Measure Theoretic Probability 1
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Lecture 36
Decomposition of Measures

Welcome to this lecture. This is the first lecture of week 8. In this lecture, we are going to study some important structural properties of measures. Using these results and consolidating whatever we have learned throughout this course, later in the week we are going to make some very interesting comments. In particular, as promised in a later lecture we are going to finally talk about Absolutely Continuous Random Variables and their expectation.

In particular, we are going to see that as seen for the case of discrete random variables the usual formulations that you have seen in your basic probability theory, agree with the major theoretical structure, but all of these are now coming under the umbrella of measure-theoretic structures and this is a very very general framework now. So let us move ahead and discuss the structural properties of measures.

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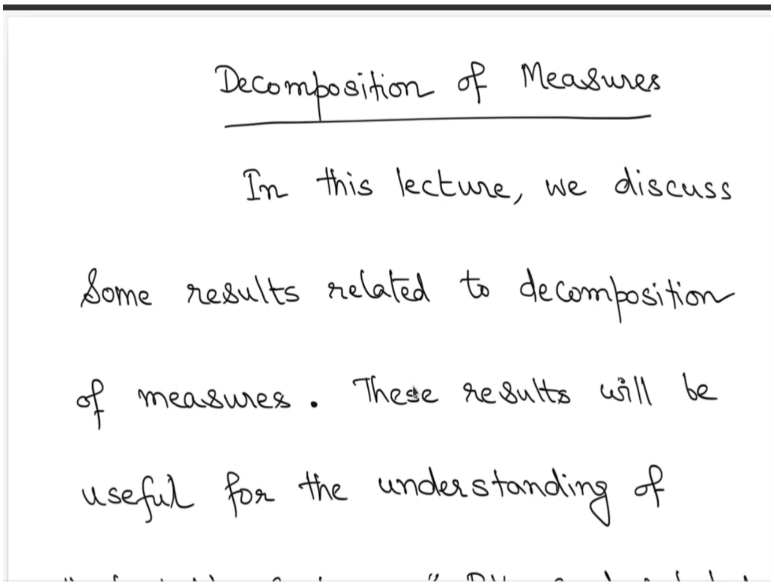

Some results related to decomposition of measures. These results will be useful for the understanding of "absolutely continuous" RVs and related computation of the expectation of such RVs.

RVs.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

Note ①: Recall from Proposition ③ (also Note ⑭ and Exercise ④) of Week 6 that the set function $\nu(A) := \int_A f \, d\mu, \ \forall A \in \mathcal{F}$, for any fixed non-negative measurable

In this lecture we are going to focus on certain results which are usually referred to as decomposition of measures. And using these results later on we are going to talk about absolutely continuous random variables and their expectations. Before proceeding let us first fix the notation, for us we will work with measure μ on some measurable space (Ω, $\mathcal{F}$), so (Ω, $\mathcal{F}$, μ) will construct a measure space.

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function f, is also a measure. Moreover, in this case,

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(i) ν is a finite measure if f is integrable, i.e. $\int |f| d\mu = \int f \, d\mu < \infty$.

(ii) ν is a σ-finite measure if μ is σ-finite.

Note ②: If a measurable function f

Let us first recall this result from week 6 that given any non-negative and measurable function f, if you fix the measure μ and fix that measurable function f but vary the sets over which this integration is being conducted, you are going to get back a measure and we have also remarked earlier that there are some special cases where if you are dealing with f to be integrable, so that integration of $|f|$ is nothing but integration of f which is finite, f is given to be non negative.

So $|f| = f$, so integration of $|f|$ equals integration of f itself and if f is integrable these integrations are finite, but this is the total mass associated to the set Ω under the measure ν . Therefore, this ν becomes a finite measure under this condition that f is integrable with respect to the measure μ.

So, here integrability refers to the original measure and moreover you can also make this statement that for a general non-negative measurable function you expect to get a measure ν to be σ-finite provided your given measure μ is σ-finite. So you have these structures for such new set function, great.

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Note 2: If a measurable function f is μ-integrable, i.e.

$$\int f^+ d\mu < \infty \text{ and } \int f^- d\mu < \infty,$$

then the set functions

$$\nu^+(A) := \int_A f^+ d\mu \text{ and } \nu^-(A) := \int_A f^- d\mu, \ \forall A \in \mathcal{F}$$

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are finite measures. Recall from Note 17 of week 3 that

$$\{\omega : f^+(\omega) > 0\} \cap \{\omega : f^-(\omega) > 0\} = \phi.$$

Then for all $A \in \mathcal{F}$, we have

(i) $\nu^+(A) = 0$ if $A \subseteq \{\omega : f^-(\omega) > 0\}$

But also consider this more general case that if you are dealing with a measurable function now, you can take positive or negative values, but if you assume that this is new integrable, so that is as per definition integrations of f^+ and f^- both should be finite. Then separately f^+ and f^- are non negative measurable functions and therefore, you can talk about this new set functions which we write as ν^+ and ν^-.

What are these? You have fixed the function f, so f^+ and f^- these functions are also fixed. They are non-negative and integrable. Now if you work with different sets coming from your domain side σ-field you are going to construct these set functions ν^+ and ν^-. So these are just integrating this non-negative functions f^+ and f^-, which are also integrable so as mentioned earlier these will be finite measures.

But recall earlier that we had mentioned that these functions f^+ and f^- cannot be simultaneously positive at a point, so the set of points ω such that f^+ takes positive values and f^- takes positive values that set is an empty set. This was mentioned in note 17 of week 3.

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Then for all $A \in \mathcal{F}$, we have

(i) $\nu^+(A) = 0$ if $A \subseteq \{\omega : f^-(\omega) > 0\}$

(ii) $\nu^-(A) = 0$ if $A \subseteq \{\omega : f^+(\omega) > 0\}$

Definition 1 (Mutually Singular measures)

let μ_1 and μ_2 be two measures on $(\Omega, \mathcal{F})$. we say that μ_1 is singular

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let μ_1 and μ_2 be two measures on $(\Omega, \mathcal{F})$. we say that μ_1 is singular
with respect to μ_2 or μ_2 is singular with respect to μ_1 or μ_1 and μ_2 are mutually singular, if there exists a set $A \in \mathcal{F}$ such that $\mu_1(A) = 0$ and $\mu_2(A^c) = 0$

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In this case, we write $\mu_1 \perp \mu_2$.

<u>Examples of mutually singular measures</u>

Therefore, you observe that ν^+ which is defined in terms of the function f^+ that will be 0 if you are working with sets where f^- is strictly positive, so their f^+ must be 0. On the other hand, if your set A is contained in the set f^+ being positive, so there the function f^- is 0, therefore, if you integrate there ν^- will give you the value 0.

So, that is what it happens that you get these two distinct sets, disjoint sets where you get these measures are providing masses separately, correct? So, with this observation, we now make this definition of mutually singular measures. So, what is the definition? That you take two measures μ_1 and μ_2 on the same measurable space.

You say that μ_1 is singular with respect to μ_2 or μ_2 is singular with respect to μ_1 or μ_1 and μ_2 are mutually singular. So, all these three terminologies are the same, so you can refer to μ_1 being singular with respect to μ_2 or you can say μ_2 is singular with respect to μ_1 or just to be very clear you can just say μ_1 and μ_2 are mutually singular.

So, all of these mean the same thing and that is defined as follows that you can find some set in your σ-field such that μ_1 associates 0 mass to that set, but μ_2 associates 0 mass to its complement so it is as if you are dividing the whole domain Ω into two parts A and A^c. On the first part μ_1 assigns 0 mass, so μ_1 is essentially focusing on A^c part.

μ_2 assigns 0 mass to A complement, so essentially μ_2 is concentrating its attention on the first part A. So, therefore, if you get any subsets of A, there μ_1 will associate 0 mass. If you get a set which is a subset of A^c there μ_2 will associate 0 mass, that is the idea. So, this is what is known as Mutual singularity and in this case so this is the perpendicular notation symbol that we are using to denote this situation.

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$\{\omega : f^+(\omega) > 0\} \cap \{\omega : f^-(\omega) > 0\} = \phi.$

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Examples of mutually singular measures

(i) In continuation of Note 2, we have $\nu^+ \perp \nu^-$.

(ii) Consider the Dirac measure δ_0 and the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

then the set functions

$$\nu^+(A) := \int_A f^+ \, d\mu \quad \text{and} \quad \nu^-(A) := \int_A f^- \, d\mu, \ \forall A \in \mathcal{F}$$

are finite measures. Recall from Note 17 of week 3 that

$$\{\omega : f^+(\omega) > 0\} \cap \{\omega : f^-(\omega) > 0\} = \phi.$$

Then for all $A \in \mathcal{F}$, we have

So, what are examples? Whatever we have mentioned earlier in note 2, we had mentioned two measures ν^+ and ν^-, so let us just go back. So, what we said was that if you look at this general measurable function f which is given to be integrable and look at its positive part and negative part. There you focus on ν^+ and ν^-.

Now, observe this we had mentioned that ν^+ assigns 0 mass to such sets and ν^- assign 0 Must to subsets and we can always ignore the set where the function the given function f takes 0 values. That anyway you can ignore because that is where there will be no contribution of the integration of f.

f takes 0 value there anyway if you wish you can include that set to any of these sets where f^+ is positive or f^- is positive. Either way you get these sets two distinct sets where on one side ν^+ assign 0 mass on another side ν^- assign 0 mass. Therefore, with this observation and this definition putting them together you will write this that ν^+ and ν^- are mutually singular. Let us see a more explicit example of mutually singular measures.

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have $\nu^+ \perp \nu^-$.

(ii) Consider the Dirac measure δ_0 and the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then for $A = \mathbb{R} \setminus \{0\}$, we have $\delta_0(A) = 0$ and $\lambda(A^c) = \lambda(\{0\}) = 0$. Thus, $\delta_0 \perp \lambda$.

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Exercise 1: (i) For $x \in \mathbb{R}$, consider the Dirac measure δ_x on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Show that $\delta_x \perp \lambda$.

(ii) Let μ be the law of a discrete RV. Show that $\mu \perp \lambda$. (Hint:

(ii) Let μ be the law of a discrete RV. Show that $\mu \perp \lambda$. (Hint:

Recall from Note 22 of week 4 that μ is a convex linear combination of Dirac measures. Look at the support of the RV.)

Consider these two measures on the real line, the first one being the Dirac measure supported at the point 0, the other one is the Lebesgue measure. Consider this set A which is the non-zero part of the real line, the complement of the singleton set 0. Here what do we get? We get delta 0 assigns 0 mass to this set but on A^c, λ assign 0 mass because λ, the Lebesgue measure assigns 0 mass to all singleton sets.

Therefore, as per definition we have split the whole real line into two parts A and A^c. On one part Dirac 0 that assigns 0 mass, on another part the other measure Lebesgue measure assigns 0 mass. As for definition the Dirac measure supported at 0 and the Lebesgue measure are mutually singular.

But you can expect now that you can extend this example to involve any arbitrary Dirac measure and say that for any such Dirac measure it will be mutually singular with the Lebesgue measure, but then more generally remember that a discrete random variable has the law which is a convex linear combination of Dirac measures. So what you can now try to claim that if μ is such a law then you can try to show that μ is mutually singular with respect to the Labesgne measure.

So that is the idea that you can extend the class of examples for mutual singularity with respect to the Lebesgue measure. So originally you had Dirac measures but we are now observing that convex linear combinations we will also do, but convex linear combinations of Dirac measures are nothing but laws for discrete random variables. You can try to work out this exercise, great!

(Refer Slide Time: 10:48)

Note 3: In the above examples, we have

used λ-null sets. Recall from Note 22

of week 5, the example of an uncountable

set A, called the Cantor set, such that

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then $\mu \perp \lambda$, since $\mu(A^c)=0$. In

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μ with the corresponding distribution

function F_μ being continuous. We do

Now in this above examples what we have actually done is that for these explicit examples we have used this Null set with respect to the Lebesgue measure. Recall from earlier discussion in week 5 in note 22 that there is an example of an uncountable set A with 0 mass, and we had called such an example to be a Cantor set. This is a very specific example of a set which is uncountable and has Lebesgue measure 0, so this is a very specific set.

Now if you have any probability measure μ such that it assigns full mass to this set then what will happen, is that it will assign 0 mass to the complement if μ is a probability measure and if it assigns full mass to the set A then on the complement of it you will assign 0 mass, and therefore, you now get this exact observation that for such a probability measure you will get that μ is mutually singular with the Lebesgue measure.

This is a somewhat of an extension of the type of mutual singularity that you have already observed. In particular there are examples of probability measures which assigns full mass to the Cantor set but its distribution function F_{μ} is continuous.

(Refer Slide Time: 12:22)

not go into the details. The interested reader may refer to the pages 13-14 and page 31 from "A Course in Probability Theory" by Kai Lai Chung, Third edition, Academic Press.

Definition ② (Absolute Continuity of measures

Third edition, Academic Press.

<u>Definition</u> ② (Absolute Continuity of measures)

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μ_2- null sets, i.e if $\mu_1(A)=0$ for Some $A \in \mathcal{F}$, then $\mu_2(A)=0$.

So, we are not going into the details if you are interested, please refer to pages 13 and 14 and page 31 from this book by Kai Lai Chung, the book title is "A Course in Probability Theory" this is the third edition of this book published by Academic Press. We are not going to go into the details it involves many technical constructions.

There is a opposite type of a structure involving measures this is something called Absolute Continuity of measures, so again start with two measures and we are going to compare them in some aspect so these two measures are defined on this measurable space $(\Omega, \mathcal{F})$. Say that μ_2 is absolutely continuous with respect to μ_1 , which we denote by this symbol μ_2 absolutely continuous with respect to μ_1 if μ_1 Null sets are also μ_2 Null sets.

So, what do I mean that if you get any Null set A for μ_1 and try to check its size under μ_2 that must also be 0 so any kind of an impossible event with respect to the first measure μ_1 must be impossible under the second measure. By impossible event we mean Null sets, sets with or events with probability 0. Here we are comparing two measures now great.

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Note 1: Recall from Proposition 3 (also Note 14 and Exercise 4) of week 6 that the set function $\nu(A) := \int_A f \, d\mu, \; \forall A \in \mathcal{F}$, for any fixed non-negative measurable function f, is also a measure. Moreover, in this case,

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(i) ν is a finite measure if f is integrable, i.e. $\int |f| d\mu = \int f \, d\mu < \infty$.

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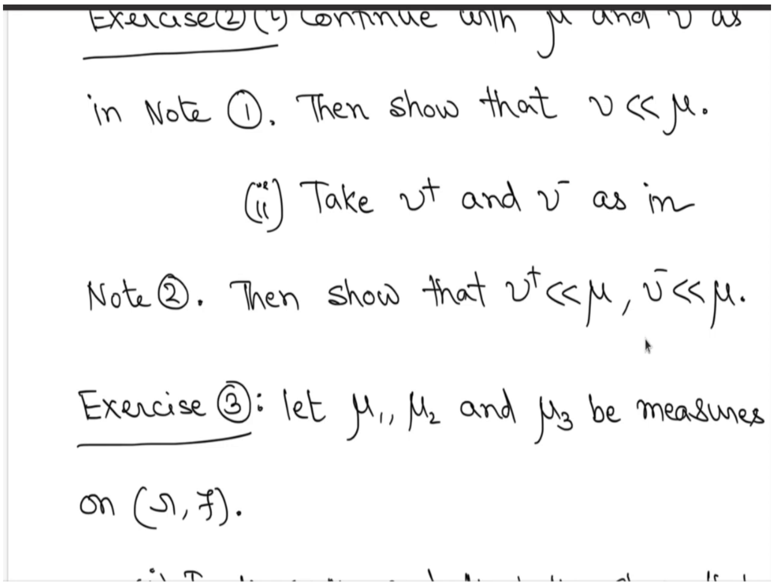

Let us try to understand what this concept actually implies, to do this there are these very simple exercises. Continue with this μ and ν as in note 1 earlier, what are this μ and ν know so let us go back to this note one, so in note one we looked at this proposition from week six which said that for non-negative measurable functions if you integrate them against these various sets you get a measure. That was this construction so remember this choice of μ and ν. Here what we now claim is that there is some such a relation between this νmeasure and the original measure Mu.

That is the concept that is being discussed here. We are claiming in this exercise part one of it that the ν measure is absolutely continuous with respect to the original measure Mu, check this. Again, in note 2 what did? We do we took a general measurable function but μ integrable. In this case what happened was that we split it into positive parts and negative parts and correspondingly defined the set functions ν^+ and ν^-, and these also turned out to be finite measures provided the original function f was μ integrable.

In this case again you can try to show that both these measures ν^+ and ν^- both are absolutely continuous with respect to the original measure μ. These are very explicit examples of absolute continuity of measures, and now you shall ask how does absolute continuity and mutual singularity relate?

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on $(\Omega, \mathcal{F})$.

(i) If $\mu_2 \ll \mu_1$ and $\mu_1 \perp \mu_3$, show that $\mu_2 \perp \mu_3$.

(ii) If $\mu_2 \ll \mu_1$ and $\mu_3 \ll \mu_2$, show that $\mu_3 \ll \mu_1$.

(iii) If $\mu_2 \ll \mu_1$ and $\mu_2 \perp \mu_1$, show that

(ii) If $\mu_2 \ll \mu_1$ and $\mu_3 \ll \mu_2$, show that $\mu_3 \ll \mu_1$.

(iii) If $\mu_2 \ll \mu_1$ and $\mu_2 \perp \mu_1$, show that

$\mu_2(A) = 0 \ \forall A \in \mathcal{F}$.

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measures. We state the result without proof.

Recall from Exercise ⑤ of

To do this there is these very simple exercises once more please try to work at it. Choose three measures μ_1, μ_2 , and μ_3 defined on the same measurable space. If it happens that μ_2 is absolutely continuous with respect to μ_1 and μ_1 is mutually singular to μ_3 then you can claim that μ_2 and μ_3 are mutually singular. That is the first statement.

The second statement says that if μ_2 is absolute continuous with respect to μ_1 and μ_3 is absolutely continuous with respect to μ_2 then if you combine them together you can claim that μ_3 is also absolutely continuous with respect to μ_1. Again, here you are just comparing the Null sets, so please check this exercise. And the final statement says forget about μ_3 look at μ_1 and μ_2 if it so happens that μ_2 is absolutely continuous as well as mutually singular with respect to μ_1 then what you can try to show is that μ_2 that measure assigns 0 mass to all the sets, so this is the 0 measure.

This you can try to check, so this is what is connecting the mutual singularity and absolute continuity of measure. These are certain important structural properties, and in the next theorem we are going to use the structural properties to discuss a very important decomposition result for measures. We are not going to state the proof we are just going to state the result.

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Recall from Exercise 5 of week 2 that addition of measures gives more examples of measures.

<u>Theorem 1</u> (Lebesgue Decomposition Theorem)

Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$. Then there exist measures

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Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$. Then there exist measures

ν_1 and ν_2 such that $\nu = \nu_1 + \nu_2$ and $\nu_1 \perp \mu$ and $\nu_2 \ll \mu$. Further, this decomposition is unique.

decomposition is unique.

Note ④: Let ν be a probability measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. By Theorem ①, we have $\nu = \nu_1 + \nu_2$ with $\nu_1 \perp \lambda^{(d)}$ and $\nu_2 \ll \lambda^{(d)}$, where $\lambda^{(d)}$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Recall from earlier that in week 2 we had mentioned that we can add measures and get examples of measures. In particular we had used such structures to compute expectations of discrete random variables by saying that the law of a discrete random variable is basically a convex linear combination of certain Dirac masses. This is a more general structure than that here we are adding up measures, so this theorem is known as the Lebesgue decomposition theorem.

Take two measures μ and ν suppose they are σ-finite, both of them are defined on the same measurable space and they are σ-finite, Now provided these two measures what you can do is that you can find out measures ν_1 and ν_2 such that ν the second measure has a decomposition like $\nu_1 + \nu_2$, so you can add them up you can split it up like this and measure ν_1 is mutually singular with respect to μ and measure ν_2 is absolutely continuous with respect to Mu.

There are these two parts; one part is absolutely continuous another part is mutually singular with respect to the first measure Mu. Moreover, this decomposition is also unique in the following sense if you can split the measure ν into two parts, let us say $\tilde{\nu}_1 + \tilde{\nu}_2$, such that $\tilde{\nu}_1$ is Mutually singular with respect to μ and $\tilde{\nu}_2$ is absolutely continuous with respect to μ then you can show that ν_1 is $\tilde{\nu}_1$ and ν_2 is $\tilde{\nu}_2$.

This decomposition of choices of ν_1 and ν_2 are fixed, given these two measures μ and ν you can split the second measure according to the information coming from the first measure.

That is the Lebesgue decomposition theorem, but then how do you apply it? Our interests are involving probability measures.

Let us work in d-dimension, this results also applies to the one dimensional case by choosing dimension d equal to 1. What do you do? Choose a probability measure ν . By this Lebesgue decomposition theorem you can split this measure ν in two parts, ν_1 and ν_2 such that the first one, the first part is mutually singular with respect to the Lebesgue measure in d dimension and the second part is absolutely continuous with respect to the Lebesgue equation.

So, remember probability measures are finite measures and in particular they are σ-finite. Lebesgue measure we have already seen to be σ-finite. So, this is where we are applying the Lebesgue decomposition theorem and splitting up a given probability measure in terms of the information coming from the Lebesgue measure. We know a lot of information about the structural properties of the Lebesgue measure and would like to use that information in analyzing the probability measure ν , so that is the idea here.

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decomposition is unique.

Note 4: Let ν be a probability measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. By Theorem 1, we have $\nu = \nu_1 + \nu_2$ with $\nu_1 \perp \lambda^{(d)}$ and $\nu_2 \ll \lambda^{(d)}$, where $\lambda^{(d)}$ denotes the Lebesgue measure on $\mathbb{R}^d$.

decomposition is unique.

Note ④: Let ν be a probability measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. By Theorem ①, we have $\nu = \nu_1 + \nu_2$ with $\nu_1 \perp \lambda^{(d)}$ and $\nu_2 \ll \lambda^{(d)}$, where $\lambda^{(d)}$ denotes the Lebesgue measure on $\mathbb{R}^d$.

The next theorem characterizes

The next theorem characterizes all measures which are absolutely continuous with respect to a given σ-finite measure. Examples have been mentioned in Exercise ②. The result below is stated without proof.

Theorem ② (Radon-Nikodym Theorem)

Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$. Then there exists a non-negative measurable function f such that

$$\nu(A) = \int_A f \, d\mu, \quad \forall A \in \mathcal{F}.$$

You can now ask, can you say something more about this absolute continuity? So, for mutual singularity you do not expect to say anything more given the information about the Lebesgue measure. So if you go back to this, ν_1 is mutually singular, Lebesgue measure will associate 0 mass to that set where ν_1 is associating possibly non-negative values possibly non-trivial values.

There λ the Lebesgue measure is not going to give you any information because Lebesgue measure is associating 0 mass there, so you cannot say anything about this but can you say anything about in the absolute continuous part of it? That is this theorem, the next theorem, this is going to characterize all measures which are absolutely continuous with respect to a given σ- finite measure.

Now, we have mentioned earlier about these examples in exercise 2. So, in exercise 2 what we essentially said was that if you are given a measure μ and if you integrate functions non-negative measurable functions you are going to get certain measures. These are some examples at hand now we would like to say are there any other examples of absolute continuous measures with respect to μ? So that is what we are trying to understand.

And the next result says that these are all the things, so this is known as the Radon--Nikodyn Theorem, and again we are just going to state it we do not go into the proof. This is again stated for the σ-finite measures case; again, take two measures on the same measurable space such that the second measure is absolutely continuous with respect to the first one.

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function f such that

$$\nu(A) = \int_A f \, d\mu, \quad \forall A \in \mathcal{F}.$$

Further, this function f is unique in the following sense: if g is another such function, then $f = g$ μ-a.e..

Definition ③ (Radon-Nikodym derivative or Radon-Nikodym density)

The μ-a.e. unique function obtained in Theorem ② is called the Radon-Nikodym derivative/density of ν with respect to μ and is denoted by $\frac{d\nu}{d\mu}$.

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Exercise ④: let μ, ν and γ be σ-finite

Now the statement says that there exists a non-negative measure of function f such that the second measure can be written as an integration involving the functions f with respect to the first measure. You can figure out such a function f which is non-negative and measurable.

Therefore, we are saying that given this measure μ which is the first measure if you ask what are all the absolute continuous measures ν are, you will immediately say that those which are given by such a expression or such a structure as an integration over some appropriate non negative measurable function f.

All you have to do is to vary this set A and you are going to get back the size of the sets under the new measure. Moreover, you can say something more about this representation that this function f is unique in this following sense. What is this idea is that if you can write it as a integration of g d μ over a set A. So if you can write it that way for another such function which is also non negative and measurable then it can be shown that $f = g$ almost everywhere with respect to the first measure μ.

Outside a μ Null set possibly these two functions may not agree but overall, almost everywhere these two functions match. That is the uniqueness of this function f. Here given these two measures if the second measure is absolutely continuous with respect to the first one you can write down such a structural property. That is the statement of this theorem.

With this theorem at hand, we now make this definition, call this function this almost everywhere unique function that you have obtained as the Radon-Nikodyn derivative or Radon-Nikodyn density of the second measure ν with respect to the first measure μ and we are going to denote this by $\frac{d\nu}{d\mu}$.

This is the idea that given this second measure ν which is absolutely continuous with respect to the first measure μ you can get this function f which is μ almost everywhere unique and you are going to call it the Radon-Nikodyn derivative or Radon-Nikodyn density of ν with respect to μ and you are going to write in symbols this.

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respect to μ and is denoted by $\frac{d\nu}{d\mu}$.

__Exercise ④:__ let μ, ν and γ be σ-finite measures on $(\Omega, \mathcal{F})$. Suppose $\nu \ll \mu$.

(i) For Borel measurable functions h on $(\Omega, \mathcal{F})$, show that

measures on $(\Omega, \mathcal{F})$. Suppose $\nu \ll \mu$.

(i) For Borel measurable functions h on $(\Omega, \mathcal{F})$, show that

$$\int h \, d\nu = \int h \frac{d\nu}{d\mu} \, d\mu,$$

if the integrals exist. (Hint: Verify the equality when h is an Indicator/

the equality when h is an Indicator/

Simple function)

(ii) If $\gamma \ll \nu$, by Exercise ③, we have $\gamma \ll \mu$. Show that $\frac{d\gamma}{d\mu} = \frac{d\gamma}{d\nu} \cdot \frac{d\nu}{d\mu}$.

(iii) If $\mu \ll \nu$, show that $\frac{d\mu}{d\nu} = \frac{1}{\frac{d\nu}{d\mu}}$.

Let us try to understand what this Radon-Nikodyn density is all about. So choose three σ-finite measures μ, ν and γ on the same measurable space and suppose you start with that setup of the Radon-Nikodyn theorem that the second measure ν is absolutely continuous with respect to the first measure μ.

Then what you can do is that for Borel measurable functions call them h, if you want to integrate it with respect to the second measure ν all you have to do is to multiply the function h by the Radon-Nikodyn density and integrate that multiplied function that function which is now measurable with respect to the first measure μ, and this equality will hold if the integrals exist.

That is the important statement that is coming out. If you want to integrate a Borel measurable function with respect to the second measure all you have to do is to go back to the first measure multiply the function h by the appropriate density function. If you do that all you have to do is to compute it in terms of the original measure μ.

How do you show this? Here is a hint for this exercise that you try to verify this equality when h is some indicator function and then go to simple functions and use the standard procedure of proving such equalities between integrations, but first try to do it for an indicator function. Now, we have taken this third measure gamma which we have not used so far so what is the use of this?

If it so happens that gamma is absolutely continuous with respect to the second measure ν then gamma is also absolutely continuous with respect to μ. In this case by the Radon-Nikodyn theorem you will get density functions of gamma with respect to μ.

And we are saying that it can be split as a product of the derivative of gamma with respect to ν and multiplied by derivative of ν with respect to the measure μ. That is the statements here again please try to check this. This will follow from the uniqueness of the density function.

We had started off with assuming that ν is absolutely continuous with respect to μ, but if it also happens the other way that two measures are absolutely continuous with respect to each other, if it so happens then you can try to show that there is a relation between the density functions of μ with respect to ν and ν with respect to μ, so that they are exactly reciprocal of each other. Please try to check this.

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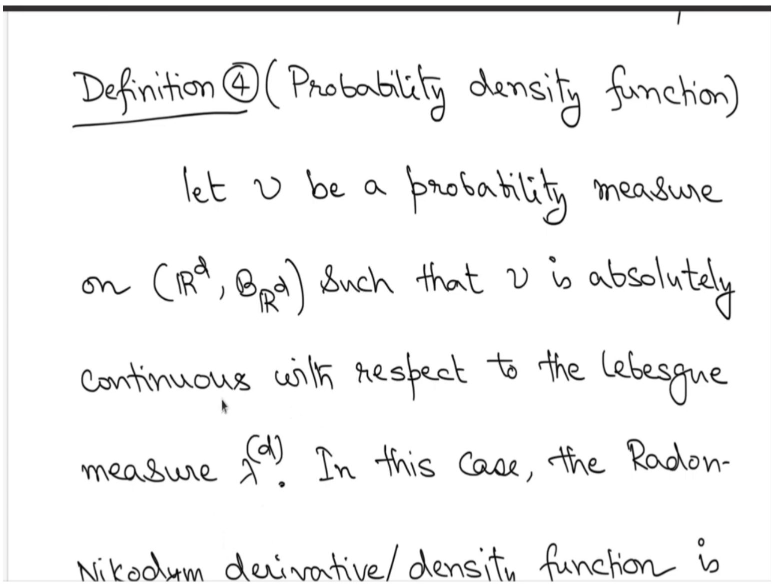

continuous with respect to the Lebesgue measure $\lambda^{(d)}$. In this case, the Radon-Nikodym derivative/density function is called the probability density function, (p.d.f, in short hand) of ν.

Note ⑤: (i) In the set-up of Definition ④,

Note ⑤: (i) In the set-up of Definition ④,

if f is the p.d.f of ν, then observe that f is non-negative, measurable with $\nu(A) = \int_A f \, d\lambda^{(d)}$, $\forall A \in \mathcal{B}_{\mathbb{R}^d}$. In particular, $\int_{\mathbb{R}^d} f \, d\lambda^{(d)} = \nu(\mathbb{R}^d) = 1$ and

hence, f is integrable with respect to $\lambda^{(d)}$.

(ii) Continue with the notations in part (i). For $x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d$, consider the set $A = \prod_{i=1}^{d} (-\infty, x_i]$. Then

$$F_\nu(x) = \nu\left(\prod_{i=1}^{d} (-\infty, x_i]\right) = \int f(y) \, \lambda^{(d)}(dy)$$

As soon as we have this Radon-Nikodyn theorem and Radon-Nikodyn densities, we are now going to apply it to probability measures on $\mathbb{R}^d$. Again, starts with the probability measure call it ν on $\mathbb{R}^d$ together with the Borel σ-field, so that is the measurable space. Suppose it happens that this probability measure is absolutely continuous with respect to the Lebesgue measure on these d dimensions. Again, what we are trying to do is to use the information that is already captured in the Lebesgue measure.

So, if it can happen, if you can do that then you get this Radon-Nikodyn derivative or density function for this probability measure with respect to the Lebesgue measure and this Radon-Nikodyn derivative or density function that you get is called the probability density function or PDF in short of the property measure ν.

If ν is absolutely continuous with respect to the Lebesgue measure you get the derivative Radon-Nikodyn derivative or Radon-Nikodyn density function and that you are going to refer to as the probability density function. This gives you existence of the probability density functions for probability measures absolutely continuous with respect to the Lebesgue measures.

In this setup if it so happens that you are going to get some function f as the PDF of this probability measure ν then observe that this function f is non-negative and measurable, so that is as part of the statement in the Radon-Nikodyn theorem and moreover you can talk about probability of events or sets of Borel sets A as an integration of this function that is this density function with respect to the Lebesgue measure that is as per the structural property of the Radon-Nikodyn theorem.

But if you choose the set to be the whole of $\mathbb{R}^d$ what happens? You are going to get back the ν of $\mathbb{R}^d$, but ν is a probability measure, therefore, the total value is 1. Therefore, all you are saying that this non-negative measurable function f integrates to 1. So, this is an integrable function with respect to the Lebesgue measure. So, this is already non negative and you have shown its integration is finite, therefore this turns out to be integrable with respect to the Lebesgue measure in this d dimensional case.

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(ii) Continue with the notations in part (i). For $x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d$, Consider the set $A = \prod_{i=1}^{d} (-\infty, x_i]$. Then

$$F_\nu(x) = \nu\left(\prod_{i=1}^{d}(-\infty, x_i]\right) = \int_{\prod_{i=1}^{d}(-\infty, x_i]} f(y)\, \lambda^{(d)}(dy)$$

$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(y_1, \ldots, y_d)\, dy_1 \ldots dy_d,$$

$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(y_1, \ldots, y_d)\, dy_1 \ldots dy_d,$$

where F_ν is the distribution function of ν and the last integral above has been written using the identification/notation as discussed in the one-dimensional case in Notes (17) and (18)

But continue with these notations a bit more you are also going to make this identification with the distribution function. What do you do choose a point x with components x 1 up to x d and look at this set A which is a default product of such kind of intervals that on the ith component you are going to take the interval $(-\infty, x_i]$. Take a look at such an interval so then what happens you are going to get the distribution function by looking at size of this product of intervals.

The size of this product of intervals under the measured ν is given by integration of the function over that set. That is as per the structural property given by the Radon-Nikodyn theorem and here you are doing the integration with respect to the Lebesgue measure in this d dimensional case but then remember in the discussion involving the Lebesgue measures and

Lebesgue integrations in connection with Riemann integrations we have seen that in one dimensional case Riemann integrations if it exists it will match with the Lebesgue integration.

In higher dimensions what will happen is that you are going to get these iterated integrals as Riemann integration. If you can make sense of these integrations as riemann integrations by the similar logic by similar arguments you can also get back the same integration in terms of the Lebesgue measure, so that is this equality that is being used here.

But more generally since the Lebesgue integration is extension of d dimensional Riemann integrations or d-dimensional such iterated integrals then all you are going to say is that using the same notational structure that was introduced in one dimensions that you are going to use the same notations for the integrations with respect to the Lebesgue measures also just for the convenience of notations.

So, you are going to write this equality. Even if the original function f is not Riemann integrable you just say that Lebesgue integration is nothing but some kind of an extension of Riemann integration so you may be able to integrate more functions so you are going to use the same notations for notational convenience. That is why we are going to write down these iterated integrals as the expression for the distribution function of the probability measure Nu. This is now your familiar expression of the distribution functions and its connection to probability density functions.

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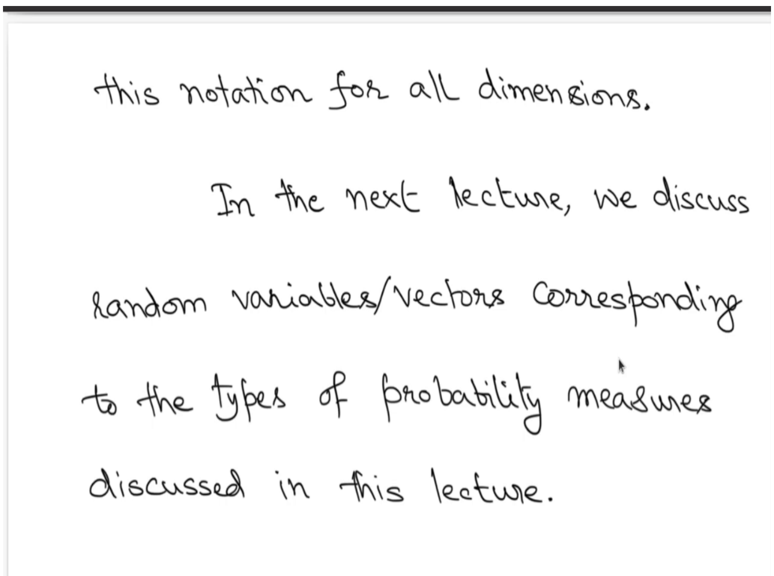

Now, in the next lecture we are going to talk about random variables and random vectors corresponding to these various types of probability measures. Remember earlier what we had discussed? We had discussed discrete random variables coming from discrete distribution functions, which connected to probability measures which were convex combinations of Dirac masses, but we have now said that convex combinations of Dirac masses are mutually singular with respect to the Lebesgue measure.

But we have also discussed this type of probability measures which are absolutely continuous with respect to the Lebesgue measure. Corresponding to every probability measure you can construct random variables, what we are going to see are these different types of random variables corresponding to these probability measures, which are absolutely continuous with respect to the Lebesgue measure.

We are also going to see random variables which are of different structures which are not falling in the structure of discrete random variables or the absolutely continuous case, it is going to follow a different kind of structure that will be connected to the type of examples that we mentioned under Cantor set that we have such probability measures where you get the distribution function is continuous.

It does not have any jumps so you cannot categorize as discrete random variables. There will be these three types of random variables corresponding to these three types of probability measures, which are classified according to its relation with respect to the Lebesgue measure. We are going to continue this discussion in the next lecture.