Measuring Theoretic Probability 1
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Lecture 35
Computations involving Lebesgue Integration

Welcome to this lecture. This is the last lecture of week 7. In this week, we have looked at the limiting behavior of measure-theoretic integration and also explored the connection between Riemann integration and Lebesgue integration.

So, in particular, the applications of this limit theorems involving integrations, and the proof of the connection between Riemann and Lebesgue integrations were incomplete. So, we will discuss this in this lecture. So, let us move ahead to the slides, and start the discussion.

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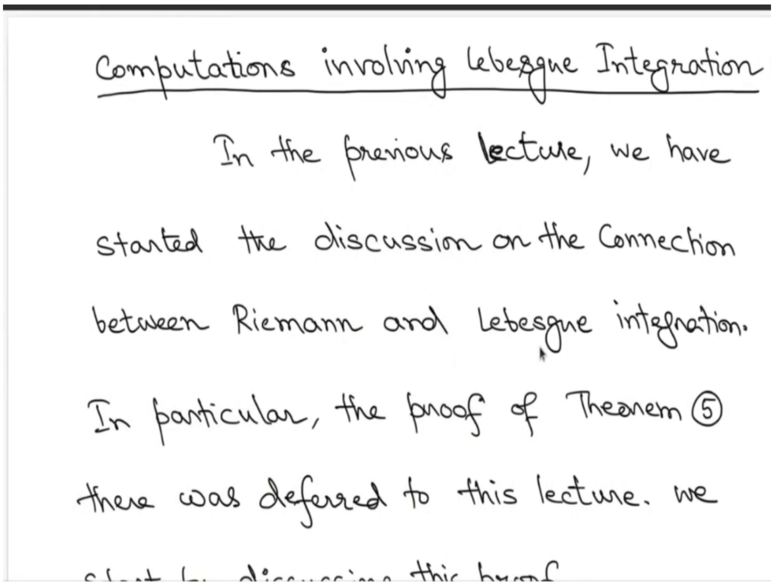

So, in the previous lecture, we started this discussion on the connection between Riemann and Lebesgue integration, and there, the main theorem was theorem 5, which said that the existence of Riemann integration will imply the existence of Lebesgue integration. So, we said that we will do it in this lecture, so let us start with that.

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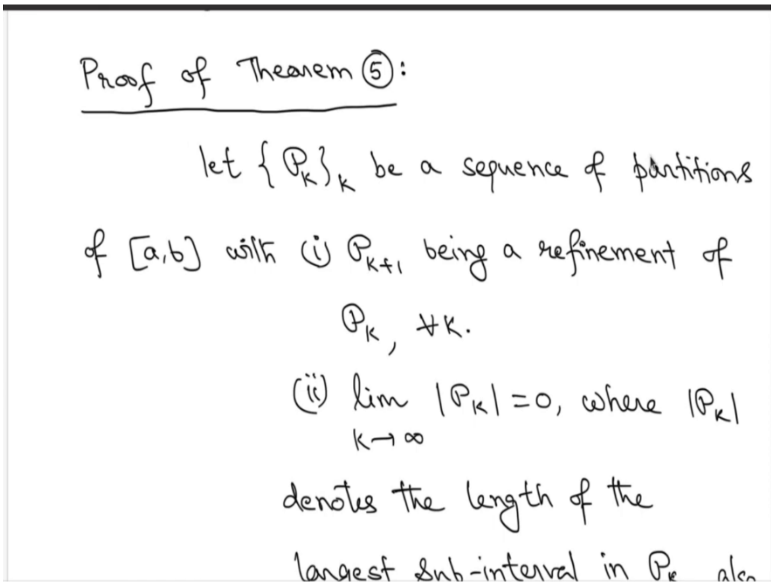

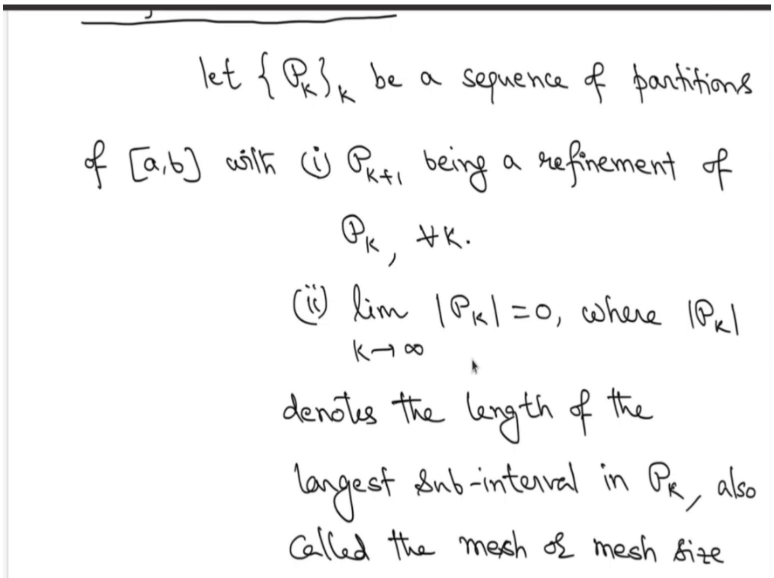

So, starting with Azure mean that you have a function defined on this closed-bounded interval $[a, b]$ a bounded function such that it is Riemann integrable. In that case, what do you do? You look at this sequence of partitions, let us denote them by $\{\wp_k\}$, such that this sequence of partitions has this property that the mesh size or the length of the largest sub interval that goes to 0 as $k \to \infty$.

So, you are taking finer and finer partitions of the domain set $[a, b]$. And you can also without loss of generality assume that $\wp_{k+1}$ is a refinement of $\wp_k$. So, this is true for all k, so you can assume that.

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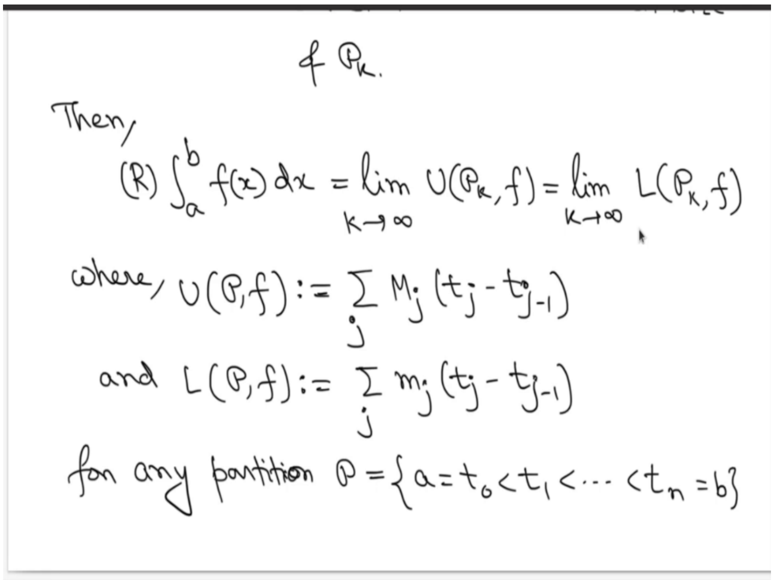

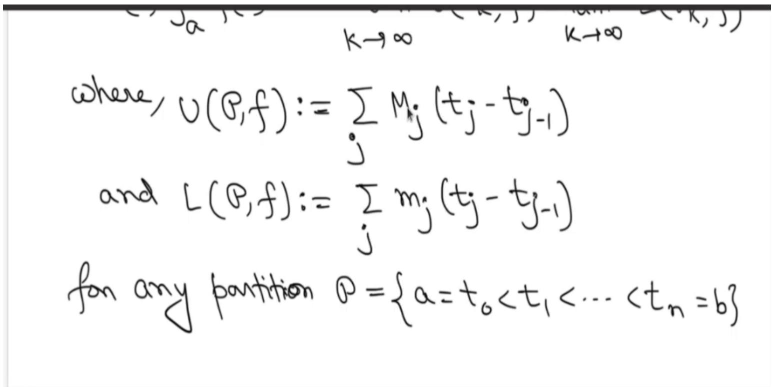

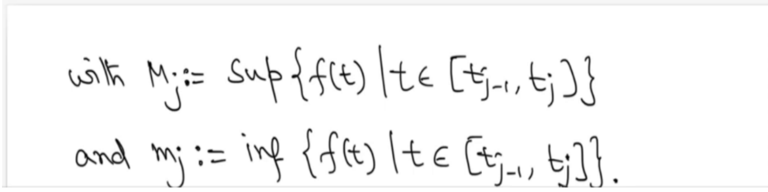

So, under this setup if provided that you have the existence of this Riemann integration, what do you do, you look at the limit of the upper Riemann sum and lower Riemann sum. So, how are they defined? So, for any partition $\wp$ you look at these expressions where you have these $M_j := sup\{f(t)\ /\ t \in [t_{j-1}, t_j]\}$ and $m_j := inf\{f(t)\ /\ t \in [t_{j-1}, t_j]\}$.

So, provided you have these two quantities M_j and m_j what do you do, you just approximate the area under the curve from above by this upper sum and from the below by this lower Riemann sums. So, for any partition $\wp$ where the points are made up of this $t_0, t_1, \ldots, t_n$

these many points, you can rewrite it in this formulation. So, these are the upper Riemann sum and the lower Riemann sum.

And provided the Riemann integration exists, so for this sequence of partitions that you have chosen, you must have that the limit of upper Riemann sum will be equal to the limit of lower Riemann sums and that should equal this value, which is the definition of Riemann integration. Fine. Now, we would like to make the connection with Lebesgue integration. So, so far, what we have discussed are recalling facts about Riemann integration. So, let us move ahead.

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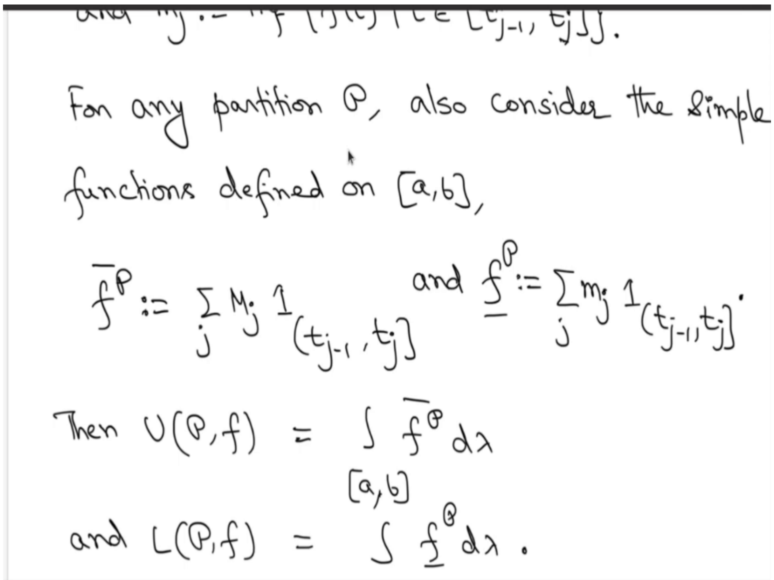

So, now, for any such partition P, you can consider a simple function defined on this closed bounded interval a, b. So, what is this? So, look at $\overline{f}^{\wp}$. So, what is this? I look at these kinds of indicators of $(t_{j-1}, t_j]$.

So, look at that indicator and multiply by the supremum value of the function within that interval, and look at their summation. Similarly, $f^{\wp}$ you would look at in terms of this combination of these indicators of the intervals multiplying it by this infimum of the function within the interval.

So, now, observe that if you were to look at these simple functions, here, if you integrate the F super bar function with respect to the Lebesgue measure, you will get back M_j multiplied by the length of the interval $(t_{j-1}, t_j]$ and that is nothing but $M_j(t_j - t_{j-1})$, so that summation.

So, therefore, you are just getting back the upper Riemann sum for that specific partition. Similarly, if you are going to look at this $\underline{f}$ that function, then if you integrate it against the Lebesgue measure, you are again going to get back that summation $m_j(t_j - t_{j-1})$. So, that is going to give you back the lower Riemann sum.

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$\overline{f}^{\mathcal{P}} := \sum_j M_j \mathbb{1}_{(t_{j-1}, t_j]}$ and $\underline{f}^{\mathcal{P}} := \sum_j m_j \mathbb{1}_{(t_{j-1}, t_j]}$.

Then $U(\mathcal{P}, f) = \int_{[a,b]} \overline{f}^{\mathcal{P}} \, d\lambda$

and $L(\mathcal{P}, f) = \int_{[a,b]} \underline{f}^{\mathcal{P}} \, d\lambda$.

Since f is bounded, the sequence of functions $\{\overline{f}^{\mathcal{P}_k}\}_k$ and $\{\underline{f}^{\mathcal{P}_k}\}_k$ are uniformly bounded.

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Moreover, $\overline{f}^{\mathcal{P}_k}$ is non-increasing in k and $\underline{f}^{\mathcal{P}_k}$ is non-decreasing in k. Therefore, the limit functions $\overline{f} := \lim_{k\to\infty} \overline{f}^{\mathcal{P}_k}$ and $\underline{f} := \lim \underline{f}^{\mathcal{P}_k}$ exist. These functions, being a limit functions $\overline{f} := \lim_{k\to\infty} \overline{f}^{P_k}$ and $\underline{f} :=$ $\lim_{k\to\infty} \underline{f}^{P_k}$ exist. These functions, being a limit of measurable functions, are also measurable.

Since the Lebesgue measure is a finite measure on $[a, b]$ by ...

Now, with this observation, what you can now claim is that you have some connection with the upper Riemann sum and lower Riemann sum and certain kind of integrations involving these kind of simple functions that you have chosen based on the partitions. But let us explore these f Superbad and $\underline{f}$ functions a bit more.

So, for that given sequence of partition $\wp_k$ look at $\bar{f}$ and $\underline{f}$, so consider this. But f is given to be bounded, so therefore, these functions by definition $\underline{f}$ and $\bar{f}$ these functions are bounded, bounded by that same constraint that bounds actual function f. So, these are bounded simple functions defined on these closed-bounded interval $[a, b]$.

Now, it can also be checked very easily that $\bar{f}^{\wp_k}$ is a non-increasing in k and $\underline{f}^{\wp_k}$ is non-decreasing in k. So, you can easily check this. Now, once you have this fact you will immediately look at the limit function. So, once you have some kind of a monotonicity behavior, you can look at the pointwise limits in k.

So, for every fixed point in the domain you let $k \to \infty$ get that limit value that will give you these functions $\overline{f}$ and $\underline{f}$. So, for that specific choice of the partitions that you have started off with look at this limit functions $\overline{f}$ and $\underline{f}$. But observe that these functions are limit of measurable functions. In particular these functions are limit of simple functions. So, therefore, this $\overline{f}$ and $\underline{f}$ must be Borel measurable.

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Since the Lebesgue measure is a finite measure on $[a,b]$, by Note 13 and using the Bounded Convergence Theorem,

we have,

$$\int_{[a,b]} \overline{f}\, d\lambda = \lim_{k\to\infty} \int_{[a,b]} \overline{f}^{\mathcal{P}_k}\, d\lambda = \lim_{k\to\infty} U(\mathcal{P}_k, f)$$

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$$= (R)\int_a^b f(x)\, dx = \lim_{k\to\infty} L(\mathcal{P}_k, f) = \lim_{k\to\infty} \int_{[a,b]} \underline{f}^{\mathcal{P}_k}\, d\lambda$$

$$= \int_{[a,b]} \underline{f}\, d\lambda.$$

But now, recall, that Lebesgue measure, if you are restricting your attention to this closed-bound interval $[a, b]$ that is a finite measure. And then, we had discussed earlier in note 13, that there is a version for bounded convergence theorem for finite measures. So, as long as you have the functions, the sequence of functions are uniformly bounded and you are dealing with integrations with respect to a finite measure, you can talk about the bounded convergence theorems.

Say in particular, here, look at the integration of $\bar{f}$ over this interval $[a, b]$. Now, this by the bounded convergence theorem, will be exactly the limit of the integration of the approximating sequence of functions. So, that is, $\bar{f}^{\wp_k}$, so pointwise it converges to this, so you can exchange the order of limits and integrations.

But observe that $\bar{f}^{\wp_k}$ integration with respect to the Lebesgue measure that we have identified as the upper Riemann sum for the partition $\wp_k$, so therefore, you have this equality now. But this, as per definition agrees with the actual Riemann integration of the given function f.

So, this is the integration with respect to the Riemann sense, but then, that also agrees with the limit of the lower Riemann sums, but then lower Riemann sums, you can identify it as the integration of f Pk with respect to the Lebesgue measure lambda. But then again, you push the limit inside, again you apply the bounded convergence theorem, you end up with the fact that this is the integration of $\underline{f}$ with respect to the Lebesgue measure.

So now, all these expressions are equal, but observe the leftmost term that is involving the $\bar{f}$ function and the rightmost term that is involving the f lower word function. By definition, $\bar{f} \geq \underline{f}$, you have that so this is by construction. And you also have that these functions have this nice integrations and they are equal.

So, you can immediately claim that the difference function $\bar{f} - \underline{f}$, if you look at that function, that is a non-negative function and that integration with respect to the Lebesgue measure is 0.

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$$= \int_{[a,b]} \underline{f} \, d\lambda .$$

Therefore, $$\int_{[a,b]} \left(\bar{f} - \underline{f}\right) d\lambda = 0 .$$

By construction, $\bar{f} \geq \underline{f}$. By Proposition 3 (ii), we have $\bar{f} = \underline{f}$ λ-a.e.

Moreover, whenever $\bar{f}(x) = \underline{f}(x)$, we have $\bar{f}(x) = f(x) = \underline{f}(x)$. Observe that

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(a) $f = \bar{f} = \underline{f}$ λ-a.e.,

(b) $\bar{f}$ and $\underline{f}$ are Borel measurable.

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technical details related to completion of the Borel σ-field, we take this as a fact.

Now you will hear about this non-negative function whose integration is 0 and by an earlier proposition you will immediately claim that these two functions must match almost everywhere. So, that is good. So, we have now made some identification about these two functions $\bar{f}$ and $\underline{f}$.

But observe that when this equality holds at the points whenever this equality holds, you have by definition that $\bar{f}$ is equal to the actual function value at x and that will agree with the f lower about function evaluated at x. So, all these three functions will agree, once you have the equality between the $\bar{f}$ and $\underline{f}$.

So, why is this? This is following the fact that by definition, the function value, the actual function value f falls between $\bar{f}$ and $\underline{f}$. So, this is easy fact to check this is by definition of the functions $\bar{f}$ and $\underline{f}$. So, once you have that, you claimed that $\bar{f}$ and $\underline{f}$ agree almost everywhere, so therefore, you have actually this equality.

So, remember, if you have so far not shown anything about the measurable structure of the function f it is just taken to be bounded function defined on the closed-end bounded interval a, b, but you are now saying that it is matching with these measurable functions almost everywhere. So, you have this function $\bar{f}$ and $\underline{f}$, these are all nice Borel measurable functions, which turned out to be limits of certain simple functions.

Now, the given function f is agreeing with Borel measurable function, Lebesgue measure almost everywhere. Provided this happens, it can now be shown that f is actually measurable with respect to the Lebesgue σ-field. So, remember, we had discussed this Lebesgue σ-field in note 8 earlier.

So, what we had mentioned was that the Borel σ-field with respect to the Lebesgue measure is not complete, meaning, it does not contain all subsets of measures 0 sets. If you include all those subsets of measures 0 sets and generate a further σ-field., you will get the Lebesgue σ -field. So, that is an enlarged σ.

And with respect to that, you can now claim that if an arbitrary function matches with a Borel measurable function almost everywhere you can actually claim that f, the given function must be measurable with respect to this bigger σ- field. So, it is not necessarily measurable with respect to the Borel σ-field but it is measurable with respect to the Lebesgue σ-field.

So, this is an important clarification that was required if you want to consider the integration of the function f with respect to the Lebesgue equation. So, remember, you can only talk about integrations of measurable functions. So, that is why this clarification was necessary.

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with respect to the Lebesgue σ-field, which was mentioned in Note ⑧. To avoid technical details related to Completion of the Borel σ-field, we take this as a fact.

Now,

$$\int_{[a,b]} \bar{f}\, d\lambda = (R)\int_a^b f(x)\,dx = \int_{[a,b]} \underline{f}\, d\lambda = \int_{[a,b]} f\, d\lambda$$

we have,

$$\int_{[a,b]} \bar{f}\, d\lambda = \lim_{k\to\infty} \int_{[a,b]} \bar{f}^{P_k}\, d\lambda = \lim_{k\to\infty} U(P_k, f)$$

$$= (R)\int_a^b f(x)\,dx = \lim_{k\to\infty} L(P_k, f) = \lim_{k\to\infty} \int_{[a,b]} \underline{f}^{P_k}\, d\lambda$$

$$= \int_{[a,b]} \underline{f}\, d\lambda.$$

But just to avoid all these technical details associated to completion of σ-fields and regarding this connection with almost everywhere identifications with measurable functions, we avoid the details we want to avoid that. And we are just saying that, this fact is true. We will just take this as true, and we will continue the discussion.

Now, the upshot is this that you have so far shown that f that general function that you have taken if it is Riemann integrable then it turns out to be measurable with respect to an appropriate σ-field, which is the Lebesgue σ. But then, you can now consider the integration of the function f with respect to the Lebesgue measure. You have a measurable function you can consider this.

But then, by the identification of the equalities that you have observed, you can now claim that since f agrees with $\bar{f}$ and $\underline{f}$ almost everywhere then their integrations will also match if you consider the integration with respect to the Lebesgue measure. So, the last equality here is true, because two functions f and $\underline{f}$ both are now measurable and both agree almost everywhere, so therefore, their integrals will match.

So, here, you are using the fact that the Borel measurable functions $\bar{f}$ and $\underline{f}$ are also measurable with respect to the Lebesgue σ-field. This is easy to check, because as long as all the pre-measures are in the Borel σ-field, it is also in the enlarged σ-field which is the Lebesgue σ-field.

So, measurability of $\bar{f}$ and $\underline{f}$ is not an issue. So, you have this equality, but you have earlier shown that the integration of $\bar{f}$ and $\underline{f}$ that agrees with the actual Riemann integration of the given function f, so that is equality that are listed right here in front. So, let us just go back. So, you had earlier approved this equality. *S*

o, you heard, just look at this part here, this part immediately tells you that integration of $\bar{f}$ and $\underline{f}$ match and that agrees with the actual Riemann integration of the given function f. So, with that, what you have managed to show is that f is now integrable in the Lebesgue sense. So, you are now considering Lebesgue integration of the function f and that is matching with Riemann integration of the given function f. So, this was the statement of the first part. So, this was the first statement in the theorem file.

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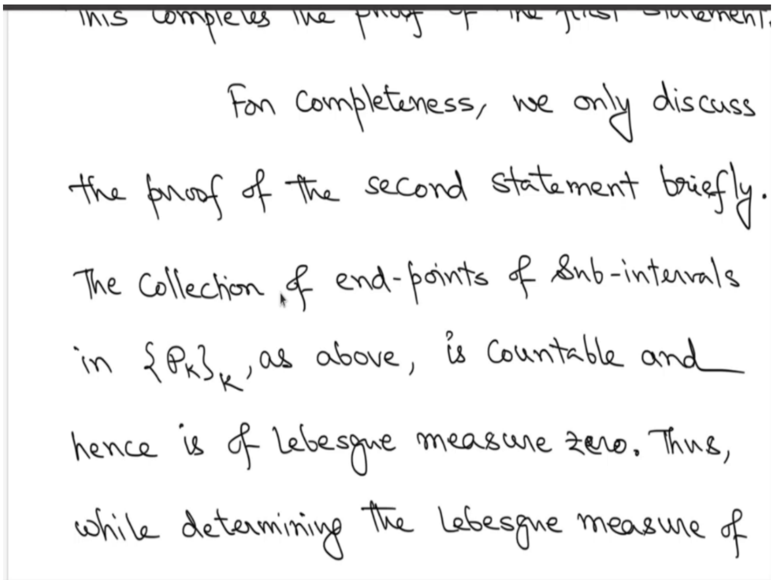

Now, the second half gave a characterization of Riemann integration. So, for completeness, we are just going to discuss this proof very, very briefly. So, again, all these verifications can be done explicitly, but let us focus on the main parts here. So again, go back to this partitions Pk. So this is this shrinking sequence of partitions. By shrinking, I mean, the mesh size decreases and decreases to 0.

So, now look at these endpoints of all the sub-intervals that are listed under these partitions. So, if you consider any fixed partition, it has finite number of intervals listed and each sub-interval there will have two endpoints. If you look at all the endpoints of all the sub-intervals in all the partitions then you end up with a countable set.

Now, here note that on this countable set, the Lebesgue measure associates 0 mass, so you do not want to care about these endpoints, and you do not want to care about the function values at these endpoints. So, while considering integrations, we do not want to care about the function values on this endpoint. So, ignore this set of measures 0 first.

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while determining the Lebesgue measure of the set of discontinuities of f, we may ignore the end-points of the sub-intervals. Now, for all other points $x \in [a,b]$, we have $\bar{f}(x) \geq f(x) \geq \underline{f}(x)$. If $\bar{f}(x) = f(x) = \underline{f}(x)$ for some x, then f is continuous

ignore the end-points of the sub-intervals. Now, for all other points $x \in [a,b]$, we have $\bar{f}(x) \geq f(x) \geq \underline{f}(x)$. If $\bar{f}(x) = f(x) = \underline{f}(x)$ for some x, then f is continuous
at x. Since $\bar{f} = f = \underline{f}$ λ-a.e, f is also continuous λ-a.e. Thus, the proof follows.

Now, while determining this set of discontinuities of this function, which is the effectively the statement that characterizes the Riemann integration, as we are trying to show, we are going to ignore the endpoints because that anyway is a set of measures 0. So, now, we ignore that set and we focus purely on the interior points of all these subintervals, and we want to check whether certain discontinuities are occurring there or not.

Now, for all the other points inside the interval $[a, b]$ you have this inequality. Again, this is by definition that a $\bar{f}$ is greater or equals to given function f is greater or equal to $\underline{f}$. But now remember that almost everywhere this equality holds. So, $\bar{f}$ f and $\underline{f}$ matches lambda almost everywhere, meaning, outside some appropriate null set the function values match.

Now, you can show that if this happens then f is actually continuous at that point. Now, that immediately tells you that on these points where these function hellos match you immediately we claim that f is continuous, and therefore, the set of discontinuity points will have measure 0. So, you have to throw out all these points where the discontinuous are not occurring.

So, these are all the points where f is continuous. So, only possible discontinuities occur when is not equal to $\underline{f}$, that is when possible discontinuities of the given function f might occur. So, this is a statement that we will take it as a fact.

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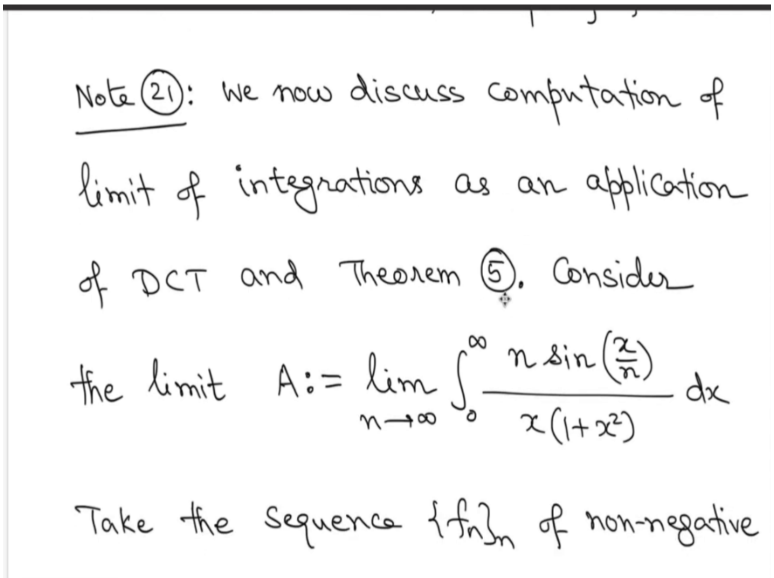

So, therefore, the proof now follows that you have started off with this Riemann integration structure, and then you have identified the Riemann integration with the Lebesgue integration, and you have also identified the characterization of human integration functions, as in terms of the set of points of discontinuance.

Now, we are going to focus on applications of this result. So, we are now using the fact that we have discussed these limiting behaviors of integrable functions, and we have also discussed these connections between Riemann and Lebesgue integration. So now, let us focus on the applications of this, and we will look at our explicit example of limit of integrations.

Look at these functions $\frac{n \sin\left(\frac{x}{n}\right)}{x(1+x^2)}$. So, look at these functions on the interval $[0, \infty]$. Look at their integration and look at $\lim_{n \to \infty} \int_0^\infty \frac{n \sin\left(\frac{x}{n}\right)}{x(1+x^2)} dx$ Call that limit as A, we want to compute this limit A. Now, what do you do first? This is a given Riemann integration for each fix at n.

You want to first go to the Lebesgue integration and that is where you are going to apply the appropriate dominated convergence here.

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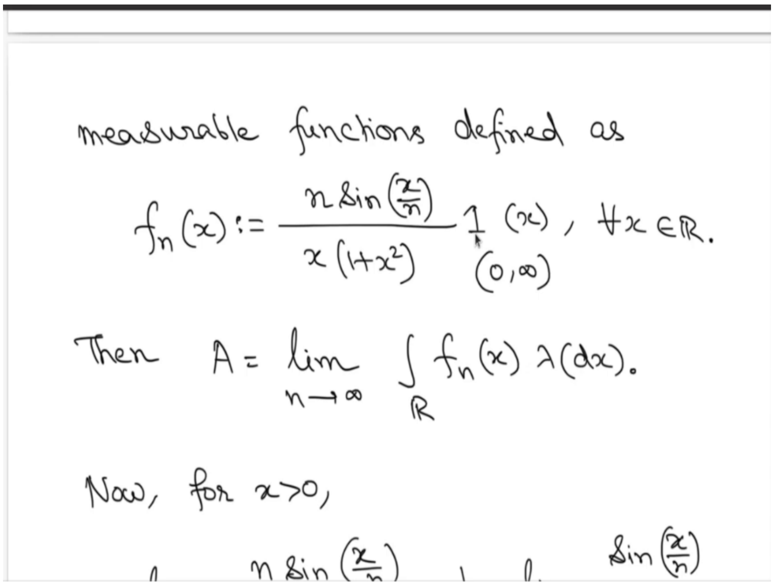

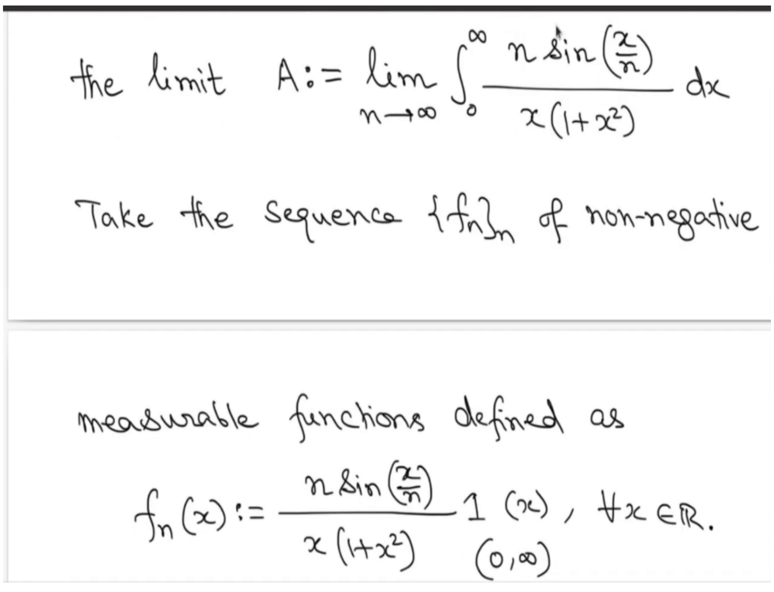

So, let us look at this identification in terms of the linear integration. So, choose a sequence fn of non-negative measurable functions given by this. So, you are looking at functions on this interval $(0, \infty)$, so here what is happening is that you ignore the value at 0, so that is fine because Lebesgue measure anyway associates 0 mass two singletons, so therefore, you ignore the function value there.

So, you are just multiplying by this indicator to get rid of all these other values. So, now look at the actual function, which is given function $\frac{n \sin\left(\frac{x}{n}\right)}{x(1+x^2)}$ look at this. Now, you want to compute that limit a, right. So, now, this Riemann integration is nothing but the integration of the function fn with respect to the Lebesgue measure.

So, therefore, you simply write that $A =$ limit of integrations of f_n with respect to the Lebesgue measure. Now, you want to apply the DCT, but before that, you would like to first compute the pointwise limit of this function f_n.

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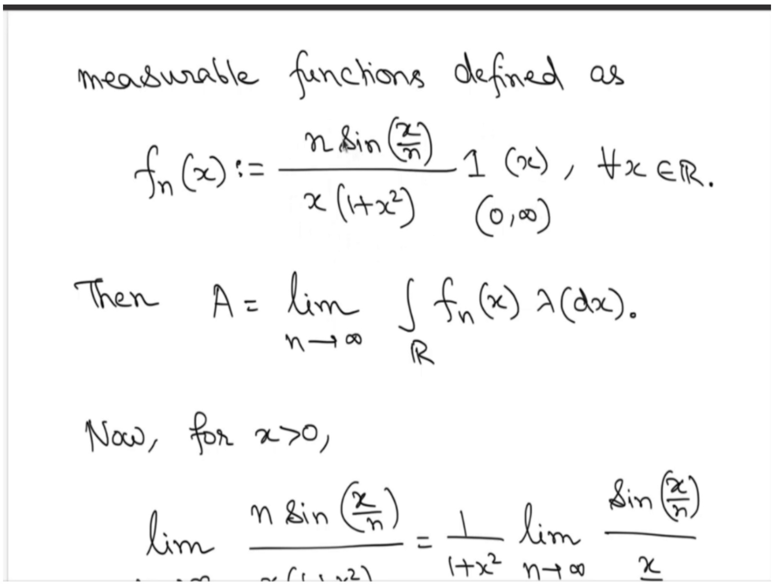
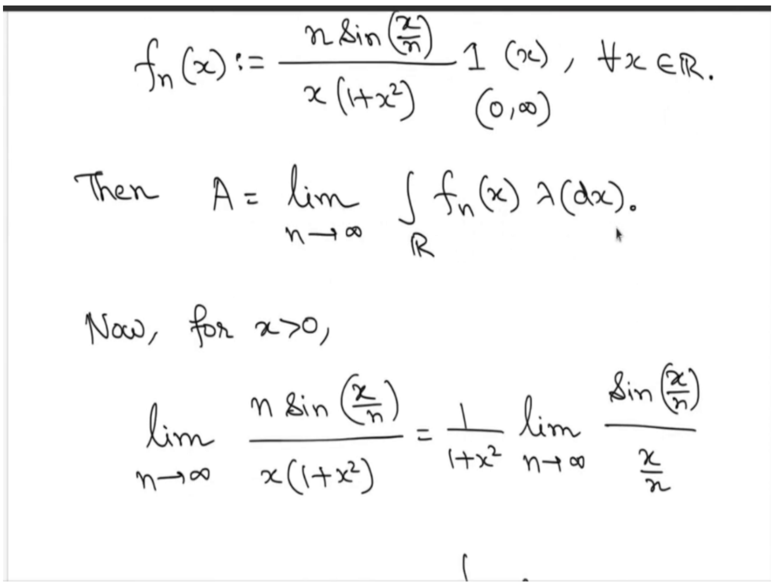
Then $A = \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)\, \lambda(dx).$

Now, for $x > 0$,

$$\lim_{n \to \infty} \frac{n \sin\left(\frac{x}{n}\right)}{x(1+x^2)} = \frac{1}{1+x^2} \lim_{n \to \infty} \frac{\sin\left(\frac{x}{n}\right)}{\frac{x}{n}}$$

$$= \frac{1}{1+x^2}.$$





Now, you are focusing your attention only on the positive part of the real line because, anyway, on the other parts, the function value is 0, so that will not contribute to this integration. So, let us look at $x > 0$. In this case, look at this expression and let $n \to \infty$, so this is the point wise limit.

So, bring out $\frac{1}{1+x^2}$ that does not depend on n anyway, now write the remaining part as $\frac{\sin\left(\frac{x}{n}\right)}{\frac{x}{n}}$. Now, for any fixed x, $\frac{x}{n}$ goes to 0. So therefore, you have this standard limit which goes to 1, so therefore, you will end up with this limit function which is $\frac{1}{1+x^2}$.

So, therefore, the limit functions of these functions fn is exactly $\frac{1}{1+x^2}$ on the interval $(0, \infty)$, otherwise, the limit function is 0.

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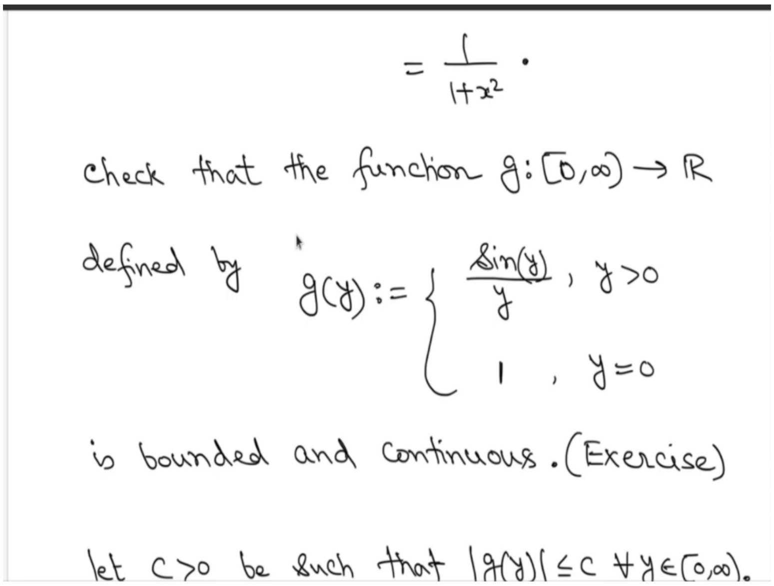

Now, our idea is to obtain a uniform bound of these functions in terms of a integrable function. In preparation for that, let us first consider this function g defined on this interval $[0, \infty)$, and taking values in the set of real numbers. So, how is this defined? So, for positive y look at the ratio $\frac{sin\,(y)}{y}$ at the point $y = 0$ assign the value 1. What you can now easily check, is that, this function is bounded and continuous. So, please take this as an exercise. This function will be used to obtain the bound for the functions f_n.

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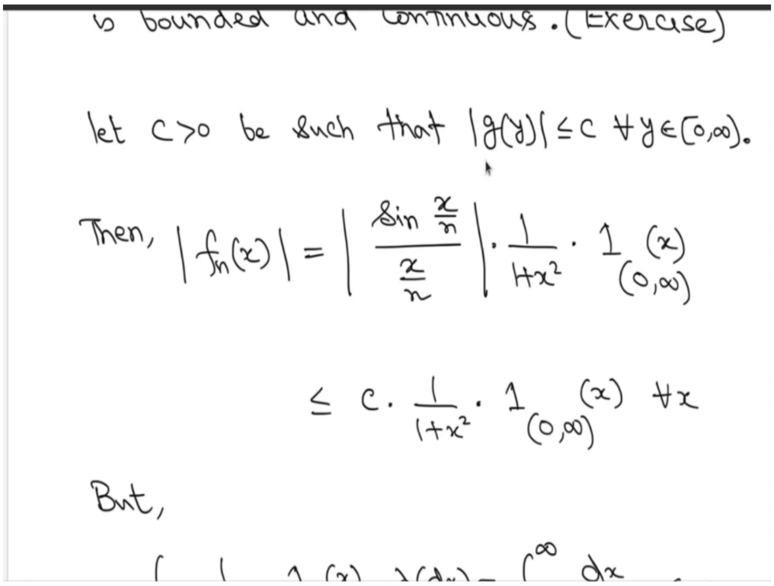

Now, let us see via positive quantity such that this function g is bounded by that constant c. So, you have proof that this g function, as defined above, is bounded and continuous, so, therefore, we will get some positive real number which will bound this. Now, look at the bound of fn given by these constants.

So, how? So, look at this modular's quantity, so that is again in the form of the function g. Now, g is uniformly bounded by this constant c, so therefore, you get the bound $c\,\frac{1}{1+x^2}$ multiplied by the indicator. Now, to apply the dominated convergence theorem, you need the integrability of the bound function.

So, this is the upper bound function, and you need the integrability of that. So, just try to integrate this. I am ignoring the constant here, it does not matter because c is some positive constant. So, now, this one is nothing but the usual Riemann integration $\int_0^\infty \frac{dx}{1+x^2}$. So, this is the usual Riemann integration.

Now, here you can use the standard trigonometric transformation that $x = \tan\theta$ and compute this. There are other methods of computing this, but this is one of the methods and that will give you some finite value. So, therefore, this non-negative function is integrable.

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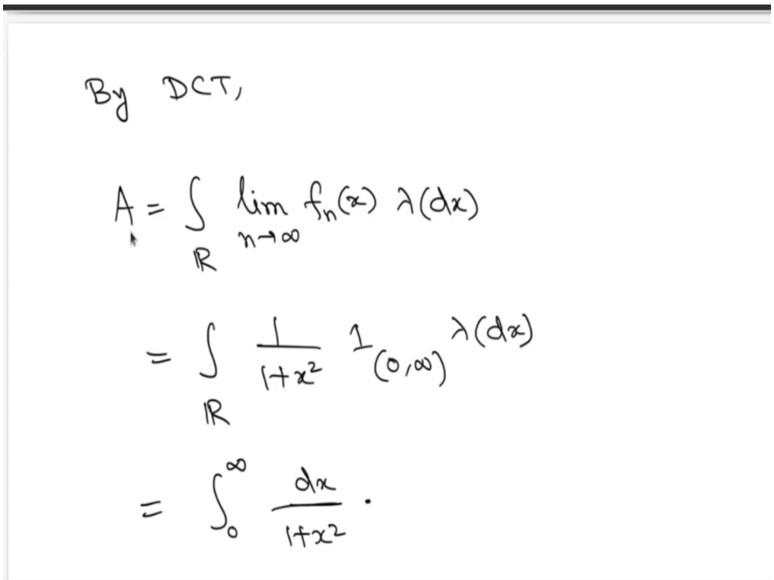

And therefore, you are now allowed to apply DCT, to push the limit insight in this computation. So, limit value that is a can be computed at integration of the limit functions. But then you have computed the limit function as $\frac{1}{1+x^2}$ multiplied by the indicator. So, therefore, again you go back to the usual Riemann integration and that will be just $\int_0^\infty \frac{dx}{1+x^2}$, which you have already computed, that will be some finite quantity anyway.

So, this is an application of the DCT, as well as the connection between Riemann and living integrations. So, you have seen that in many of the steps we have moved back and forth between Riemann and Lebesgue integrations. So, as soon as you can compute this final integration in the Riemann sense, you will get back the value of the limit, but in between we had applied this measure theoretic arguments to apply the DCT. That allowed us to push the limit inside.

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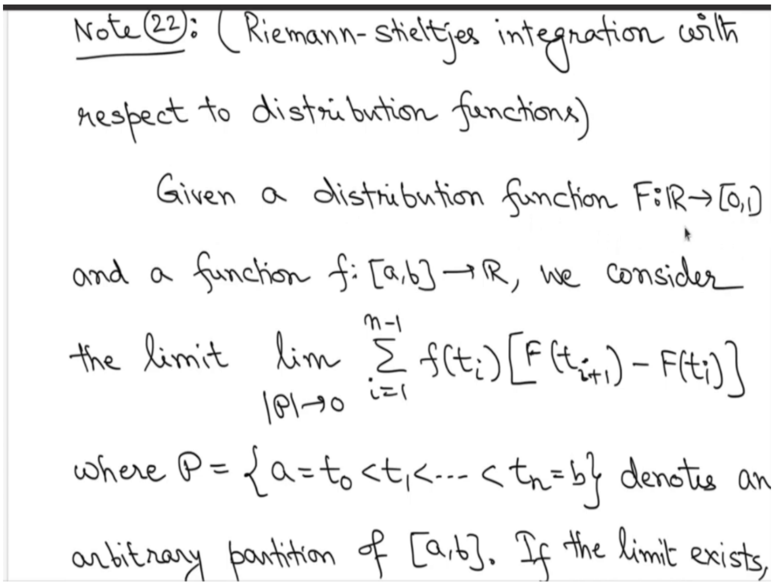

Note (22): (Riemann-stieltjes integration with respect to distribution functions)

Given a distribution function $F: \mathbb{R} \to [0,1]$ and a function $f: [a,b] \to \mathbb{R}$, we consider the limit $\lim\limits_{|P| \to 0} \sum\limits_{i=1}^{n-1} f(t_i)\left[F(t_{i+1}) - F(t_i)\right]$ where $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ denotes an arbitrary partition of $[a,b]$. If the limit exists,

So, now, we have seen this applications of this limit theorems. But there are some few comments about certain more general notions of integrations. So, the first comment here is about Riemann stieltjes integrations with respect to distribution functions. So, if you take a distribution function so, that is a function defined on the real line taking values between 0 and 1 and it is nondecreasing light continuous limit at ∞ is 1 limit at $-\infty$ 0 to take such a function F.

Now, take another function f, which is defined on the interval $[a, b]$ taking real values. And then, consider this limit for any partition like this. So, choose a partition of the closed-bounded interval $[a, b]$ and look at this quantity. So, this is the usual Riemann sum except that in Riemann sum you will look at this $t_{i+1} - t_i$, but here you are looking at the increment of the distribution function.

So, instead of looking at this $t_{i+1} - t_i$, you are looking at the increment of the distribution function that is the only difference, which you are considering here. So, with some appropriate hypothesis you can establish the existence of the limit for appropriate functions F and f.

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where $\mathbb{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ denotes an arbitrary partition of $[a,b]$. If the limit exists, we define this limit to be the Riemann-Stieltjes integral $\int_a^b f(x)\,dF(x)$. We may define $\int_{-\infty}^{\infty} f(x)\,dF(x)$ in the way similar to the Riemann integration.

Note (23): (Lebesgne-stielties integration with

So, now here what is happening is that once you get this limit this is what is known as the Riemann stieltjes integration. So, we are not going into the exact definition of the Riemann stieltjes integration we are just indicating the idea. Now, once you have this integration from a to b, you can also possibly consider integration so far the whole real line and this will be similar in definition, as considered for the Riemann integration case. So, you can consider integrations of functions f with respect to distribution functions.

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Note (23): (Lebesgne-stieltjes integration with respect to a distribution function)

Given a distribution function $F: \mathbb{R} \to [0,1]$ and a measurable function $f: \mathbb{R} \to \mathbb{R}$, we consider the integral $\int_{\mathbb{R}} f(x)\, \mu_F(dx)$, where μ_F is the probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Now, there is a general version of this which is called the Lebesgue stieltjes integration. So, again, you start with a distribution function, but now you start with a measurable function as the integrand. And then given this function F you get a probability measure μ_F defined on the, this measurable space real line together with the production method.

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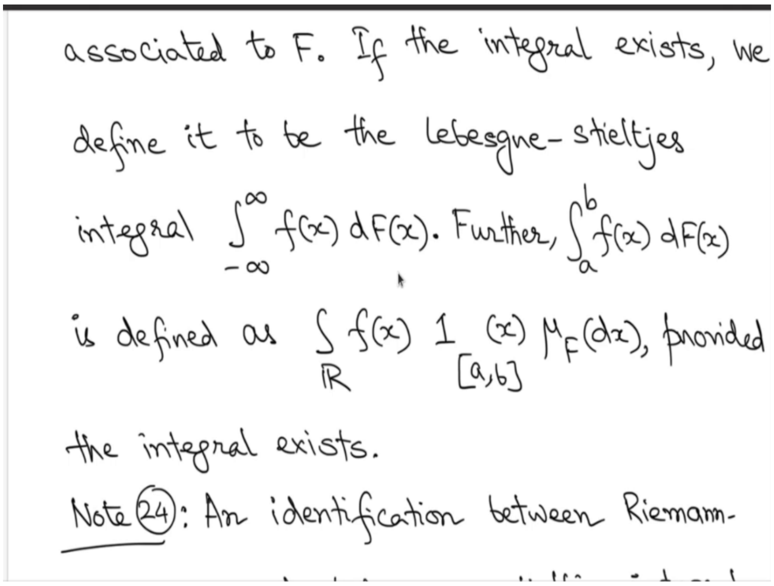

Now, try to integrate this measurable function with respect to this probability measure. If you can define it then you call it the Lebesgue stieltjes integration of f with respect to the distribution function F. So, this is the notation that we are introducing. This is nothing, but the integration of the f with respect to the corresponding probability measure.

Now, once you have considered this $-\infty, +\infty$ integration that integration you can also consider integrations over intervals like, so, that will run similar to the earlier discussions. Here once you have these measure theoretic structures, it is just defined as integration of $f(x)$ multiplied by this indicator function here. So, that is just cutting off the function values outside this interval.

So, outside this interval this product is 0, so that does not contribute to the integral. So, that is basically the idea behind this definition, so this is what we had also considered in Lebesgue integrations or measure theoretic integrations.

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the integral exists.

Note (24): An identification between Riemann-stieltjes and Lebesgue-stieltjes integrals can be stated and is similar to as the result in Theorem (5).

Note (25): (Expectation of RVs as a Lebesgue-Stieltjes integral)

Now, we have this identification between Riemann integrations and Lebesgue integration, so that was discussed in theorem 4. But you can also get a identification between Riemann stieltjes integration and Lebesgue stieltjes integrations. So, you can state this in a similar fashion and prove it with appropriate justifications, so we are not going into the details.

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Note (25): (Expectation of RVs as a Lebesgue-Stieltjes integral)

let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be an RV.

Then its law is $\mathbb{P} \circ X^{-1}$ and denote the corresponding distribution function as F_X.

Then for any measurable $g : (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Then for any measurable $g:(\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

we have,

$$\mathbb{E}\, g(X) = \int_{\mathbb{R}} g(x)\, d\, \mathbb{P} \circ X^{-1}(x) = \int_{-\infty}^{\infty} g(x)\, dF_X(x),$$

where the last integral is a Lebesgue - Stieltjes integral, provided $\int_{\mathbb{R}} g(x)\, d\,\mathbb{P}\circ X^{-1}(x)$ exists.

Note 26: (Characteristic function of an RV)

But here is something, which is quite interesting, since we are interested in expectations of random variables. So, you can identify the expectation of random variables as some kind of a a Lebesgue stieltjes integration. So, how do you do this? So, look at a random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then its law is $\mathbb{P} \circ X^{-1}$ and you can also consider the corresponding distribution function F_X.

Then for any measurable function, you can now consider $E(g(x))$ and that is, by an appropriate change of measure argument can be written as integration of the function g with respect to $\mathbb{P} \circ X^{-1}$. So, this we had discussed earlier. But now, as per the discussion in this lecture $\mathbb{P} \circ X^{-1}$ is exactly the probability measure corresponding to the distribution function, and therefore, you will immediately write it as a Lebesgue stieltjes integration of the function g with respect to the distribution function.

So, this last integration is being interpreted as a Lebesgue stieltjes integration, and that will exist provided you can make sense of this integration with respect to this appropriate probability measure $\mathbb{P} \circ X^{-1}$. So, you can write down expectations as some kind of a limit such an integration with respect to distribution functions.

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Stieltjes integral, provided $\int_{\mathbb{R}} g(x)\, d\mathbb{P}\circ X^{-1}(x)$ exists.

Note (26): (Characteristic function of an RV)

Given an RV $X:(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$,

Consider the function $\phi_X : \mathbb{R} \to \mathbb{C}$ defined by

$$\phi_X(u) := \mathbb{E}\, e^{iuX}, \quad \forall u \in \mathbb{R}$$

$$= \int_{\Omega} e^{iuX(\omega)}\, d\mathbb{P}(\omega)$$

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$$+ i \int \sin(uX(\omega))\, d\mathbb{P}(\omega)$$

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$$= \int_{\mathbb{R}} \cos(ux)\, d\,\mathbb{P}\circ X^{-1}(x)$$

$$\Omega$$

$$= \int_{\mathbb{R}} \cos(ux) \, d\mathbb{P}\circ X^{-1}(x)$$

$$+ i \int_{\mathbb{R}} \sin(ux) \, d\mathbb{P}\circ X^{-1}(x)$$

$$= \int_{\mathbb{R}} e^{iux} \, d\mathbb{P}\circ X^{-1}(x).$$

Since the RVs $\cos(ux)$ and $\sin(ux)$ are bounded for each $u \in \mathbb{R}$ they

Now, an important function in these discussions about random variables is called a characteristic function. So, now that we have discussed about integrations with respect to measures, you can now come back to this specific type of function. So, what do you do? Again, continue with a random variable X and consider this function denoted as F_X.

So, that is a function defined on the real line, and takes values in the set of complex numbers. So, this is defined as follows. So, for any fixed u in the real line, you will look at expected value of e^{iux}, and that is by definition $\int_{\Omega} e^{iuX(\omega)} \, d\mathbb{P}(\omega)$.

But, now, what does this function look like, so split it into real part and imaginary part. So, here this is nothing but in terms of cosines and sins. So, this is a familiar formula to you. Now, once you look at this formula, these are now integrations of nice measurable functions with respect to a probability measure. So, that is what we are looking at here.

So, these are now nice random variables these compositions are nice random variables and you are integrating it with receptor probability measure. So, therefore, these things are well defined. So, integrations existence is fine, but then you change variables and you go to integration with respect to $\mathbb{P} \circ X^{-1}$.

Once you do that, you get back this familiar formula that this is integration of e^{iux} with respect to the law. So, you will see more calculations involving this in later, but what you can now try to do, is that, you can take discrete random variables and try to compute these characteristic functions and this will match with whatever you have earlier seen in your basic probability course.

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$$+ i \int_{\mathbb{R}} \sin(ux) \, d\mathbb{P}\circ X^{-1}(x)$$

$$= \int_{\mathbb{R}} e^{iux} \, d\mathbb{P}\circ X^{-1}(x).$$

Since the RVs $\cos(uX)$ and $\sin(uX)$ are bounded for each $u \in \mathbb{R}$, they are integrable and hence ϕ_X is well-defined. This function is called the

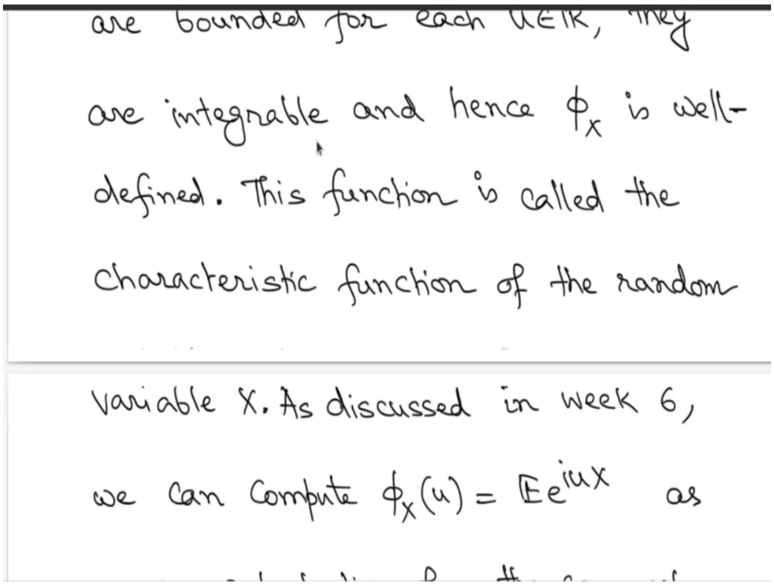
Given an RV $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$,

Consider the function $\phi_X : \mathbb{R} \to \mathbb{C}$ defined by $\phi_X(u) := \mathbb{E}\, e^{iuX}, \ \forall u \in \mathbb{R}$

$$= \int_{\Omega} e^{iuX(\omega)} \, d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \cos(uX(\omega)) \, d\mathbb{P}(\omega)$$

$$+ i \int \sin(uX(\omega)) \, d\mathbb{P}(\omega)$$


are bounded for each $u \in \mathbb{R}$, they are integrable and hence ϕ_X is well-defined. This function is called the characteristic function of the random variable X. As discussed in week 6, we can compute $\phi_X(u) = \mathbb{E} e^{iuX}$ as

And an important comment here is that these random variables $\cos(u\,X)$ and $\sin(u\,X)$, which we had used in these integrations, they are actually bounded random variables. So, for any fixed u, these are bounded random variables and therefore, they are integrable with respect to the probability measure.

And hence for any fixed u this quantity is well defined. So, this always exists. By definition, this is taking values in the set of complex numbers. So, this is called the characteristic function of the random variable X.

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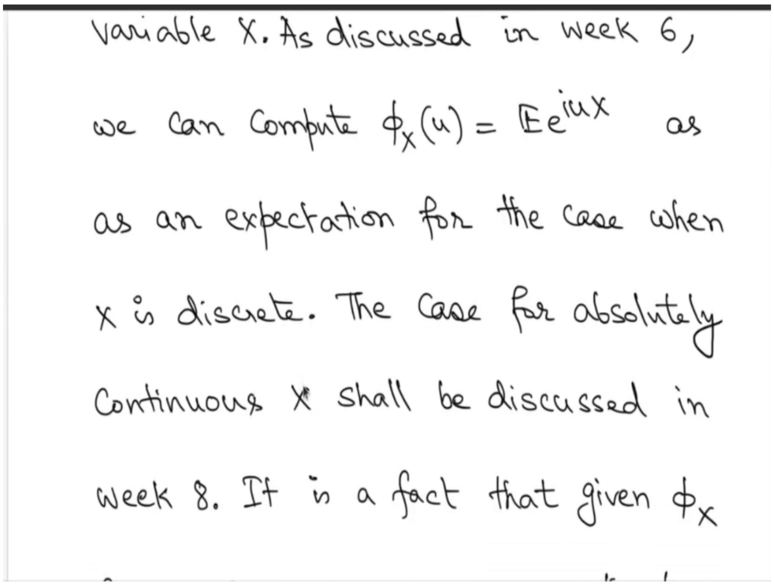

And as discussed earlier, as we have just mentioned that for the case of discrete random variables, you can write down expressions or computations of these characteristic functions. So, the competitions will run similar to whatever we have discussed earlier in week 6.

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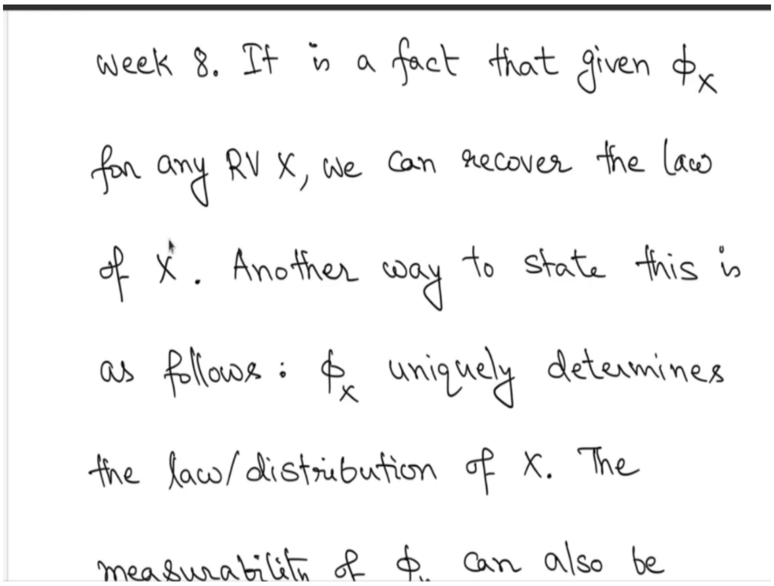

Now, the case for absolutely continuous random variables will be discussed in next week. But it is a fact that given this characteristic function, you can recover the law of the random variable so, this is a very, very important fact. Another way to state this is that the characteristic function uniquely determines the law or distribution of the random variable X. So, this is a very important statement.

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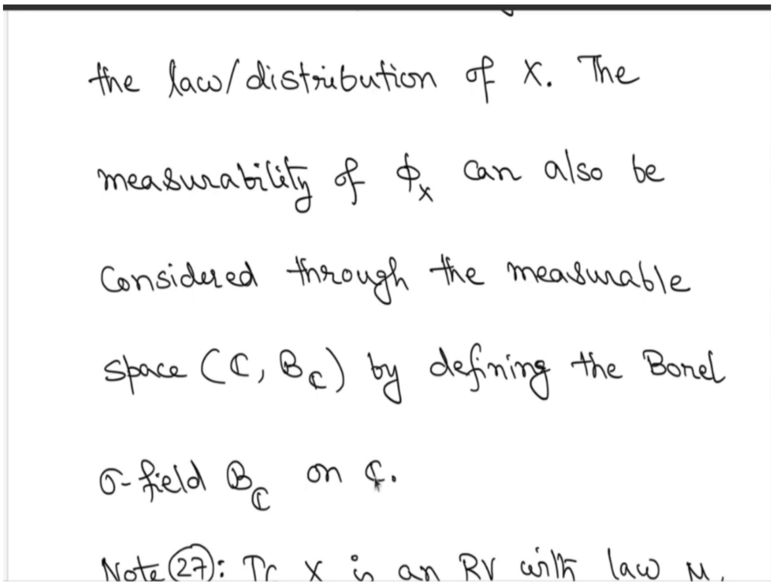

Now, another point to note is that we have defined this cost function, as taking values in the set of complex numbers. You can actually consider the measurable structure with respect to the complex numbers together with the Borel σ-field by appropriated defining the Borel σ -field on the set of complex numbers.

But we are not going into the details here. We take the easier route, we go via the real part and imaginary part and make sense of their integrations.

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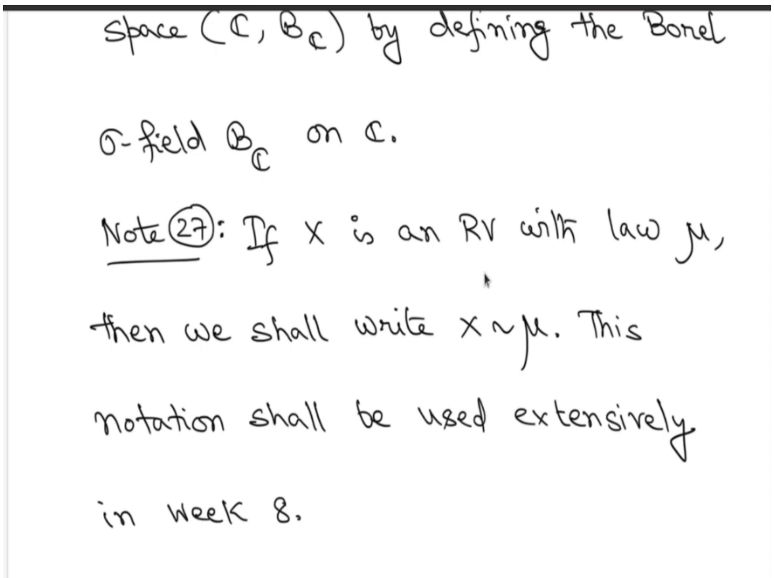

So, final comment as we stop, is that, if are talking about these random variables with specified law, we are going to use this notation X, meaning X has this law μ or $X \sim \mu$. So, this symbol refers to the fact that X is a random variable with law μ. These notations shall be used extensively in our discussion next week.

So, there, we will put everything together that you have learned in this course so far. And we will also discuss about the absolutely continuous random variables. So, that discussion we will do in the next week's discussion. We stop here.