**Measuring Theoretic Probability 1**
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**Lecture 34**
**Riemann and Lebesgue integration**

Welcome to this lecture. In the past few lectures, we have been extensively discussing about measure theoretic integration. We have seen many of the important properties involving certain limiting behaviors. So, with that setup, it is now a good time to understand where these integration procedures stand with respect to our usual integration procedure, which we call as Riemann integration.

So, in this lecture, we are going to start exploring the connection between Riemann integration and measure theoretic integration. We move on to the slides for the discussion.

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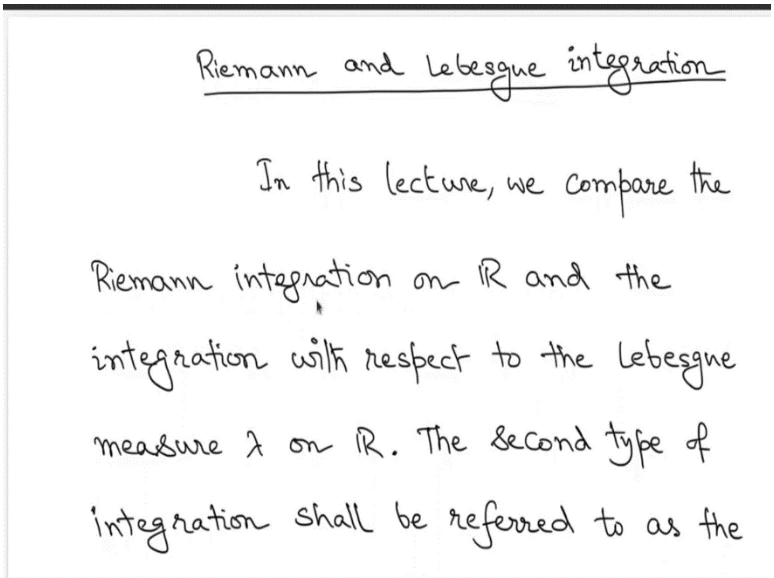


So, you are working with Riemann integration on the real line that we know about. And what we are going to consider, as a special type of measure theoretic integration is the integration with respect to the Lebesgue measure $\lambda$ on the real line. So, we are going to compare these two things. Now, the second type for ease of communication shall be referred to as the Lebesgue integration. So, any integration of a measurable function with respect to the Lebesgue measure will be referred as the Lebesgue integration.

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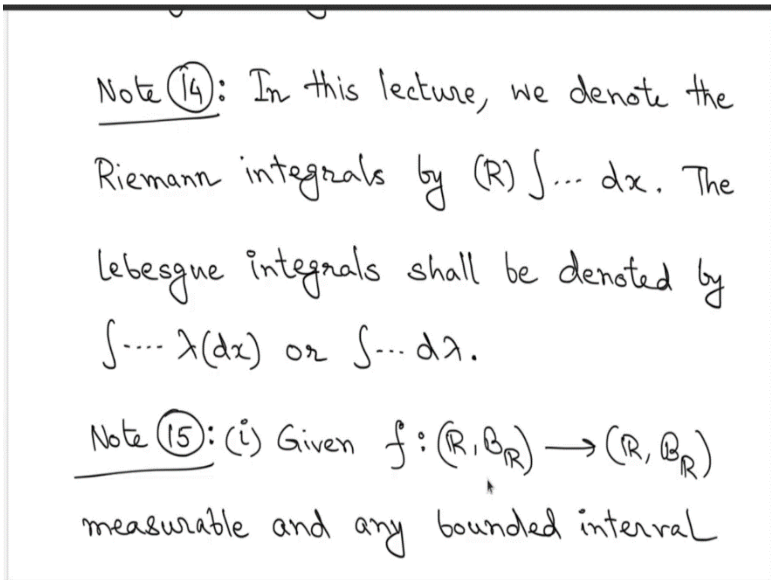


Now, another important notational issue, just to clarify, is that, in this lecture there will be two types of integrals appearing. One, will be this Riemann integrations, the second one will be Lebesgue integrations. Now, just to distinguish, which integration is being interpreted in the Riemann sense, which integration is being interpreted in the Lebesgue sense, we want to clarify certain notations.

So, for Riemann integrations, what we will do, we will write it in the usual integration of our integrand with respect to dx, but in front we will write $(R)$ to denote Riemann integration. The Lebesgue integrations will be denoted, as integration with respect to the major λ that is as per the usual notation that has been used so far in this course.

And more explicitly, sometimes λ dx may be explicitly written, otherwise, it will be simply the integrand that function dλ, that integration symbol will refer to Lebesgue integrations.

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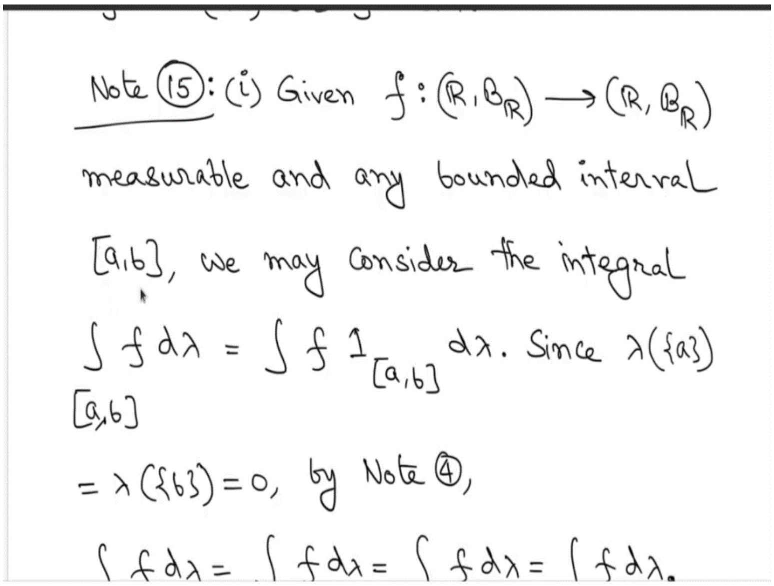


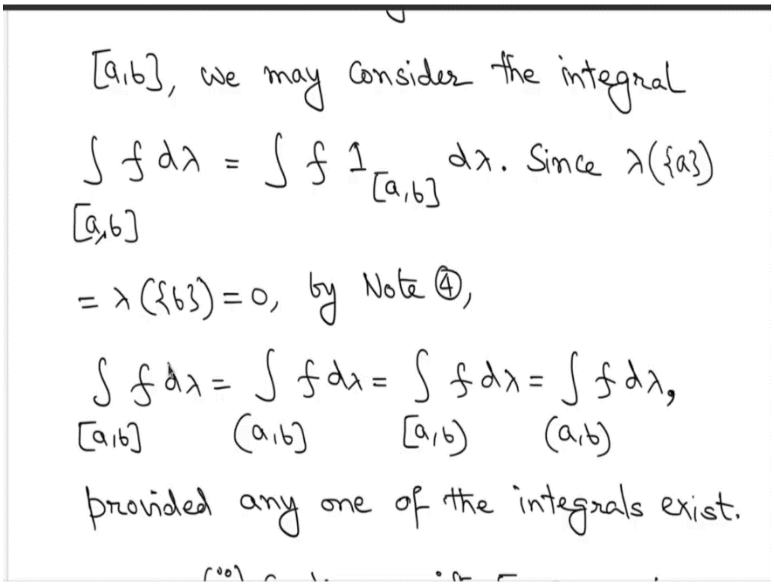


So now, a few comments about Lebesgue integrations that requires justification. First start with a measurable function on the real line, taking real numbers, as their values, and you also fix some bounded $[a, b]$. Now, consider this $\int_{[a, b]} f \, d\lambda = \int f \, 1_{[a, b]} \, d\lambda$. with respect to the Lebesgue measure. But now, recall, that Lebesgue measure associates mass 0 to singletons and in particular, here, the boundary points $a$ and $b$ those singleton sets will be associated 0 mass by the Lebesgue measure.

Now, by an earlier discussion on the Lebesgue measure, you can now claim that $\int\limits_{[a,\, b]} f\, d\lambda$ will be the same, as integration of the function over this $(a, b]$, $[a,\ b)$, $(a,\ b)$. So, whichever type of intervals you prefer, as long as, these sets differ only by sets of measures 0, the integrations will not change. So, that was the discussion made in note 4 earlier.

So, again, just to be clear, if one of these integrations exists, so does the others and the equality holds, so that is the meaning of this here.

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provided any one of the integrals exist.

(ii) Continue with $[a,b]$ as above.

using the idea of restrictions of the Lebesgue measure on Borel subsets in Note (24) of Week 5, we may consider the integral $\int\limits_{[a,b]} g \, d\lambda|_{[a,b]}$ for any $g$:

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$([a,b], \mathcal{B}_{[a,b]}) \longrightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable. We shall write $\int g\, d\lambda$ here to simplify

But this is for function defined on the real line, taking real numbers as their values. But now, there is a related concept of restrictions of functions and measures. So, what is this? So, continue with that interval as above. Now, what you can consider is that you can look at the Lebesgue measure restricted to interval $[a,\ b]$.

So, this was discussed earlier in note 24 of week 5. Just to recall, there, we had considered intervals of length 1, but in general, you can look at any such closed-bonded interval $[a,\ b]$ and restrict the Lebesgue measure to it. So, it does the same construction all over again.

But here you might not get a probability measure, you will get a finite measure. So, suppose you take any measurable $g$ defined on $[a,\ b]$ taking real numbers as their values. Then for all such $g$ you can try to $\int_{[a,\ b]} g\ d\lambda|_{[a,\ b]}$.

So, now living measure restricted to $[a,\ b]$ is a measure on this measurable space $[a,\ b]$ together with the Borel σ- field. And here, the integration of $g$ is over the actual domain, which is $[a,\ b]$. So, here, we are not looking at a subset of the domain here the domain is $[a,\ b]$ itself. So, here you can consider this type of integration.

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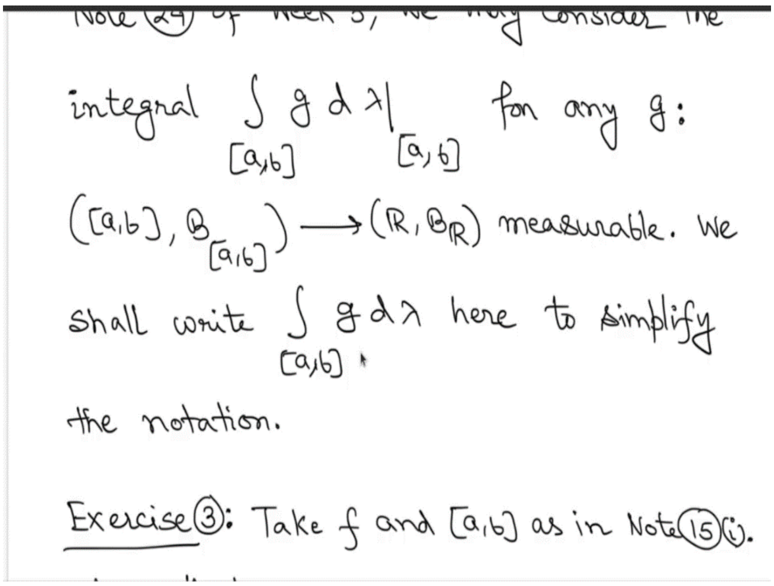

Note (24) of Week 5, we may consider the integral $\int_{[a,b]} g \, d\lambda|_{[a,b]}$ for any $g$: $([a,b], \mathcal{B}_{[a,b]}) \longrightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable. We shall write $\int_{[a,b]} g \, d\lambda$ here to simplify the notation.

Exercise (3): Take $f$ and $[a,b]$ as in Note (15)(i).

Now, just to avoid any further confusions we use this simplified notation that, we will take it to be the limit measure itself on the closed interval $[a,\ b]$. So, just to stop writing this restriction symbol here, we will simply write $d\lambda$ here, but we will understand, as long as $g$ is given to be defined on this closed interval $[a, b]$ this is the integration of $g$ with respect to the restriction of the Lebesgue measure to this interval.

So, that will be the understanding, but here we are going to use this simplified notation. But with this as the motivation, let us look at our few properties that are left as exercise for your verification.

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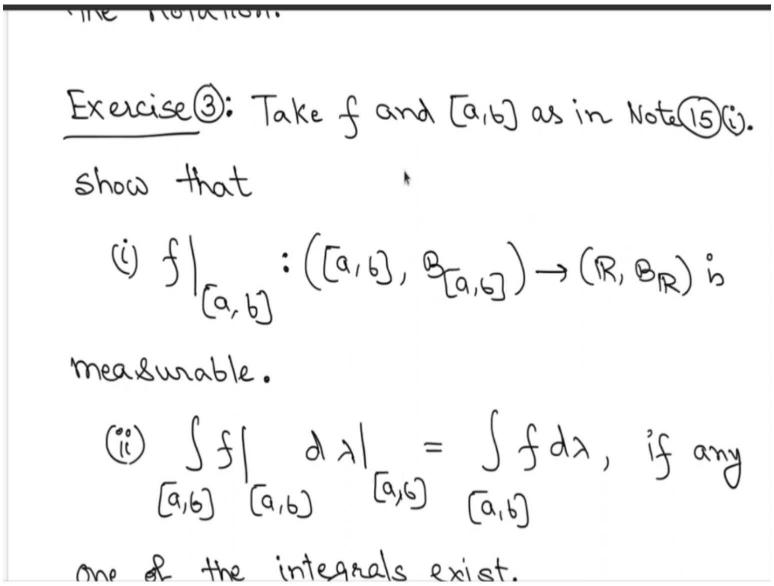


So, continuing with the function $f$ as described above, so here, we are taking the function $f$ to be defined on the real line and taking values within the real line, and you want it to be measurable. Now, if you continue with that same interval closed-bounded interval $[a, b]$ then you can show that restriction of this function to this interval will be measurable.

So, here you are putting the appropriate Borel's sigma fields on both sides, on the range and as well as the domain. So, with respect to these appropriate sigma fields, these restriction functions becomes measurable, you can check this. But then, once you have restricted this function, you can now consider the integration of the function the restricted function with respect to the restricted Lebesgue measure, and that integration will be over the actual domain here now, though the actual domain is $[a, b]$.

But you can identify or show that these value is nothing but $\int f\, 1_{[a,\, b]}\, d\lambda$. So, this is in the sense of integration of our real-valued function defined on the real line. So, that means, on the right-hand side it is taken, as $\int_{[a,\, b]} f\, d\lambda$.

Now, this equality will hold if any one of the integrals exists, you can show the existence of the other and you can show the equality. So, that is the idea. So, this is why, we are not focusing on restriction of λ, we will in general write λ itself, we are not going to carry around these restriction functions.

It will be clear from the context which limit measure is getting used, but the understanding will be that the measure has to be on the appropriate domain spaces.

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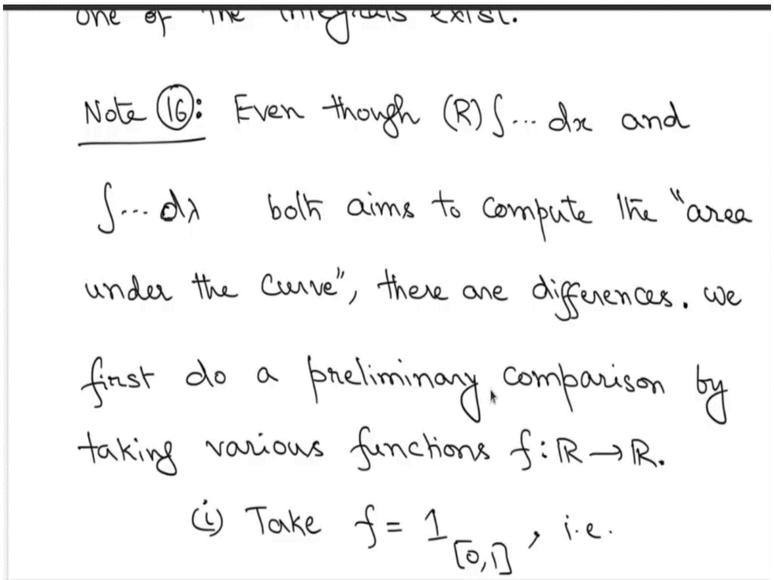


Now, here we start the discussion about the connection between Riemann integrations and the Lebesgue integrations. Remember, that Riemann integration was motivated, as the area under the curve. Similarly, when we started the discussion about measured theoretic integration, we started by looking at area under the curve of indictor functions.

And then, both these integrations took over from there from these basic definitions, and then we tried to extend these notions of integrations to the appropriate class of functions where the integrations can be defined. So, this is true for, both, Riemann integration and integration with respect to the Lebesgue measure.

Now, both these integrals are trying to compute the area under the curve for any general function in a general notion, there are certain differences. So, before we go into the actual theoretical results, let us just do some preliminary comparisons by looking at various forms as examples and see if there are any differences.

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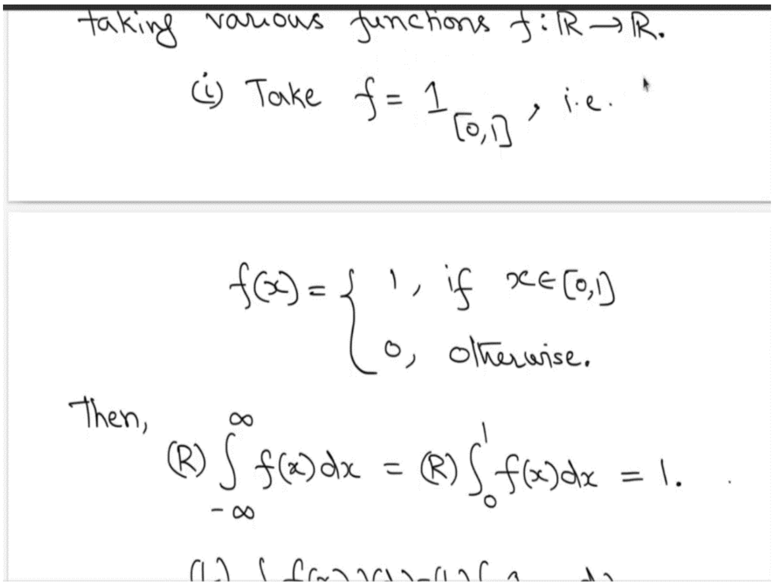


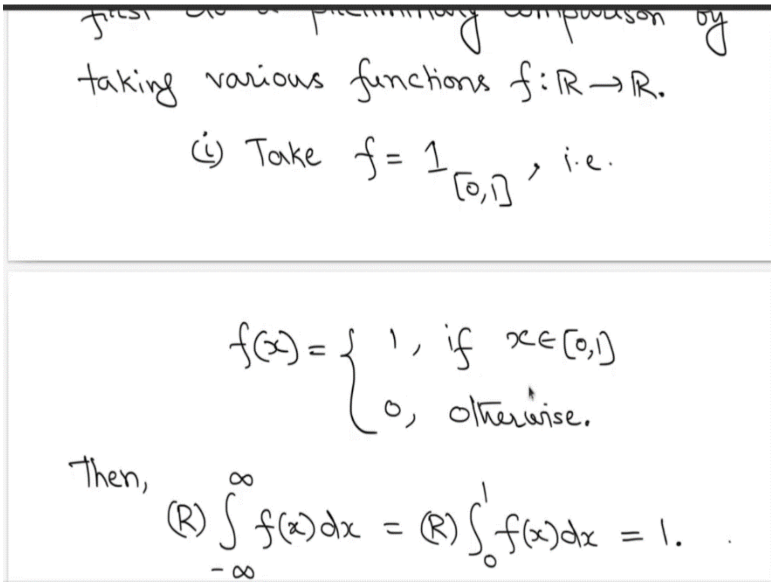


So, start with if defined on the real line and taking values in the real line. So, while talking about Riemann integrations on these closed-bounded intervals you typically work with bounded functions, while you are talking about the Riemann integrations, you would need these functions to be measurable.

So, a priori there are no direct connections between the hypothesis So, we will see the connection when we do the theoretical analysis, but start with $f$ which is a indicator function of $[0,\ 1]$, $1_{[0,\,1]}$. So, this is a closed bounded interval $[0,\ 1]$. More explicitly this is the function that takes value 1 on this interval otherwise it takes the value 0.

So, remember this function has finitely many discontinuities, discontinuities appear at this boundary point 0 and 1, so this will be Riemann integrable, and therefore, you can also compute the Riemann integration of this function that will be 1. That is just a simply the integration of the constant function 1 over the interval $[0,\ 1]$, so that will give you the value 1.

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$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$(R)\int_{-\infty}^{\infty} f(x)\,dx = (R)\int_0^1 f(x)\,dx = 1.$$

$$(L)\int_{\mathbb{R}} f(x)\,\lambda(dx) = (L)\int_{\mathbb{R}} 1_{[0,1]}\,d\lambda$$

$$= \lambda([0,1]) = 1 - 0 = 1.$$

Again,

$$(R)\int_0^1 f(x)\,dx = (R)\int_0^1 dx = 1,$$

What happens to the living measures situation So, if you want to integrate the function against the Lebesgue measure observe that this is a genuine measurable function it is just a indicator of this measurable set $[0,\ 1]$. So, as per definition this will be the living integration. So, we are writing (L) just to highlight this distinction.

So, here, what is happening, is that, you are integrating this indicator function against the Lebesgue measure so you get the length of the interval as the value of the integral. This is as per definition, and that is nothing but 1. So, here you get that the Riemann integration and the Lebesgue integration match.

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$$= \lambda([0,1]) = 1 - 0 = 1.$$

Again,

$$(R)\int_0^1 f(x)\,dx = (R)\int_0^1 dx = 1,$$

$$(L)\int_{[0,1]} f\,d\lambda = \int f\,1_{[0,1]}\,d\lambda$$

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Here, the integrals match.

(ii) Let $\mathbb{Q}$ denote the set of rational

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taking various functions $f : \mathbb{R} \to \mathbb{R}$.

(i) Take $f = 1_{[0,1]}$, i.e.

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$(R)\int_{-\infty}^{\infty} f(x)\,dx = (R)\int_0^1 f(x)\,dx = 1.$$

Now, you can try to check a different quantity now, you can try to integrate the function over this interval $[0,\ 1]$. So, earlier this integration was over the whole real line, but now, you are doing the integration only over this closed-bounded interval $[0,\ 1]$. And as per definition, this is the integration of $f\ 1_{[0,\ 1]}$.

And again, if you do the computation, it again gives you the value 1. So, for this specific function $1_{[0,\ 1]}$ if you do the Riemann integration or the Lebesgue integration do integrations over these intervals $[0,\ 1]$ or these intervals $[0,\ 1]$ here, all are matching in this specific situation. So, let us now take a slightly more general example.

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Here, the integrals match.

(ii) Let $\mathbb{Q}$ denote the set of rational numbers. Take

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

$$= 1_{[0,1] \cap \mathbb{Q}}(x).$$

So, suppose, you look at a set of rational numbers and look for all rational numbers contained inside the closed interval $[0, 1]$. So, look at all sorts of rational points there. So, then define the function to be 1 on those points otherwise you define it to be 0. So, this is the indicator of the set the set of rationales contained within the closed bounded interval $[0, 1]$ so that is this indicator function.

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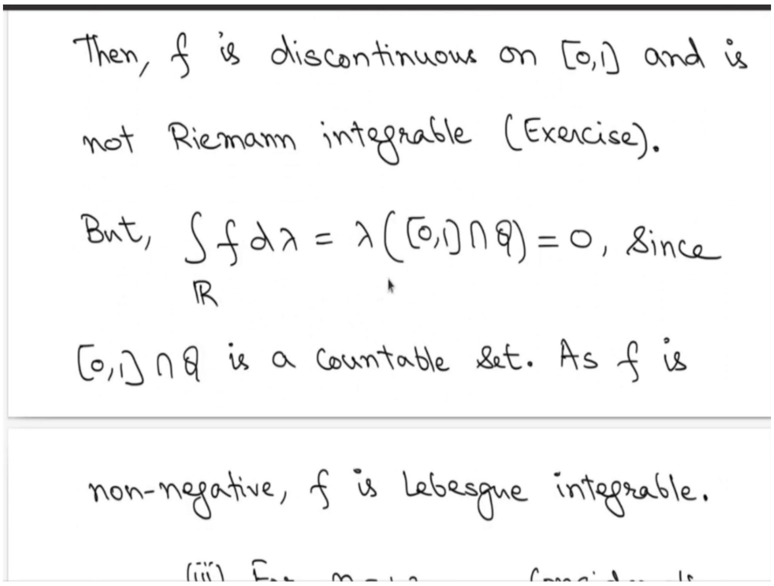


Now, observe that $f$ is discontinuous on these closed contour intervals $[0, 1]$ and you can show that this will not be Riemann integrable, you can take it as an exercise, but then this function is a genuine indicator of a measurable set a Borel subset of $\mathbb{R}$ and therefore, you can consider its integration and that is nothing but the length of this set.

And then, note, that this set here, this is a subset of the rational numbers. And since rational numbers are countable, this subset also is countable and Lebesgue measures assigns 0 mass to all countable sets. So, therefore, the area under the curve here in the sense of Lebesgue integration, this is 0 here, so that is interesting.

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non-negative, $f$ is Lebesgue integrable.

(iii) For $n = 1, 2, \cdots$, Consider the

Set $$\mathbb{Q}_n = \left\{ p \in \mathbb{Q} \;\middle|\; \begin{array}{l} r = \frac{p}{q} \text{ with } q = 1, 2, \ldots, n \\ \text{and } p = -q, -(q-1), \ldots \\ -1, 0, 1, \cdots, (q-1), q \end{array} \right\}$$

Note that $\mathbb{Q}_n$ is finite for each $n$ and

Then, $f$ is discontinuous on $[0,1]$ and is not Riemann integrable (Exercise).

But, $\int_{\mathbb{R}} f \, d\lambda = \lambda([0,1] \cap \mathbb{Q}) = 0$, Since $[0,1] \cap \mathbb{Q}$ is a countable set. As $f$ is

non-negative, $f$ is Lebesgue integrable.

and $p = -q, -(q-1), \ldots$
$-1, 0, 1, \cdots, (q-1), q$

Note that $\mathbb{Q}_n$ is finite for each $n$ and

$\mathbb{Q}_n \uparrow \mathbb{Q}$. Then $[0,1] \cap \mathbb{Q}_n \uparrow [0,1] \cap \mathbb{Q}$. Here,

$f_n := \mathbb{1}_{[0,1] \cap \mathbb{Q}_n} \uparrow f := \mathbb{1}_{[0,1] \cap \mathbb{Q}}$. Check that $f_n$'s are both Riemann and

Now, here $f$ is non-negative, so you can talk about Lebesgue integrability here. So, here the function is not Riemann integrable, but it is Lebesgue integrable. So, this is an important example. Now, we make the previous example slightly more complicated. So, consider this specific set of rational numbers such that it has a form $\frac{p}{q}$ with $q$ taking values between these natural numbers 1, 2 up to n, but $p$ can vary from $-\ q$ to $+\ q$ including 0.

So, $\frac{p}{q}$ only $p$ can take since $q$ is a natural number. So, you are looking at those type of rational numbers, which can be written as $\frac{p}{q}$ form which these range of $p$ and $q$. So, here this range of $q$ is dependent on this natural number $\mathbb{N}$ and therefore, you look at this set of rational and denoted by $Q_n$.

Now, it is easy to check that $Q_n$ is finite because you are only allowing finitely many choices of $q$ and for each choice of $q$ you are allowing finitely many choices of $p$. So, you have that this rational numbers which are of the form $\frac{p}{q}$ for this range of $q$'s and $p$'s this set of rationales is finite. But then, observe that as $n$ increases, this $Q_n$'s also increase they get larger and they will eventually cover the rational numbers set meaning their completely union countable union is the rational number $\mathbb{Q}$ itself.

So, therefore, you can immediately claim that $[0,\ 1]\ \cap Q_n$ meaning this specific type of rational numbers within $[0,\ 1]$ will increase an increase to all rational numbers within closed $[0,\ 1]$. But then, you can also look at the similar feature in terms of the indicators. As long as these sets increase to the set you have the indicators corresponding indicators increasing to this functions, which is also an indicator.

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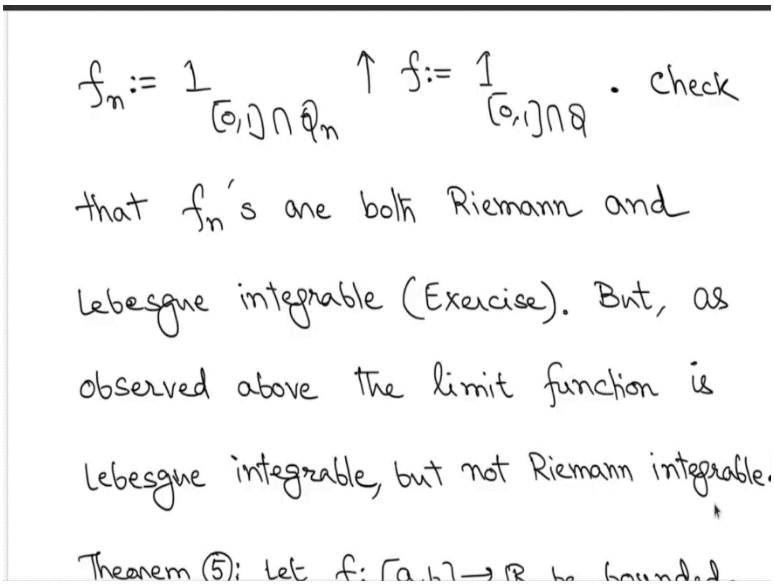


Now, here, it is interesting that the functions $f_n$ for each fixed $n$ are Riemann integrable and Lebesgue integrable, $f_n$'s are Riemann integrable, as well as Lebesgue integrable. But the Lebesgue function, which happens here this function $f$ indicator of rational numbers within $[0, 1]$ that indicator function this limit function is Lebesgue integrable but not Riemann integral. So, limit of Riemann integrable functions need not be Riemann integrable. So, this is a important comment about Riemann integrable functions.

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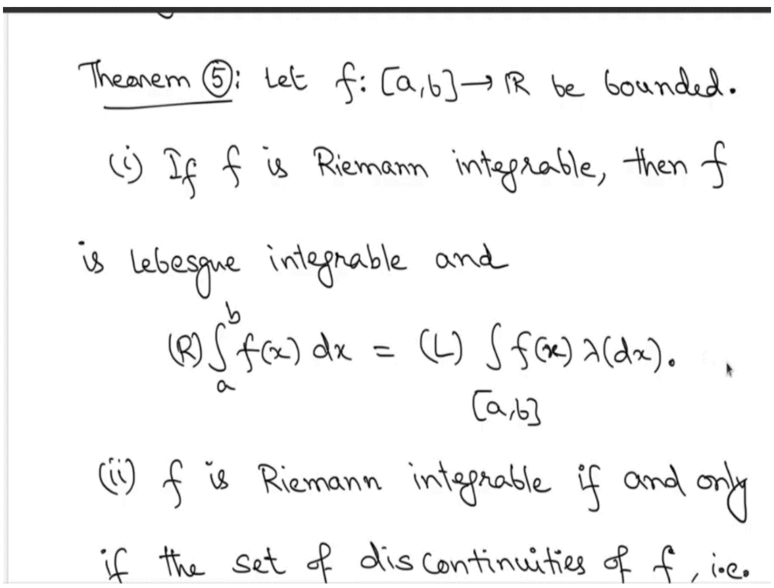


So, with these observations at hand we have already noted certain important distinctions between Lebesgue integrations and Riemann integrations. So, now, here is the concrete theorem which we want to focus on. So, look at a bounded function defined on this closed bounded interval $[a,\ b]$ taking values in the real line.

Now, start by assuming that $f$ is Riemann integrable then you can show that $f$ is Lebesgue integrable and the Riemann integration whatever the value will match with the Lebesgue integration. So, again, we are highlighting (L) here just to denote that this is Lebesgue integration. Of course, the Lebesgue measure is appearing here, but just to clarify, we are writing (L) to denote this Lebesgue integration.

So, what we are saying is that if you have Riemann integrability then you get Lebesgue integrability and the values match. So, therefore, if you get a function for which you can compute the Riemann integration then you do not need to separately compute the integration with respect to the Lebesgue measure, you can just take them value that appeared from the Riemann integration. So, this is an important observation.

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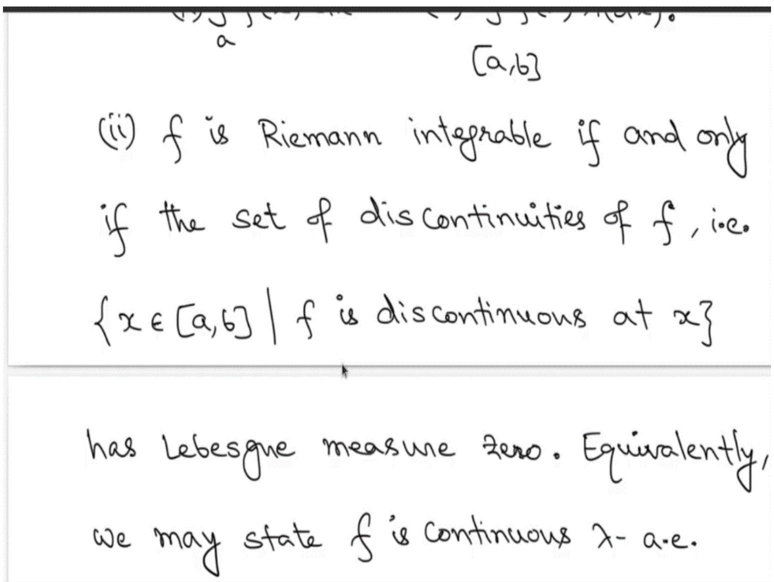


And here is an important classification or characterization of Riemann integrable functions $f$ is Riemann integrable then it happens if and only if the set of discontinuities meaning the set of points where $f$ is discontinuous that set of points has Lebesgue measure 0. So, this is an important identification of Riemann integrable functions.

Now, if it So, happens in this case, if $f$ is Riemann integrable then you get the statement that $f$ is continuous living measure almost everywhere. So, by that I mean that outside some appropriate null set, which contains this set of discontinuities of $f$ is continuous. So, here this is an important characterization of Riemann integrable functions.

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we may state $f$ is continuous $\lambda$- a.e.

We shall discuss the proof of Theorem 5 in the next lecture. In particular, the measurability of $f$ in part (i) needs justification. we now focus on the interpretation of this result.

Note 17: Keeping the part (i) of Theorem 5

Theorem 5: Let $f: [a,b] \to \mathbb{R}$ be bounded.

(i) If $f$ is Riemann integrable, then $f$ is Lebesgue integrable and

$$(R)\int_a^b f(x)\,dx = (L)\int_{[a,b]} f(x)\,\lambda(dx).$$

(ii) $f$ is Riemann integrable if and only if the set of discontinuities of $f$, i.e.

So, this proof will be discussed later in the next lecture, but there are a couple of important comments here. In particular, let's focus on part 1. So, in part 1 we are saying that $f$ is Riemann integrable implies $f$ is Lebesgue integrable. But remember, before even you talk about integrability you have to justify that f is measurable.

So, this will require proper justification and this will be included in the proof. So, that is including certain appropriate technical statements. So, all these details will be discussed in the next lecture. But now, we are going to focus on this interpretation of this result.

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Note (17): Keeping the part (i) of Theorem (5) in mind, we now write "$\int_a^b f(x)\,dx$" for $(L)\int_{[a,b]} f\,d\lambda$, provided the integral makes sense. We treat $(L)\int_{[a,b]} f(x)\,dx$ as an extension of Riemann integration. Even

$(L)\int_{[a,b]} f\,d\lambda$, provided the integral makes sense. We treat $(L)\int_{[a,b]} f(x)\,dx$ as an extension of Riemann integration. Even
though all Riemann integrable functions are Lebesgue integrable, the converse is not

though all Riemann integrable functions are Lebesgue integrable, the converse is not true (see Note 16 above)

<u>Note 18</u>: We now discuss the connection between (R) $\int_{-\infty}^{\infty} f(x)\, dx$ and (L) $\int_{\mathbb{R}} f(x)\, \lambda(dx)$.

If $f$ is bounded and Riemann integrable,

So, keeping this idea at hand that whenever you can compute the Riemann integration, you immediately get the value of the Lebesgue integration. So, that is the main content of the previous theorem. So, with that in mind, then, what you can say is that the Lebesgue integration as written here is essentially some kind of an extension of the Riemann integration.

So as long as Riemann integration exists you will get the same value for the living integrations, but there are functions, remember, there are functions which are Lebesgue integrable, but not Riemann integrable. So, this is important. So, just to identify that living integration is essentially an extension of the Riemann integration, we will use this simplified notation for the Lebesgue integration.

Now, that Lebesgue integration over $[a,\ b]$ will be taken as $\int_a^b f(x)\, dx$. So, we will suppress the symbol λ we will simply write this. So, if you can compute it in terms of the Riemann integration the same value will be given to the Lebesgue integration, but if you cannot compute it through Riemann integration, try it directly through the Lebesgue integration.

Now, it is important that we are using this symbol, but the justification, is that, this is a simplified symbol saying that Lebesgue integrations match with Riemann integrations provided the Riemann integrations exist. But, remember, there are Lebesgue integrable functions, which are not Riemann integrable. So, this is very important.

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**Note (18):** We now discuss the connection between $(R)\int_{-\infty}^{\infty} f(x)\,dx$ and $(L)\int_{\mathbb{R}} f(x)\,\lambda(dx)$.

If $f$ is bounded and Riemann integrable, then by Theorem (5)(i) above,

$$(R)\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} (R)\int_{-n}^{n} f(x)\,dx$$

then by Theorem (5)(i) above,

$$(R)\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} (R)\int_{-n}^{n} f(x)\,dx$$

$$= \lim_{n\to\infty} (L)\int_{[-n,n]} f(x)\,\lambda(dx)$$

$$= (L)\int_{\mathbb{R}} f(x)\,\lambda(dx),$$

$$= \lim_{n\to\infty} (L)\int_{[-n,n]} f(x)\,\lambda(dx)$$

$$= (L)\int_{\mathbb{R}} f(x)\,\lambda(dx),$$

provided the limits exist. We shall now write $\int_{-\infty}^{\infty} f(x)\,dx$ for $(L)\int_{\mathbb{R}} f\,d\lambda$.

**Note (19):** If a function $f$ has a singularity

So, now, what happens, if you want to discuss demand integrations with improper limits, meaning infinite limits possibly about the whole real line. Now, here, we are focusing on these types of integrations. Now, if $f$ is bounded and Riemann integrable in this situation then you can possibly try to justify this set of equalities that Riemann integration over $[-\ \infty,\ \infty]$ is essentially $\lim_{n\to\infty} (R) \int_{-n}^{n} f(x)dx$.

But then on this closed bounded intervals you have identified that the Riemann integration and the living integrations match, so you just replace the Riemann integration by the Lebesgue integration with respect to the measure λ. So, this is $(L) \int_{[-n,\, n]} f(x)\, \lambda dx$. But then, if you can justify the existence of this then this functions point-wise approximate the function $f$ what the whole real line.

So, $f$ multiplied by the indicator of this interval as $n$ increases will approximate the function $f$ point-wise. So, as long as these limits exist, you can justify all these equalities, then Riemann integration and the Lebesgue integration will match over the whole real line now. But for that you have to justify all these limits and the value that appears in the limit. So, you have to justify all those things, but this is the formal equalities that we can use.

So, therefore, again, exactly as done for finite intervals bounded intervals, you can also do the same for integration over the whole real line for the Lebesgue integrations that you write $\int_{-\infty}^{\infty} f(x)\, dx$, as a shorthand notation, just to identify that whenever Riemann integrations exists, it will match there. So, this is just an extension of the usual Riemann integrations.

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Note (19): If a function $f$ has a singularity at any interior/boundary point of the interval and provided the "Riemann integral" can be defined, then we can consider the connection with Lebesgue integrals. To reduce technical details, we shall assume this equality in

interval and provided the "Riemann integral" can be defined, then we can consider the connection with Lebesgue integrals. To reduce technical details, we shall assume this equality in
this case.

Note (20): (A pictorial comparison)

But then there are also improper integrations where you deal with singularities of the function. So, singularities may appear, as an interior point of the domain or at the boundaries. Now, here, if you can identify the Riemann integration, then it will require certain technical justification. So, there are appropriate ways of defining Riemann integrations.

We are not going to recall this, but once you can define Riemann integration or your function is Riemann integrable in this situation, you can consider the connection with the Lebesgue integrations once more. What you can say is that the equalities will still continue to hold, but to avoid these technical details, we are going to assume this equality, we are not going to discuss in complete detail the actual equality or the proof.

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this case.

Note (20): (A pictorial comparison)

For simplicity, we work with bounded and non-negative $f : [a,b] \to \mathbb{R}$.

If $f$ is Riemann integrable, then the "area under the curve" $(R)\int_a^b f(x)\,dx$ is

Now, we finish the discussion with a pictorial comparison of the ways of Riemann integrations and Lebesgue intuitions. And for simplicity, we are going to work with bounded and non-negative function $f$ defined on such close bounded interval $[a,\ b]$ and taking values in real line. So, assume that $f$ is Riemann integrable.

So, here, is where you can now compare the area under the curve in these two different ways. One is through Riemann integration another is through Lebesgue integration. So, let us try to look at in the Riemann integration setup.

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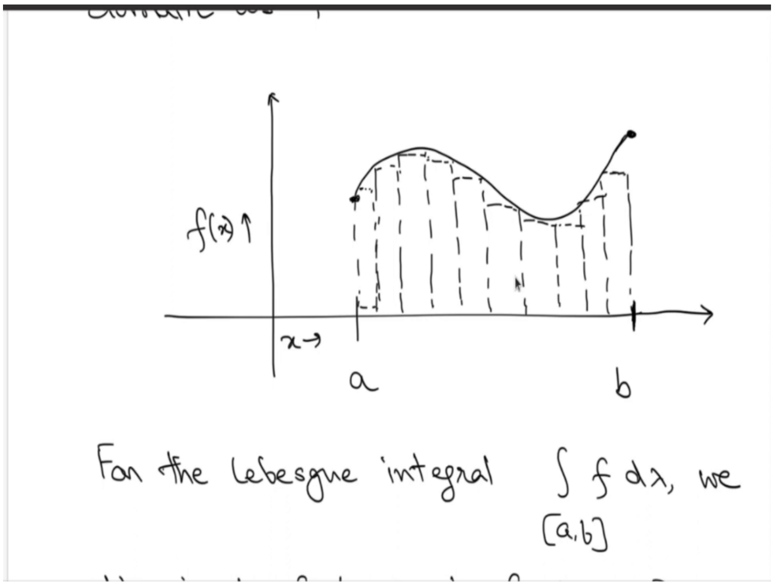
bounded and non-negative $f : [a,b] \to \mathbb{R}$.

If $f$ is Riemann integrable, then the "area under the curve" $(R)\int_a^b f(x)\,dx$ is approximated by "vertically splitting" the

domain as follows




This is what we do, we vertically split the domain side. So, if your function graph is something like this, you look at this domain $[a, b]$, you split it vertically and try to fit in some appropriate rectangles, which will approximate the area under the curve. So, you vertically split the domain and try to compute the sum of these areas of these rectangles. So, as long as your partitions on the domain, side shrink you get finer and finer approximations to the actual area under the curve.

So, that is what Riemann integration is all about, but what happens for the Lebesgue integrations? Remember, if you are dealing with this non-negative and bounded function, let us say you have justified integrability, then what he will do, you will approximate it through simple functions and you will use the area under the curve for simple functions as a approximation for the area under the curve of the additional function $f$.

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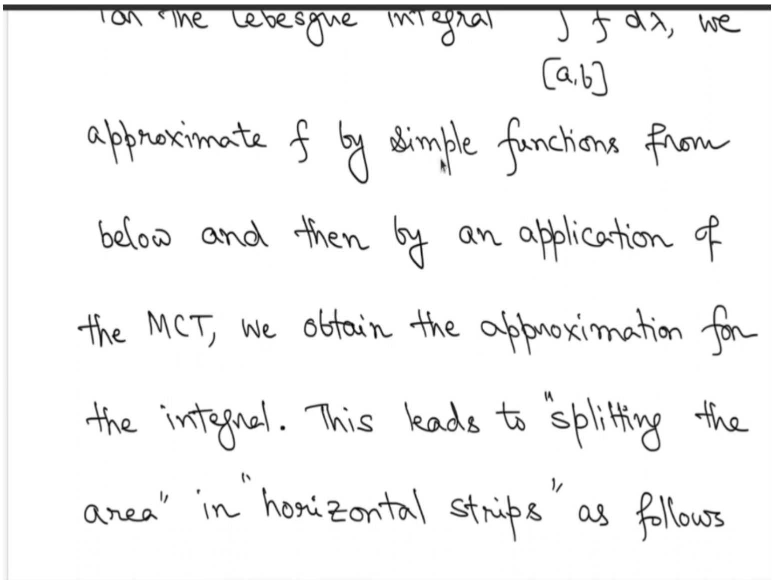


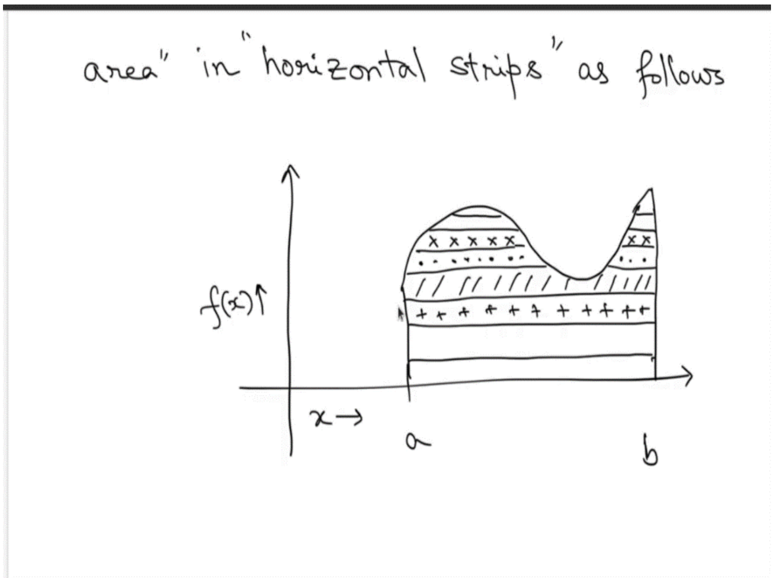


So, the idea is this. So, as soon as you are approximating this non-negative function f by simple functions from below then what you are actually obtaining is a different type of splitting the area under the curve. So, here, you are looking at range of values for the functions within those specific range of values, these are now horizontal strips, so these will denote specific range of values small range of values.

And for these strips you are approximating it through the appropriate simple function, which takes some boundary value as the approximation of the function. So, go back to that approximation by simple functions, try to look at the definition of the simple functions there and you will immediately observe that the lower boundary value for the function is taken as the approximation in simple functions well.

So, you are splitting the area under the curve in a horizontal sections now, horizontal strips, and now you are putting them together. So, what do you do here observe that for this example here the area here is the length, which is given by the appropriate measure on the domain side multiplied by the function value. So, that will be the area under the curve for that simple function that contribution.

Again, remember, that there might be situations where there are two parts in the domain, which take the same range of values. So, for example, this dotted part here is two possibly disjoint parts where the same range of values are being taken. Again, the values for the function will be approximated by a lower a point here and you just look at the total length this part dotted and this part dotted as given by the domain side measure.

So, that will give you an approximation for this area, that is the idea. Now Lebesgue measure or Lebesgue integrations what they are doing, they are splitting the area under the curve in horizontal strips, and in Riemann integrations you are splitting it into vertical strips. So, that is basically the pictorial comparison of Riemann integration and Lebesgue integration.

But it so happens that all Riemann integrable functions are Lebesgue integrable but the converse is not true. So, what is happening is that in some sense splitting it through this horizontal strips is allowing you to integrate much more functions much more bad functions, which need not be integrable in the Riemann sense.

So, this is a very important clarification regarding the Lebesgue integration. And this discussion, we will formalize in the next lecture, when we prove the exact connection between this Riemann integration and the Lebesgue integration as stated here in theorem 5 discussed earlier. We will continue this discussion in the next lecture.