentrating on within the real line. So, therefore, $\pm\infty$ these points are not within the sets that we consider.



So now, in our last lecture, we had talked about sigma fields generated by different collections of sets. And in this regard, we had proved this proposition 4. So, please go back and check this from the previous lecture. But, what we said was that, if you have the fact that if you have two collections subsets, one is let us say $\mathcal{C}$ and one is $\mathcal{D}$ if the sigma field generated by $\mathcal{C}$ contains the collections $\mathcal{D}$ and vice versa, then the sigma fields generated by $\mathcal{C}$ and $\mathcal{D}$ must be the same.

So, this is something was the statement of Proposition 4 in the previous lecture, and using this, we are arriving at a very important theorem,

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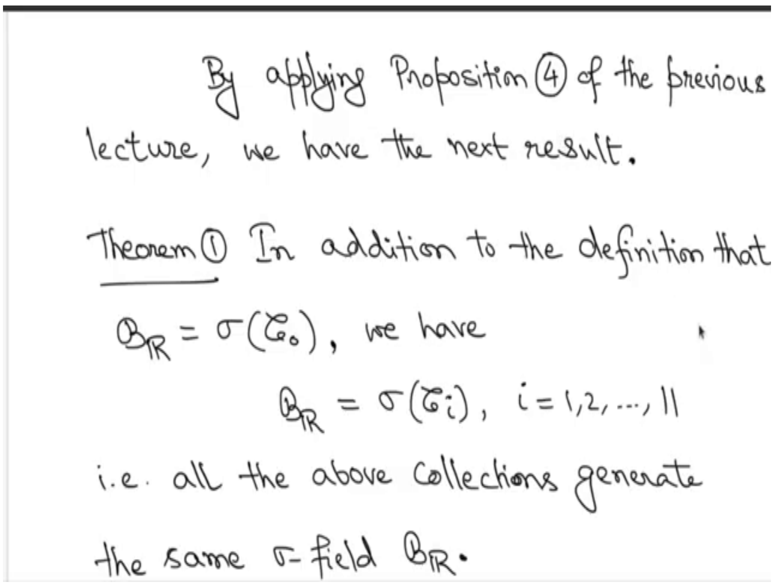

Theorem (1) In addition to the definition that

$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{C}_0)$, we have

$$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{C}_i), \quad i = 1, 2, \ldots, 11$$

i.e. all the above Collections generate the same σ-field $\mathcal{B}_{\mathbb{R}}$.

Note (19): Before proving the theorem, we make several observations.

Theorem 1: All these collections subsets that we have introduced above, all of these separately generate the Borel sigma field.

So, for example, $\mathcal{C}_1$ was the collection of all closed sets. So, if you generate the sigma field with all closed sets, you will again end up with the Borel sigma field. So, we will not get a different collection of sets, but we will only get the same sigma field as the Borel sigma field originally generated by open sets. Similarly, we will again get the same sigma field for all the types of intervals that we have considered. So, this is a very, very important property of the Borel sigma field. So, all in all, these collections generate the same sigma field. Now, there are several comments before we go forward and prove this. So, the comments are like this.

(i) So, first of all, remember that the generating sets belong to the generated sigma field. So, whatever generating sets you look at must belong to the end of that sigma field you have constructed. So, therefore, as a consequence, what you can now immediately say is that if you assume that theorem 1 if you know that all these collections generated the same sigma field, you can immediately claim that all these sets that you have already considered this type of intervals, closed sets, open sets and so on, all of these must be there in the Borel sigma field. So, the Borel sigma field typically constitutes all these nice sets you need to look at. Later on, we will see some examples of subsets of the real line, which are not in the Borel sigma field or which is not a Borel subset of the real line, we will see that. So, the Borel sigma field is strictly smaller than the power set. Still, for all our practical purposes, we will typically work with open sets open intervals, closed sets closed intervals or left open right closed intervals, and so on. All of these are included in the Borel sigma field. So, that is something very much is needed, keep it in your mind.

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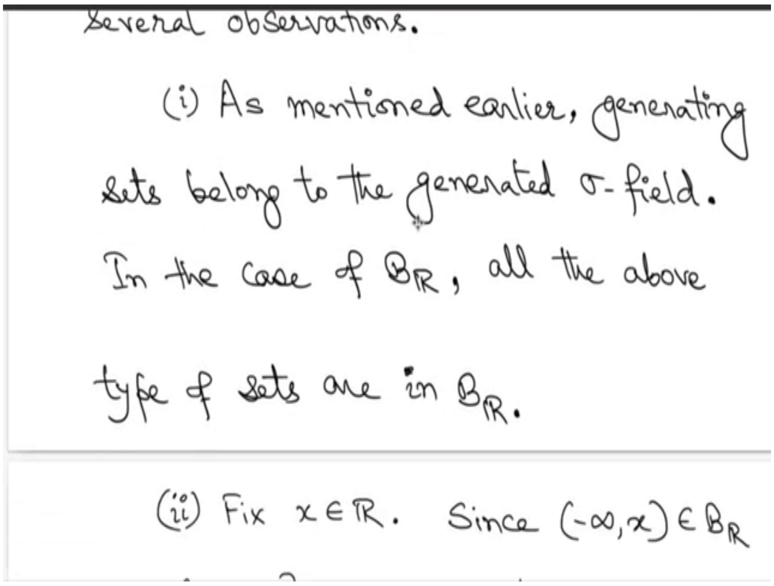

(ii) So, the next observation is that if you fix a point x on the real line, you have both types of sets in the Borel sigma field. So, this $(-\infty, x)$ and $(-\infty, x]$. If you have both these sets on the Borel sigma field, then what do you have immediately is that the

$$\{x\} = (-\infty, x] \cap (-\infty, x]^c.$$

So, you have this first set, subtract the second set, which is nothing, but take the first set to intersect with the complement of the second set. There is a set minus notation, which you have talked about earlier. Now, in the end, it gives you only $\{x\}$. That is, therefore, in this in the Borel sigma field following the properties of a sigma field. So, you have every singleton subset of the real line in the Borel sigma filter, which is great.

(iii) But then something nice happens, if you take any finite set or countably infinite set, you can write it as a finite union or a countable union of singleton sets. So, for example, if you look at the set of all positive integers $\{1, 2, 3, \dots\}$, you can write it as a $\{1\} \cup \{2\} \cup \dots$. And therefore, since you already know that these singletons are in the Borel sigma field, you already claim that all these countable sets including the set of positive integers or the natural numbers are in the Borel sigma field. Similarly, you will make the same comment about the set of rational numbers because this is a countable set.

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(ii) Fix $x \in \mathbb{R}$. Since $(-\infty, x) \in \mathcal{B}_{\mathbb{R}}$ and $(-\infty, x] \in \mathcal{B}_{\mathbb{R}}$, we have

$$\{x\} = (-\infty, x] \setminus (-\infty, x) \in \mathcal{B}_{\mathbb{R}}.$$

Hence, every singleton subset is in $\mathcal{B}_{\mathbb{R}}$.

(iii) Since any finite or countably infinite set can be written as a finite or countable union of singleton sets, all

countable union of singleton sets, all
finite or countably infinite subsets of $\mathbb{R}$ is in $\mathcal{B}_{\mathbb{R}}$. For example, $\mathbb{N} = \{1, 2, \cdots\}$ and the set of rationals $\mathbb{Q}$ is in $\mathcal{B}_{\mathbb{R}}$.

<u>Note</u> (20) In all of the generating classes in Theorem (1), most often, we shall use $\mathcal{C}_0$, $\mathcal{C}_4$, $\mathcal{C}_7$ and $\mathcal{C}_{11}$.

Note 20: Now, it so happens that we have looked at many, many different classes of sets, $\mathcal{C}_0$ to $\mathcal{C}_{11}$, but most often, we will use these four types of things. So, $\mathcal{C}_0$, $\mathcal{C}_4$, $\mathcal{C}_7$ and $\mathcal{C}_{11}$.

So, let us just go back to that list or the description given at the beginning of this lecture and see what these collections are; $\mathcal{C}_0$, $\mathcal{C}_4$, $\mathcal{C}_7$ and $\mathcal{C}_{11}$. So, what are these? So, let us go back. So, $\mathcal{C}_0$ just to recall, this was the collection of all open sets.

Next was $\mathcal{C}_4$ this was the left open right closed intervals you have already used to construct these nice fields. You will also see them when we later discuss distribution functions. Then what is $\mathcal{C}_7$ that is $(-\infty, a]$. So, that type of sets also appears in connection with distribution functions that we shall see later on.

And then $\mathcal{C}_{11}$ is that the example of the field that was not a sigma field. So this was the finite disjoint unions of left open right closed intervals. So, that was the example. So, let us go back and start with the proof of that theorem that all these different sets generate the same sigma field. So, how do you prove this?

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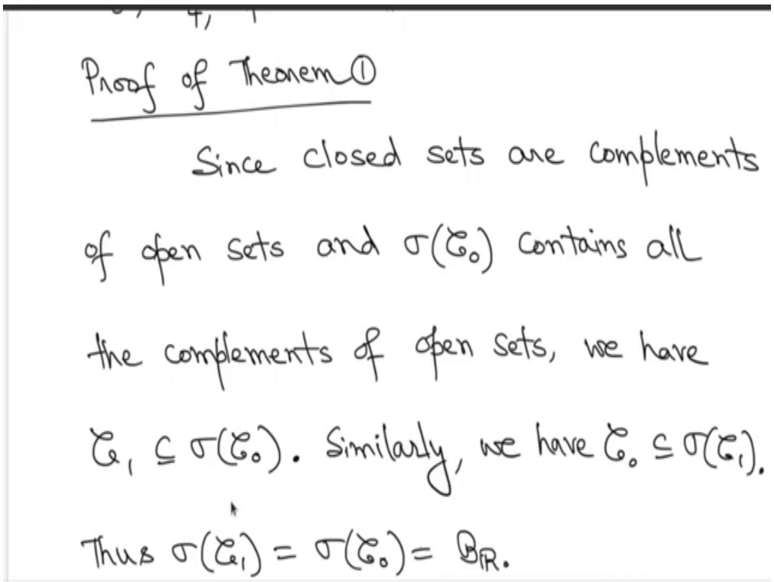

Proof of Theorem 1: So, you start with the first statement, that you want to show that the collection of all open sets that generate the Borel sigma field will be the same if you generate it with only the closed sets. So, for the closed sets that collection, we denoted it by $\mathcal{C}_1$. So, applying proposition 4 from the previous lecture, we need to show two-sided inclusions that close sets are contained within the sigma field generated by open sets.

But this is pretty simple to prove. Once you observe, the closed sets are nothing but the complement of open sets. And the sigma field, of course, sigma field generated by all open sets necessarily contains all its complements of the open sets. So, therefore, closed sets are there. Therefore, $\mathcal{C}_1 \subseteq \mathcal{C}_0$. Similarly, if you observe that open sets are nothing but the complement of closed sets.

So, open sets or the collection of all open $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$. So, therefore, you have both sides' inclusions as required, and you will immediately claim that both these sigma fields generated by one by the open sets and one by the all closed sets must match. So, that gives you that connection.

So, let us try to go to a different one. For simplicity, let us take an example involving the compact set. So, this was $\mathcal{C}_{10}$. So $\mathcal{C}_{10}$, what was that set? So, this was a list of all compact sets or the collection of all compact sets in the real line. And we want to claim that all compact sets that will generate a sigma field will be the same as the Borel sigma field. So, what is how do you prove this? How does the proof go?

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Thus $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_0) = \mathcal{B}_{\mathbb{R}}$.

Recall that compact sets in $\mathbb{R}$

are closed and bounded. Hence $\mathcal{C}_{10} \subseteq \mathcal{C}_1$

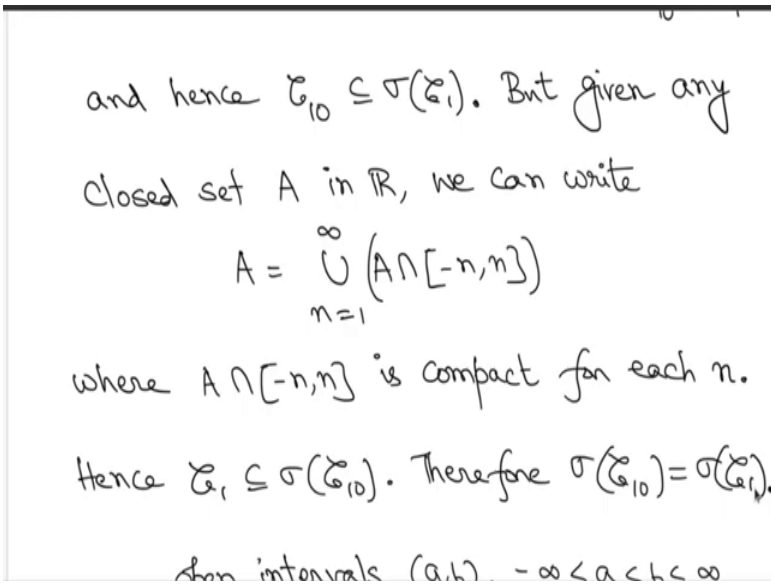
and hence $\mathcal{C}_{10} \subseteq \sigma(\mathcal{C}_1)$. But given any

closed set A in $\mathbb{R}$, we can write

$$A = \bigcup^{\infty} \left(A \cap [-n, n]\right)$$

So, you start with all compact sets. Recall that all compact sets are closed and bounded. And hence, you have that all compact sets are contained in the collection of all closed sets in particular, and therefore, all compact sets are contained in the sigma field generated by all closed sets. Fine, but given any called closed set, let us take the set to be A. You can write it as a countable union like this.

So how? You look at the interval $[-n, n]$, take the intersection with the set A and take the union about all possible n. Now, what happens is that in $A \cap [-n, n]$, both of the sets A and $[-n, n]$ are close. So the intersection of that is close, but this set is also bounded because it is contained within $[-n, n]$. It is a bounded set. So, $A \cap [-n, n]$, that set is closed and bounded and, hence, a compact set.



So, therefore, what you have been able to write is that the originally closed set A can be written as a countable union of compact sets. Therefore, you have that all closed sets are contained in the sigma field generated by all compact sets because the sigma field by compact sets will contain countable unions of compact sets and, therefore, all closed sets.

Now, you have shown that the collection of all compact sets is contained in the sigma field generated by all closed sets and vice versa. Therefore, the sigma field is generated by $\mathcal{C}_{10}$ and $\mathcal{C}_1$match. And since you are already shown that the sigma field generated by the compact sets is the same as the sigma field generated by the closed sets, which is the same as the Borel sigma fields that connect the connection with the compact sets.

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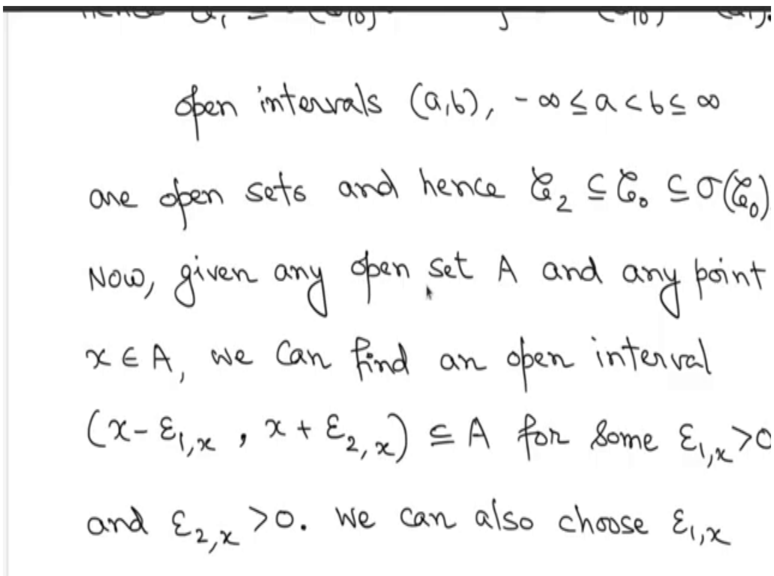

So, let us now move on to different types of sets as open intervals. What are these again, so you look at open intervals of the form (a, b), for a, b like this. Now, these are all open sets. So, therefore, if you look at this collection of all open intervals, which is $\mathcal{C}_2$ that is contained in all open sets, which is $\mathcal{C}_0$ and therefore contained in the sigma field generated by all open sets. So, therefore, all open intervals are contained in the sigma field generated by all open sets.

Conversely, you can show that all open sets can be written as some countable union of open intervals. How do you show this? So, this is similar to what we have shown for the connection with the compact sets. So, this will require some argument like this again. So, how do you do this? So, first of all, the first question is that you want to write an open set as a union of open intervals to say that you have first to figure out the open intervals within our open set.

But that is easily obtained because if you take any point x, you can find the open interval something like this. Let us say you do not need to get a symmetric interval. You can get intervals of this form. So, $(x - \epsilon_{1,x}, x + \epsilon_{2,x})$. So, you will get some interval containing that point x and that open interval should be completely contained within the set A.

$(x-\varepsilon_{1,x}, x+\varepsilon_{2,x}) \subseteq A$ for some $\varepsilon_{1,x}>0$ and $\varepsilon_{2,x}>0$. We can also choose $\varepsilon_{1,x}$ and $\varepsilon_{2,x}$ large enough such that the interval $(x-\varepsilon_{1,x}, x+\varepsilon_{2,x})$ is the largest open interval contained in A and containing the point x. Then, $A = \bigcup (r-\varepsilon_{1,r}, r+\varepsilon_{2,r})$, where

So, if you take any open set and take any point within, there is an open interval containing that point x and the open interval must be contained in the original set A. So here are these choices of $\epsilon_{1,x}$ and $\epsilon_{2,x}$ is dependent on x. So, therefore, that is why it is explicitly written that these numbers $\epsilon_{1,x}$ and $\epsilon_{2,x}$ depend on x.

But what you can do, you can choose this $\epsilon_{1,x}$ and $\epsilon_{2,x}$ these numbers are large enough that this interval that you obtain is the maximum such thing is the largest search open interval that is contained in A and the property that the open interval contains x. So, if you choose a point, you, of course, get open intervals but enlarge it on both sides, and you will get a maximal such open interval, which is still containing it.

and containing the point x. Then,

$$A = \bigcup_{r \in \mathbb{Q} \cap A} (r - \varepsilon_{1,r}, r + \varepsilon_{2,r}), \text{ where}$$

$\mathbb{Q}$ denotes the set of rational numbers.

Since $\mathbb{Q}$ is countable, the above union is a countable union of open intervals.

Hence, we have the inclusion $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_2)$.

If you go beyond that, it will the open interval will go outside the set A. So, this is what you can try to visualize. So, in that case, I try to show that A is made up of this type of open intervals, where you just look at open intervals of these types that we have just obtained, but only around the rational points within the set A. So, here the set $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{Q} \cap A$ denotes the set of all rational numbers within the set A.

And therefore, all you look at are all the rational numbers within the set A, look at those intervals around the rational points smaller, take their union that will turn out to be the whole set A you can easily check this try to work this out. But then what will happen is since $\mathbb{Q}$ is countable is above the union of all rational points within the set A, that is also a countable set, and hence these above union is countable. Great. So, therefore, you have written an open set A as a countable union of open intervals.

And hence you have the inclusion that all open sets are contained in the sigma field generated by all open intervals, and therefore,e you have both-sided inclusion as required. And you can easily claim this equality applying proposition 4.

So, let us now move on to a different type of interval one side open, one side closed. So, let us take a left open right closed. So, you have to connect with all types of other generating classes that we have already seen. So, what you do here, here is that you take $(a, b]$.

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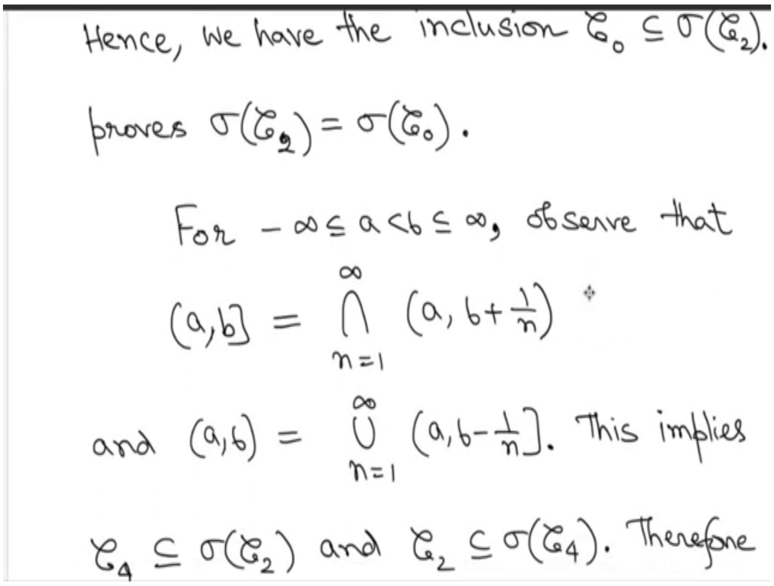

Then you can write it as an intersection of these things, left intersection of this type of an interval open interval. How? So, you take $\left(a, b + \frac{1}{n}\right)$. Now, if you take the intersection of these, you can easily check that $(a, b]$ is exactly these countable intersections. On the other hand, if you take an open interval (a, b), you can write it as a countable union of such intervals left open, right closed. You can check this.

This implies that the sigma field is generated by all open intervals, which is C_2 contains all left open right closed intervals and all open intervals, which is the collection C_2 that is contained in the sigma field generated by all left open right closed intervals, which is C_4. Therefore, these two must match, but you also have that inclusion since we have already proved that the sigma field generated by only the open intervals is nothing but the Borel sigma field. Great.

Now, let us come to C_7. So, what was C_7? C_7 was intervals of the form $(-\infty, x]$. Now, of course, if you look at such intervals, they are already left open, right closed. So, therefore, they are contained in the sigma field generated by left open right closed intervals. On the converse, if you take our interval of the form $(a, b]$, you can write it as a set like this. So, you take this infinite interval $(-\infty, b]$, but subtract out $(-\infty, a]$.

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$\sigma(\mathcal{C}_4) = \sigma(\mathcal{C}_2)$.

By definition, $\mathcal{C}_7 \subseteq \mathcal{C}_4 \subseteq \sigma(\mathcal{C}_4)$.

Again for $-\infty \leq a < b \leq \infty$,

$$(a,b] = (-\infty, b] \setminus (-\infty, a].$$

Therefore $\mathcal{C}_4 \subseteq \sigma(\mathcal{C}_7)$. This implies

$\sigma(\mathcal{C}_4) = \sigma(\mathcal{C}_7)$.

Recall that this is nothing but the intersection between $(-\infty, b]$ and the complement of $(-\infty, a]$. So, therefore, you get this and subsets this left open right closed intervals are contained within this left open right closed infinite intervals, which is the collection $\mathcal{C}_7$. So, you have this connection, this equality and therefore, from the previous results, you immediately claim that these are all generating the Borel sigma field.

Therefore $\mathcal{C}_4 \subseteq \sigma(\mathcal{C}_7)$. This implies

$\sigma(\mathcal{C}_4) = \sigma(\mathcal{C}_7)$.

By definition, $\mathcal{C}_{11} \subseteq \sigma(\mathcal{C}_4)$ and

$\mathcal{C}_4 \subseteq \mathcal{C}_{11} \subseteq \sigma(\mathcal{C}_{11})$. Hence $\sigma(\mathcal{C}_4) = \sigma(\mathcal{C}_{11})$.

Rest of the proof is left as exercise.

And finally, let us come to $\mathcal{C}_{11}$. So, that is our familiar field, which is not a sigma field, and this is made up of the finite disjoint union of left open right closed intervals. But then these left open right closed intervals are contained in the sigma field generated by only the left open right closed intervals, which is fine. And therefore, you have that inclusion. But conversely, if you look at only left open right closed intervals, they are, of course, contained in the collection in that field generated by this finite disjoint union of such things.

And therefore, you $\mathcal{C}_4 \subseteq \mathcal{C}_{11}$, but then you immediately claim that okay fine, $\mathcal{C}_7$ is contained within the sigma field generated by the field $\mathcal{C}_{11}$. Therefore, you have both-sided inclusions as required in the proportion 4 earlier, and therefore, you have the equality that the sigma field generated by $\mathcal{C}_4$ and the sigma field generated by $\mathcal{C}_{11}$ both must match.

Similarly, you can try to connect all the other types of sets that are left out. You can try to check this. The rest of the proof is left as an exercise for you. Please check this. So, we have already shown that many different classes or different types of sets generate the same sigma field.

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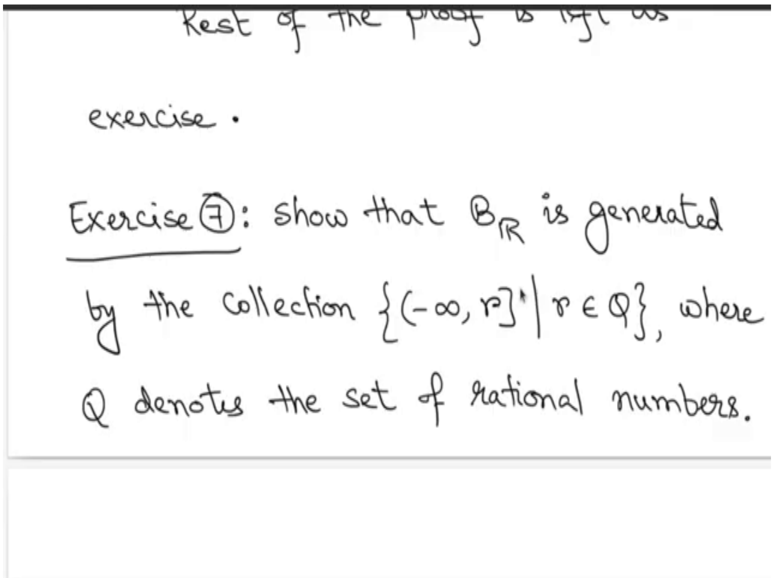

You can also try to look at other types of special types of sets.

Exercise 7: In particular, you can look at all infinite intervals of this type with rational endpoints smaller. So, again $\mathbb{Q}$ denotes the set of rational numbers and chooses all those form intervals $-(\infty, r]$. Try to check that this collection of intervals generates the Borel sigma field.

So, you are not considering points, we are not considering points, irrational points, and we are not considering these types of sets where the right side bound, right side limit, or smaller is irrational we are not considering those types of sets. We are only considering rational endpoints here and try to see that these will generate all types of sets and give you the Borel sigma field. Try to see this, try to work this out.

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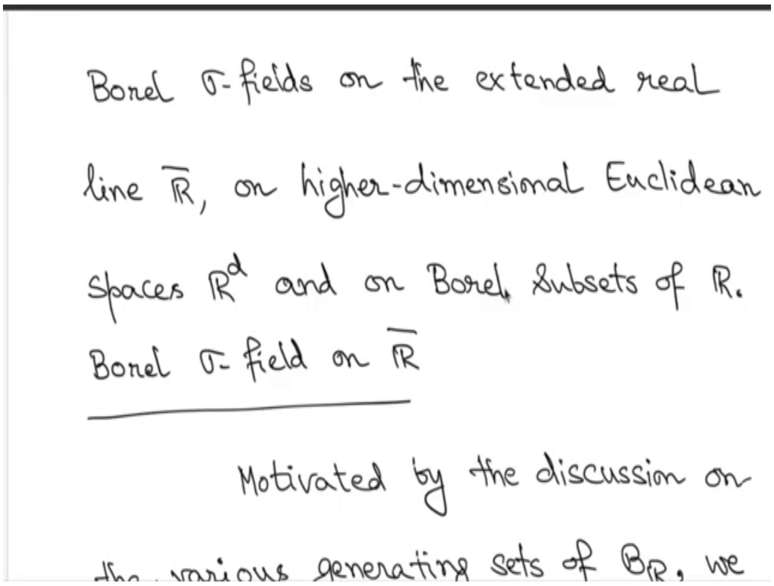
Borel σ-fields on other sets

In the above discussion, we have looked at various generating classes of subsets of $\mathbb{R}$ which generate the Borel

In the above discussion, we have looked at various generating classes of subsets of $\mathbb{R}$ which generate the Borel σ-field $\mathcal{B}_{\mathbb{R}}$. We now look at the Borel σ-fields on the extended real

But then, we can now move on to a different type of set, which is the Borel sigma field on other types of sets. Now, we have already looked at these generating collections on $\mathbb{R}$. With this motivation at hand, we now look at several different sets. For example, the extended real line, denoted by $\overline{\mathbb{R}}$, higher dimensional Euclidean space some $\mathbb{R}^d$ and some specific Borel subsets of $\mathbb{R}$.



We will look at similar fields or Borel sigma fields on Borel subsets. So, on the Borel subset, there are also non-empty sets. You can ask whether there are sigma fields, and we shall construct that. So, let us go to the extended real line and see the notion of Borel sigma fields there.

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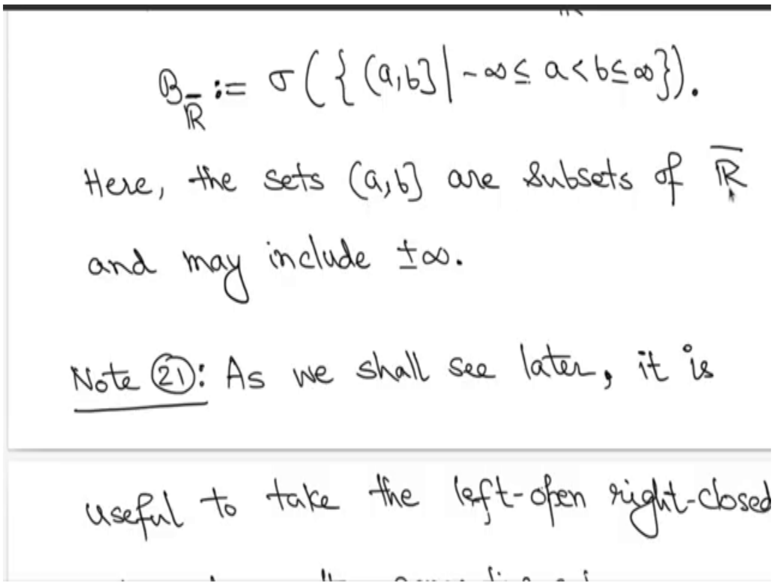

So, again on the real line, we saw that many different sets generate the sigma fields, that Borel sigma field. And again, just taking that motivation, what we do is that we will focus our attention on this left open right closed intervals. So, take this $(a, b]$, but here remember that we have to take, we can take the points a, b to be ∞ or $-\infty$, they can be like that.

So, in that case, what you take in the extended real line, you take this type of open intervals, left open right closed intervals and generate a sigma field, whatever you get, I will call that

Borel sigma field on the extended real line. So, here again, just to repeat, the sets $(a, b]$ are subsets of this extended real line and, therefore, may include the points $\pm\infty$.

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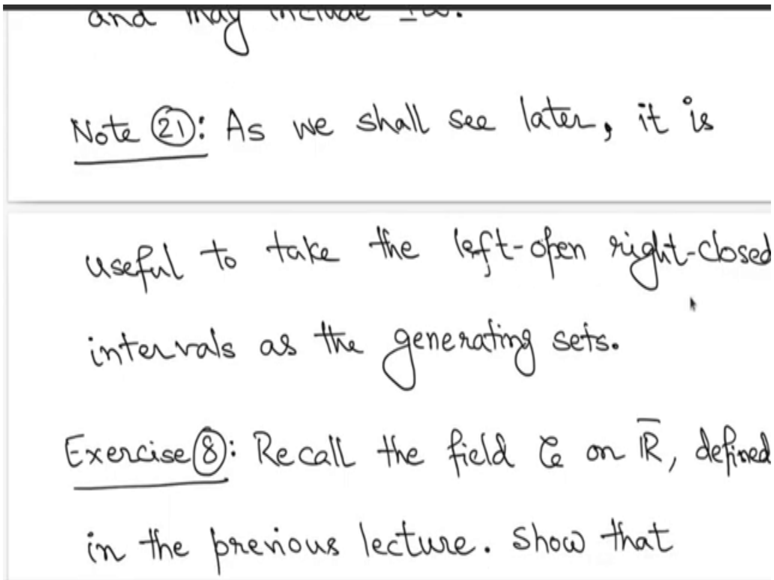

Note 21: So, you might ask, what about the usefulness of left open right closed intervals? Why not look at other types of sets? But we shall see the usefulness later on in connection with some distribution functions when we discuss that later on.

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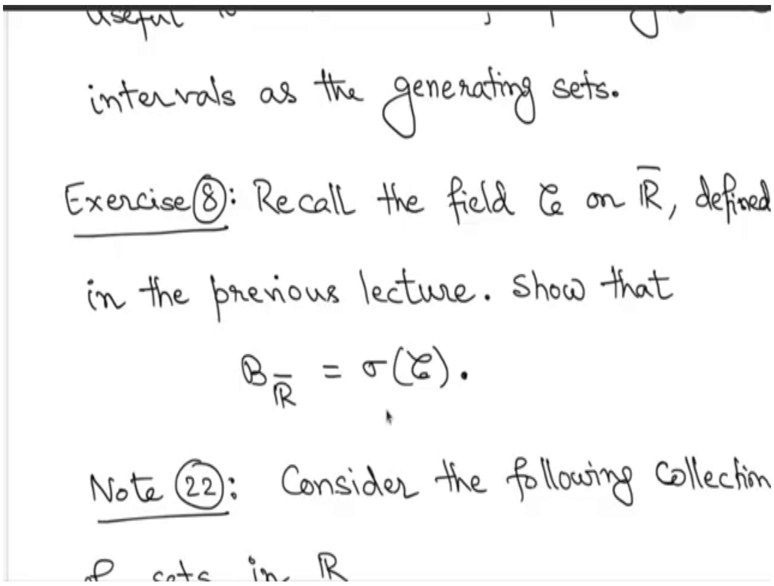

Exercise 8: Remember that we have also discussed a field $\mathcal{C}$ not a sigma field on the extended real line. So, that was defined earlier. Yes, you can check that this also generates the same sigma field, the Borel sigma field, as defined above.

So, you have only taken the definition of the Borel sigma field on the extended real line as the sigma field generated by left open right closed sets, but now we are saying take the field which is not a sigma field. That example on the extended real line and try to see that it will also generate the same sigma field. So, the argument goes as the same argument that was done on the real line side, but it can again be the same argument that will work here. Try to write it down.

So, now, there is a very interesting observation. Let us look at the Borel sigma field on the extended real line that we have observed. So, the sets that you see there, the Borel sets that you see on the extended real line, may contain $\pm\infty$ because these are subsets of the extended real line. But take the intersection with the real line. So, this is a notation for a collection of sets.

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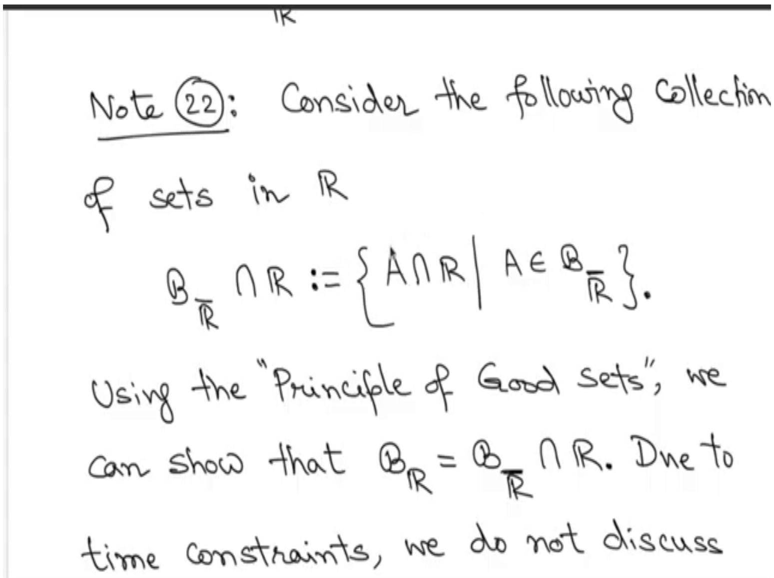

So, we mean that on one side of this intersection, you will see that we are looking at a collection of sets of the collection of subsets of the extended real line, and we are intersecting that with a real line. What does it mean? It simply means that we take sets A and intersect to the real line. So, these are the other types of sets that I am going to consider. And that is a notation that we have written intersection with ℝ, which is the notation we are interested in.

Now, we are saying that take Borel sets A, take their intersection with the real line. Now, what is happening, is that if the set, if the Borel set A contains $\pm\infty$, you are simply removing that. It may so happen that the set A does not contain $\pm\infty$. In that case, it is already within the real line, and it will be a subset of the real line. If there is $\pm\infty$ there, you are simply removing that and taking the rest of the set. So, that is the collection that you are considering now.

$$\mathcal{B}_{\overline{\mathbb{R}}} \cap \mathbb{R} := \{ A \cap \mathbb{R} \mid A \in \mathcal{B}_{\overline{\mathbb{R}}} \}.$$

Using the "Principle of Good Sets", we can show that $\mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\overline{\mathbb{R}}} \cap \mathbb{R}$. Due to time constraints, we do not discuss this principle/method in this course.

<u>Borel σ- field on $\mathbb{R}^d$</u>

Now, there is a general principle called the principle of good sets. We can show that this collection that you have just talked about will match the Borel sets on the real line or the Borel sigma field on the real line. So, therefore, what we are saying is that if you take any Borel set on the real line, you can write it as an intersection of Borel set on the extended real line intersection with the actual real line. So, that is the description here.

But this principle of good sets, it is a bit technical, and due to certain time constraint, we are not going to discuss this principle or this methodology in this course, this is for your information that there is some general methodology called the principle of good sets that will allow you to prove such equalities.

So, with that at hand, now we move forward to the Borel sigma field on the higher dimensions. And for simplicity, do not jump from one dimension to a very arbitrary higher dimension. Let us go to two dimensions. So, on dimension $d = 2$ using the ideas discussed above, what do we do? We again look at left open, right closed sets, but we are now going to look at a certain special type of thing; rectangles and rectangles will be made up of such things.

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this principle/method in this course.

Borel σ-field on $\mathbb{R}^d$

For simplicity, consider $d=2$.

Following the ideas discussed above,

we define the Borel σ-field $\mathcal{B}_{\mathbb{R}^2}$ on $\mathbb{R}^2$

by

$$\mathcal{B}_{\mathbb{R}^2} := \sigma\left(\left\{(a,b]\times(c,d] \;\middle|\; -\infty \le a<b\le \infty \right.\right.$$

So, you take $(a,b]$ and another $(c,d]$. Take their product, so I mean that I take points on the two dimensions whose first coordinate is lying within the set $(a,b]$ and the second coordinate is lying within the set $(c,d]$. So that is how these points are described. So, that is kind of a rectangle that you see on the two dimensions.

Now, look at such rectangles and look at this collection, then generate a sigma field. So again, the generation of sigma fields that can be done on any non-empty set, you can always generate a sigma field if you have a specific collection of sets. So, we are looking at this special choice of rectangles on two-dimensions and generating a sigma field, and whatever we get, we define it as a Borel sigma field on $\mathbb{R}^2$.

we define the Borel σ-field $\mathcal{B}_{\mathbb{R}^2}$ on $\mathbb{R}^2$

by

$$\mathcal{B}_{\mathbb{R}^2} := \sigma\left(\left\{(a,b]\times(c,d] \;\middle|\; \begin{array}{l} -\infty \le a<b\le \infty \\ -\infty \le c<d \le \infty \end{array}\right\}\right),$$

More generally, the Borel σ-field $\mathcal{B}_{\mathbb{R}^d}$ on $\mathbb{R}^d$

is defined by

So, again the motivation is simply coming from one dimension. So, on one dimension, we focused our attention to $(a, b]$ this type of left open right closed intervals, and the counterpart of that in two dimensions is this product type sets.

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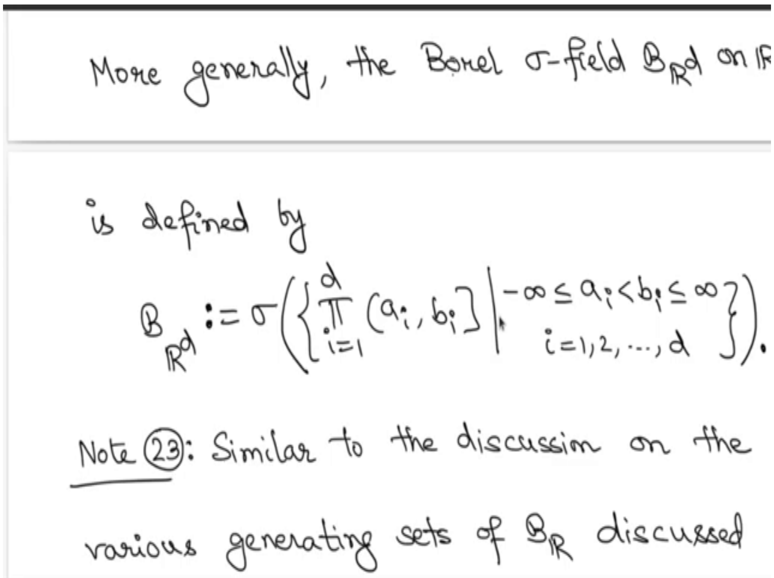

More generally, if you go to higher dimensions, d dimensions arbitrary dimensions, all you have to look at is this d fold product of such intervals. So, by that I mean you look at intervals $(a_i, b_i]$, so that the if component lies within the set $(a_i, b_i]$ and take the d fold product, whatever that rectangle or the cube or general set that you get in d dimensions, you look at the collection of sets, collections of cubes, d dimensional cubes and generate a sigma field. Whatever you get, we define it as the Borel sigma field on $\mathbb{R}^d$.

So, with this motivation that we have already taken from the real line, we are extending this notion of Borel sigma fields to the extended real line and the higher arbitrary dimensional, higher arbitrary dimensional Euclidean spaces.

Note 23: But, you can now ask that we started this lecture with different generating sets for the Borel sigma field on the real line. But you can now ask, are there any other types of sets that you can consider, for example, on R2 on two-dimensional Euclidean space? Are there other generating sets that will generate the same Borel sigma field on R2? The answer is yes, but we in this is possible, you this again you can take the motivation from one-dimensional arguments and extend these notions appropriately.

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$\mathbb{R}^d \quad (\lbrack i=1 \quad \mid \quad i=1,2,\dots,d \; \rbrace).$

Note 23: Similar to the discussion on the various generating sets of $\mathcal{B}_{\mathbb{R}}$ discussed above, other generating sets of $\mathcal{B}_{\mathbb{R}^d}$ may be discussed. In this course, we shall not require these generating sets.

But in this course, again, due to time constraints, we will not discuss this. Again we will not require those other generating sets if we even if we go to higher dimensions. This is for your information. You can still work out other generating sets on the extended real line or the high dimensional Euclidean spaces.

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Borel σ-field on Borel Subsets of $\mathbb{R}$

let E be a Borel set in $\mathbb{R}$.

Motivated by Note 22, we consider the following collection of subsets of E,

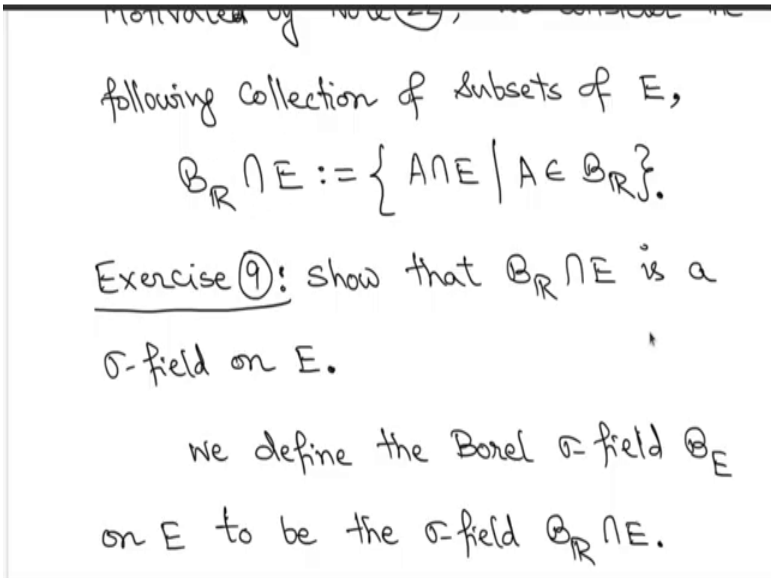
$$\mathcal{B}_{\mathbb{R}} \cap E := \{ A \cap E \mid A \in \mathcal{B}_{\mathbb{R}} \}.$$

Exercise 9: Show that $\mathcal{B}_{\mathbb{R}} \cap E$ is a

And now, we come to the final topic of this lecture. So, look at we now look at Borel subsets, let E be the Borel subset. Now is a non-empty set. Now, what do you want is that we want to define a sigma field of subsets of E_0 and we want to get the notion of Borel sigma fields here and motivated by this previous notion in note 22 what we have observed there was that the

Borel sigma field on the extended real line if you take their intersection with the real line, you get back the Borel sigma field on the real line.

So, let us repeat, the real line was a subset of the extended real line. You have a Borel sigma field on the extended real line, the bigger set. Take intersection with the smaller sets that is the real line. You end up with the Borel sigma field on the smaller set the real line. So, therefore, with this motivation, you can now say, okay, since E is a subset of the real light, R is the real line is now a bigger set. We already have a Borel sigma field on the real line.



following collection of subsets of E,

$$\mathcal{B}_{\mathbb{R}} \cap E := \{ A \cap E \mid A \in \mathcal{B}_{\mathbb{R}} \}.$$

Exercise 9: Show that $\mathcal{B}_{\mathbb{R}} \cap E$ is a σ-field on E.

We define the Borel σ-field $\mathcal{B}_E$ on E to be the σ-field $\mathcal{B}_{\mathbb{R}} \cap E$.

Exercise 9: What happens to the collection of sets if you take the intersection with E? And that this collection of sets, so we look at all Borel sets, A, Borel sets of the real line A, E take their intersection with E, and you can now try to show that it this intersection is yes, this intersection is a sigma field on E.

And just to clarify, whatever operations set operations that you do here, it is within the set E that means you take union subsets within E, so $A \cap E$ these types of sets are already subsets of E. So, if you take in your answer intersections, they only lie within the set E. So, that is not a problem. And if you are considering complementation, you have to be careful. You are taking complements within the set E.

So, if you take a subset of E which is of the form $A \cap E$, you look at its complement. The complement is taken within the set E, because E is now the big set you will consider. So, try to check that these collection subsets $B_{\mathbb{R}} \cap E$, that will be a sigma field.

Note 24: With that motivation at hand, we define the Borel sigma field, which we write as B_E to the sigma field, this intersection that was obtained from the Borel sets on the real line. So, that is our motivation, and that is what we define. But using certain notions of topology, it is good to know that you can define certain notions of open sets E. So, there is some notion of topology on subsets. You can talk about that subspace topology and things like that, but let us not go into it.

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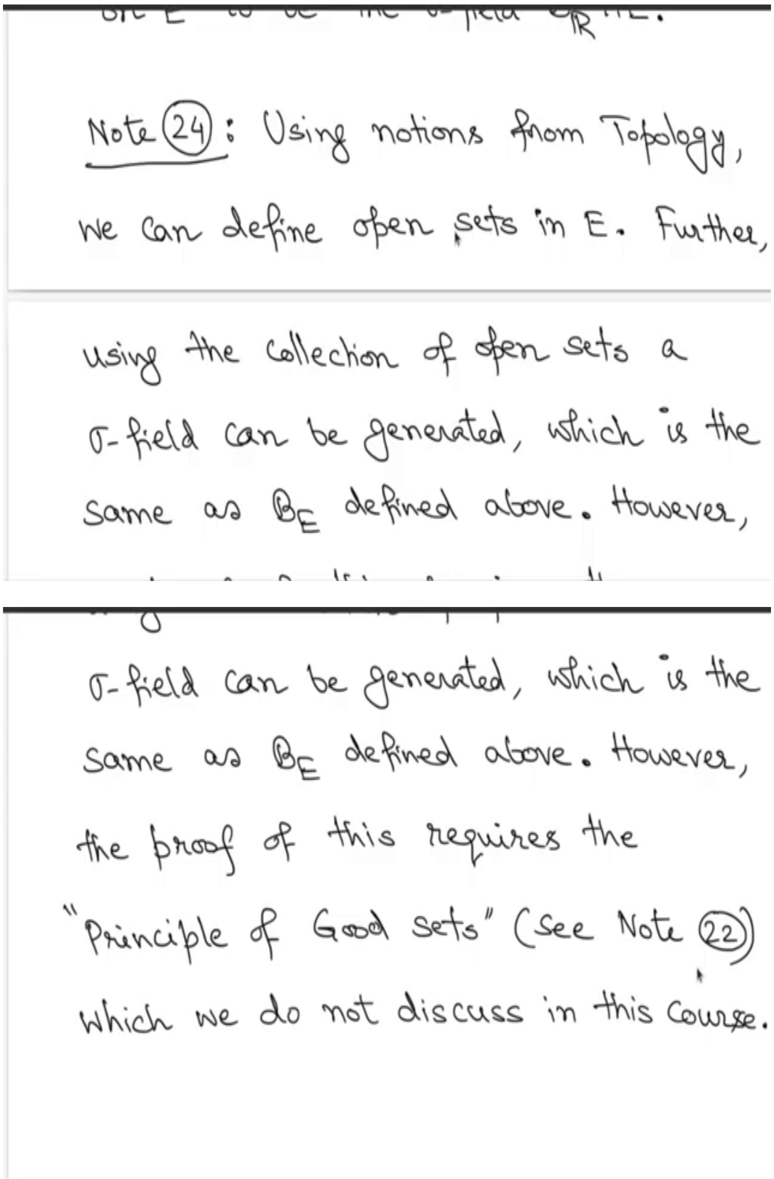

But if you have a collection of open sets within these sets E, you can talk about generating a sigma field out of it. And what you can show is that the collection of all such open sets in E will be the same as the sigma field that was just defined above. So, this in sense matches with the notion of the Borel sigma field originally defined on the real line anyway.

So, this is good to know that the Borel sigma field, in a sense, is generated by all open sets. Still, for simplicity, we will work with this notion of intersection with the Borel sigma field on the real line, because we know the Borel sigma field on the real line very well, we have many, many explicit sets, which we know about which are in the Borel sigma field. So, that is why we will concentrate our attention on the Borel sigma field on the real line, and if required, we shall go to d dimensional Euclidean spaces $\mathbb{R}^d$.

Again, this one-dimensional idea multiplying the sets appropriately or going to subsets E or going to the extended real line. But just to comment, this identification that the open sets in E will generate the same sigma field as that intersection with the Borel sigma field on the real line will require an argument involving this principle of good sets that again was mentioned earlier in note 22. Again just to repeat, this is a general technical result. And again, due to time constraints, we are not going to discuss this. Take this as the fact that this can be done. So, we stop the lecture here.