Measuring Theoretic Probability 1
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Lecture 33
Fatou's Lemma and Dominated Convergence Theorem

Welcome to this lecture. In this week, we are looking at limiting behavior of measure theoretic integration. One of the major results that we has seen has been the monotone convergence theorem, which was property involving this limiting behavior of measurable functions and corresponding limiting behavior of the integrals. So, but we have also seen that as a consequence of monotone convergence theorem we had the additivity property. Now, we are going to discuss further general results that apply in limiting behaviors.

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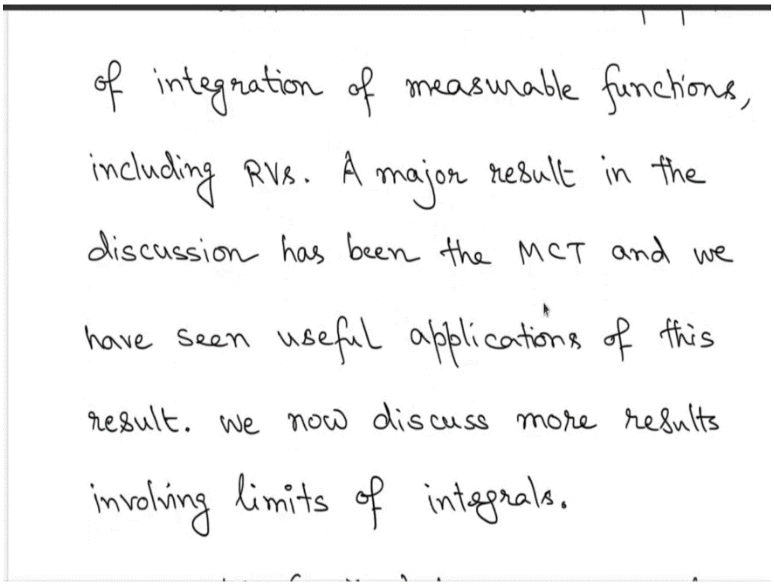

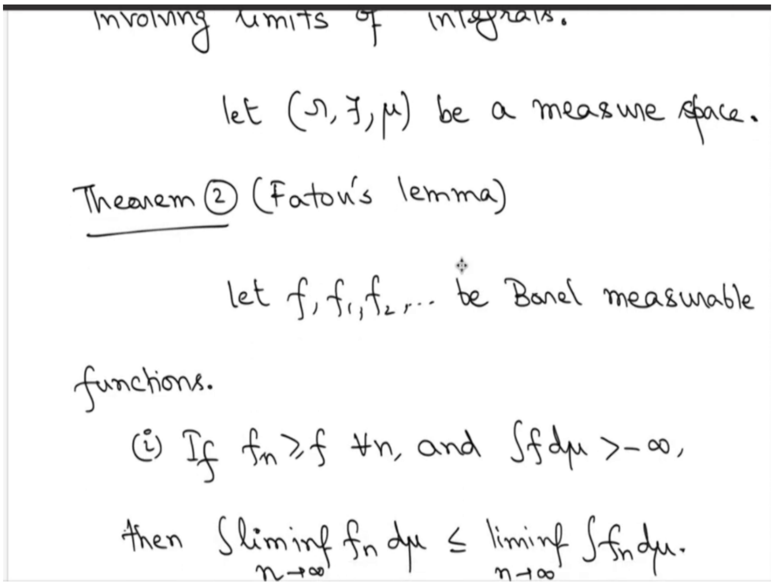

So, let us move ahead to the slides and discuss. So, we have already seen this in monotone convergence theorem and seen these useful applications, we are now focusing towards results specifically involving limits of this integrals.

Now, important comment is that in all of our discussion, we are going to fix this measure space $(\Omega, \mathcal{F}, \mu)$. Now, here is a very important result that is called the Fatou's Lemma and this is applicable to a quite a general situation, and that is why, this has been referred to it as theorem even though it is called as the Fatou's Lemma.

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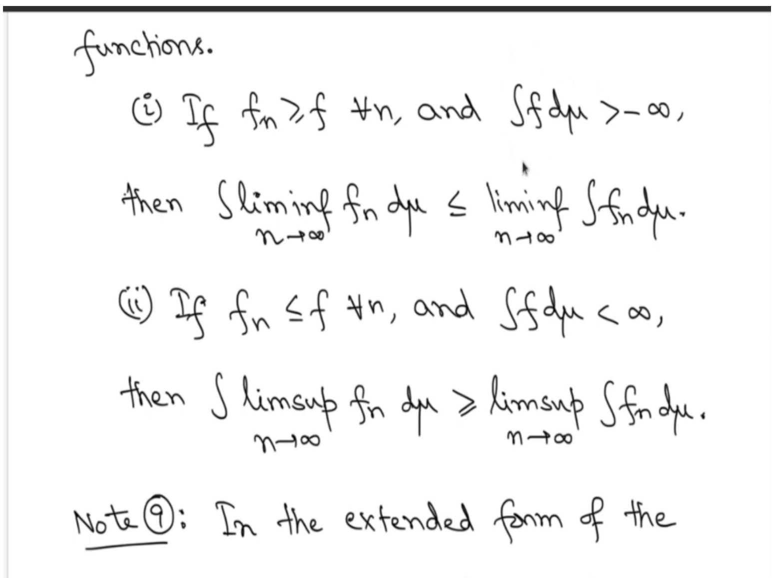

Note (9): In the extended form of the Monotone Convergence Theorem (Theorem (1)), we have stated that the sequence of functions

should be monotone. In Fatou's lemma, we do not assume any monotone behaviour of the sequence and as such

existence of $\lim_{n\to\infty} f_n$ is not guaranteed. We can, of course, talk about $\limsup_n f_n$ and $\liminf_n f_n$. These functions now appear in Fatou's lemma.

Proof of Fatou's lemma:

Consider the functions $g_n := \inf_{k\geq n} f_k$; $n=1,2,\ldots$

So, let's consider this situation that you are looking at this Borel measurable functions $f_1, f_2, \ldots, f_n$ so on that is a sequence, and then you consider a function f. So, these are some collection of Borel measurable functions. Now, if it so happens that f_n's this functions are dominated from below by this function f for all n and if the integration of f is avoiding the value $-\infty$. So, first of all, integration of f should exist and avoids the value $-\infty$.

In this situation you can get a statement involving integration and lim inf of the sequence of functions. And this says that you can bring out the lim inf from the integral, but you will end up with an inequality. So, therefore, integration of $\lim\inf$ of functions is less or equal to $\lim\inf$ of the integrations. So, this is the first part of the statement.

The second part says that if you have a bound going in the other direction, so, that means that if f_ns are dominated from above by this function f and if it so happens that the integration of f exists and avoiding the value $+\infty$ then you claim the other sided inequalities, but it involves limits superior of the functions and their integrations.

So, here the statement says that integration of the limit superiors is at least limit superior of the integrations. So, that is the statement here. You might recall that a similar statement was considered for the extended version of the monotone convergence theorem. So, therefore, we have seen such lower bounds given by a certain type of a function and we had considered the limiting behaviors.

But here it is important that the sequence of functions in the extended version of the monotone convergence theorem should be more monotone meaning either it should be increasing or decreasing. But here in Fatou's lemma we are not assuming any mode on behavior of the sequence of the functions. And as such, it is important to note that limit of the functions pointwise limit of the functions need not exist.

However, what we can always talk about are these functions limit superior of the sequence and limit inferior of the sequence? These are the things that are appearing in the statement of the Fatou's lemma.

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Consider the functions $g_n := \inf_{k \geq n} f_k ; n = 1, 2, \ldots$

observe that $f \leq g_n \leq f_n \ \forall n$ and

$g_n \uparrow \liminf_k f_k$.

Hence, by the extended form of MCT (Theorem 1), we have

$$\int g_n \, d\mu \uparrow \int \left(\liminf_k f_k\right) d\mu.$$

So, how do you prove this Fatou's lemma. So, consider this sequence of functions f_n and using these sequence of functions consider another sequence of functions call it g_n. So, how these $g_n s$ are defined. So, you look at for integers in 1,2,3,4 so on, you look at the infimum of the remaining functions. So, that means, that if your $n = 10$, look at f_{10}, f_{11}, and so on. Look at all those functions, look at the infimum of that.

So, pointwise infimum will define for you the corresponding function g_n, so that is how these functions are defined. But then remember, infimum over a countable set of functions if the functions $f_n s$ are given to be Borel measurable, then you immediately get the functions g_n to be measurable. So, infimum will give you measurable functions. But, moreover, there are certain interesting properties of this function g_n.

So, here f_ks case when you are considering the monotone behavior, here, the infimums are always less equals to the individual f_m. So, $g_n \leq f_n$ by definition, but you have also assumed here in the first part that f_ns have a lower bound given the above by the function f. But now, you also have that the g_ns are increasing in n and it will approximate the limit inferior of the functions f_n.

So, here this is an important comment that this infimum of the functions that is how the g_ns are defined, they increase in n and increase to the limit inferior of the functions. Now, here what you do, you have constructed a sequence of functions g_n, which are now monotone, monotone increasing, and are bounded from below of function f with integration of f avoiding the value $-\infty$.

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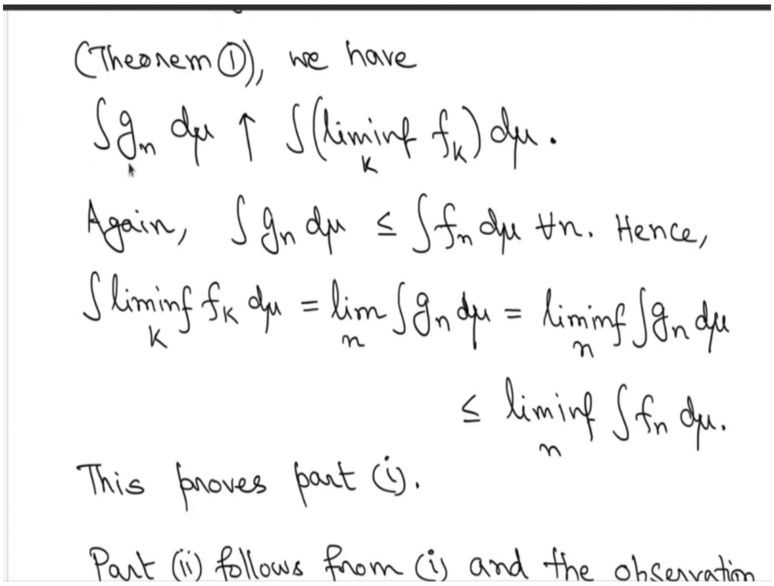

So, therefore, you can apply them extended form of the monotone convergence theorem, and you immediately get that $\int g_n$ will increase and increase to the $\int \ lim\, inf\ (f_k)$. So that is a statement here that applies from the extended form of the modern convergence theorem.

Now, you would like to make this connection with the actual sequence of functions f_n's, and they are integrals. So, let us just try to compare the $\int g_n$ and $\int f_n$. So, you want to bring in f_n now, and you want to connect it with the integration of the limit inferior, which appears as a limit of these integrations of g_n's.

Now, for every fix at n, you have this inequality because $g_n \leq f_n$. Now, you observe this set of arguments, that $\int\ lim\, inf\ (f_k) = lim\ \int g_n = lim\, inf\ \int g_n$, because the limit exists here, so therefore, it is equal to the limit inferior. So, that is all we are using.

But now observe that individually g_ns integrations of that is less equals to integration of $\int f_n$. So therefore, $lim\, inf\ \int g_n \leq\ lim\, inf\ \int f_n$. And, therefore, you get the required inequality, that integrations of limit inferiors of the functions is less equal to limit inferior of the integration of the functions. You get the inequality, required inequality. This proves the part 1.

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$$\leq \liminf_{n} \int f_n \, d\mu.$$

This proves part (i).

Part (ii) follows from (i) and the observation that $\limsup_{n} f_n = -\liminf_{n} (-f_n)$.

Note (10): In part (i) of Fatou's lemma, if $f_n \geq 0\ \forall n$, then we may take $f=0$.

let $f, f_1, f_2, \ldots$ be Borel measurable functions.

(i) If $f_n \geq f\ \forall n$, and $\int f \, d\mu > -\infty$, then $\int \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu$.

(ii) If $f_n \leq f\ \forall n$, and $\int f \, d\mu < \infty$, then $\int \limsup_{n \to \infty} f_n \, d\mu \geq \limsup_{n \to \infty} \int f_n \, d\mu$.

But now from part 1, you can prove part 2, by observing that limit superior of the functions and limit inferior functions have this interesting connection, that limits superior of the functions is equal to minus or the limit inferior of minus efference. So, now, let us go back to that assumption in part 2. So, we are going to consider the minus f_n's now.

So, if you consider the minus f_n's, you immediately get the lower bound given by minus f, and four minus f, the integration of that avoids that value $-\infty$, so that is exactly in the set off of part 1, and therefore, you can consider $lim\,inf\,(-f_n)$, and therefore, by part 1, you get $lim\,inf\,(-f_n) \leq lim\,inf\,(-\int f_n)$. And just multiply both sides by -1, you will immediately get the required relation involving lim sup as stated in part 2.

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Consider the functions $g_n := \inf_{k \geq n} f_k$; $n=1,2,\ldots$

observe that $f \leq g_n \leq f_n$ $\forall n$ and

$g_n \uparrow \liminf_k f_k$.

Hence, by the extended form of MCT (Theorem 1), we have

$\int g_n \, d\mu \uparrow \int (\liminf_k f_k) \, d\mu$.

Again, $\int g_n \, d\mu \leq \int f_n \, d\mu$ $\forall n$. Hence,

that $\limsup_n f_n = -\liminf_n (-f_n)$.

Note 10: In part (i) of Fatou's lemma, if $f_n \geq 0$ $\forall n$, then we may take $f=0$.

Note 11: (An example of strict inequality in Fatou's lemma)

Consider the Lebesgue measure

Note (11): (An example of strict inequality in Fatou's lemma)

Consider the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Let $\{f_n\}_n$ be the sequence of functions on $\mathbb{R}$ given by

$$f_n(x) := n \mathbb{1}_{(0, \frac{1}{n}]}(x), \quad n = 1, 2, \ldots \quad x \in \mathbb{R}$$

So, the proof of Fatou's lemma is pretty simple, as long as you choose the right function, right sequence of functions and you will immediately get it. So, therefore, that proves Fatou's lemma. But it is important to note that in part 1 of Fatou's lemma, if all the functions are non-negative, then we may as well take the lower bound function to be identically 0. So, this is used sometime.

So, you are having this general form of monotone convergence theorem now, that, if all the functions are non-negative, even though the functions are not convergent to a limit, you can consider limiting the failure of the functions and limits superior of the functions and you can claim certain inequalities involving the integrations of the limit inferior and limits superior.

Now, in the Fatou's lemma, you have observed that we have only managed to prove an inequality. So, you might ask whether we can actually improve it to an equality. See it so happens that you will actually get a inequality you cannot improve it further. So, we are showing this by an example of a strict inequality that might appear, so you cannot get an equality statement. So, consider the Lebesgue measure lambda on the real line, and consider a sequence of function f_n.

So, again the set of a Fatou's lemma you do not expect these functions to be monotone, but you should expect to get some kind of a limit inferior or limit superior. So, let us try to look at this kind of functions given by n times indicator of $(0, \frac{1}{n}]$. So, these functions takes them value n on the interval $(0, \frac{1}{n}]$, everywhere else this function takes the value 0, so that is how the functions are defined.

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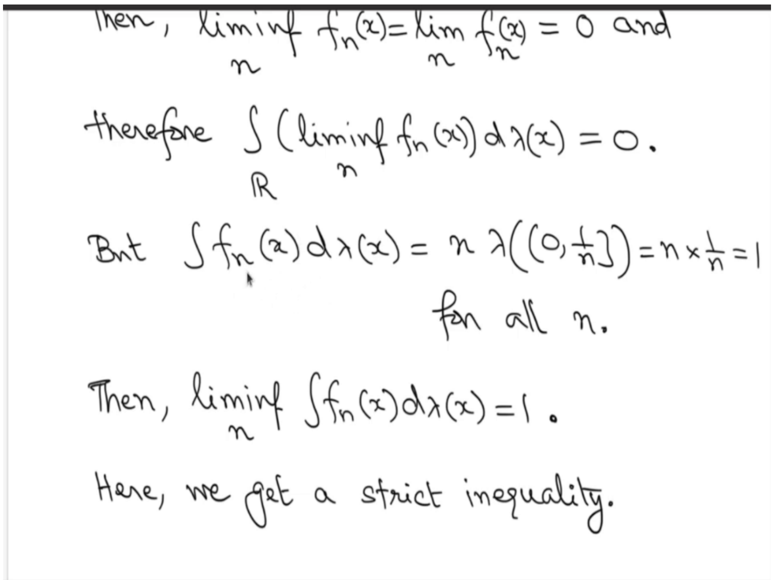

Now, it is easy to check that the limit function actually exists here, and therefore the limit inferior will agree with that limit function. So, here, for any point x you can show that the limit function is identically 0. So, therefore, integration of the limit inferior of the functions is exactly 0. So, if you integrate this limit inferior with respect to the Lebesgue measure, limit infinity is identical is 0, so therefore, the integration is identically 0.

But what happens to lim inf of the integrals. So, let us first compute the integrals of f_ns. So, here f_n s, remember, they are n times the integrator of $(0, \frac{1}{n}]$. So, therefore, follow the definition of the integral with respect to the measure, lambda the Lebesgue measure that will be nothing but n times the Lebesgue measure of the $(0, \frac{1}{n}]$, but that is nothing but n times $\frac{1}{n}$ which is 1.

So, for all positive integers n, you immediately computed that the integration of f_n is exactly equal to 1. So, therefore, $lim\, inf \int f_n$ is exactly equal to 1. Here you are getting a strict inequality that $\int lim\, inf\, f_n$s is exactly equal to 0, but limited inferior of the integrations is equal to 1. So, this is a strict inequality. So therefore, you cannot improve the Fatou's Lemma statement further, and put it as equality, you cannot do that.

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We now discuss a very important result in measure theoretic integration.

<u>Theorem ③</u> (Dominated Convergence Theorem)

Let $f, f_1, f_2, \ldots$ be Borel measurable functions such that $f_n \xrightarrow{n\to\infty} f$ μ- a.e. (i.e.

there exists a μ-null set N such that for

Let $f, f_1, f_2, \ldots$ be Borel measurable functions such that $f_n \xrightarrow{n\to\infty} f$ μ- a.e. (i.e.

there exists a μ-null set N such that for

all $\omega \in N^c$, $\lim\limits_{n\to\infty} f_n(\omega) = f(\omega)$.

If there exists an integrable function g, with $|f_n| \leq g \; \forall n$, then f

And now we go to a more general result, and this is a very important result in measure theoretic integration. This is called the **dominated convergence theorem**. So, again, we are now going to consider limit functions and their integrations. And this is going to give us some ideas on computing integrations by approximations.

So, here, we are considering a sequence of functions f_1, f_2, so on, and considered a function f. So, suppose that sequence of functions converges to f almost everywhere. So, see, we have now relaxed the convergence to involve this almost everywhere statement. So, what does it mean? So, it means, that they exist a μ- null set N such that outside the μ- null set N

on the complement of the μ- null set the convergence holds and the function values f_n will convert the function value of f.

So, for all those points outside the null set, you will get the convergence. So, we do not care what happens on the null set, because we have already remarked that values on a null set will not affect the integral's values. So, we are here interested in the integration's values.

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If there exists an integrable function g, with $|f_n| \le g \ \forall n$, then f is integrable and

$$\int f \, d\mu = \int \lim_n f_n \, d\mu = \lim_n \int f_n \, d\mu.$$

<u>Proof:</u> Given that $0 \le |f_n| \le g, \forall n$. Since, $f = \lim_n f_n$ μ-a.e, we have $|f| \le g, \mu$-a.e.

So, if it so happens that the functions, the original functions are dominated by a fixed function g this is very important. You need a fixed function g that dominates $|f_n|$. If it so happens for all n and if g is integrable, then you claim that the limit function f is also integrable and you get this exchange of limit and integration.

So, by that, I mean, $\int f d\mu$ is which is exactly equal to $\int lim\, f_n = lim \int f_n$. So, you are allowed to exchange the order of limit, and integration here. And that will agree with the integration of the limit function, even though it is up almost everywhere convergence. So, again, remember, here, the first equality immediately follows because these two functions agree almost everywhere.

Only on a null set these functions might differ, but it does not affect the values of the integral. Now, as for the conditions f_n's are dominated by a integrable function g, and therefore, f_n's are already integrable, so that is not an issue. So, but here you need to show that integration of f that is well defined and in particular you are claiming as a part of the statement of the theorem that f is integrable. So, you have to somehow establish that.

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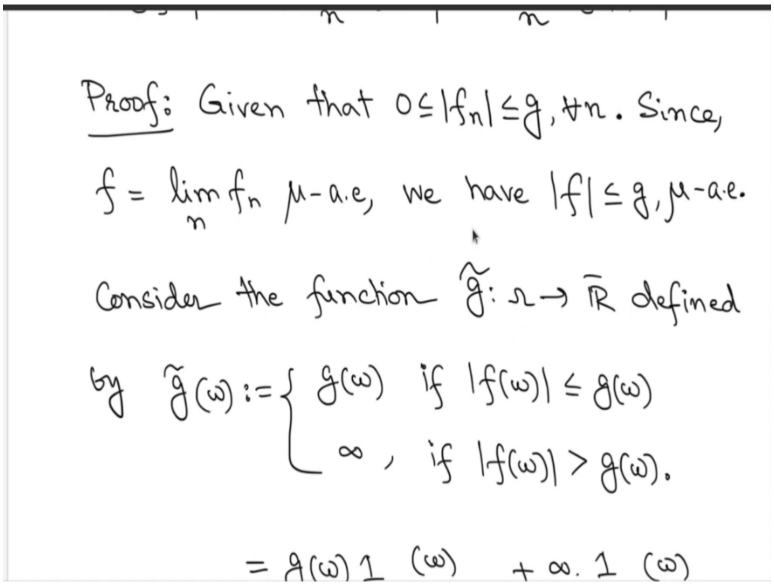

So, how do you prove this? So, start with this bound that is given to you. So, now these functions more different they are non-negative functions and they are dominated from above by these integrable function g. Now, here, you get the pointwise convergence of a f_ns to f, so therefore, you get that $|f| \le g$ is almost everywhere, outside this null set where possibly the convergence does not hold.

So, on the null possibly the convergence may not occur. So, outside that null set you get this point is convergence, so, therefore, you get this bound upper bound given by the function g. Now, you consider a different function $\tilde{g}$, which we are going to use as a upper bound on the function $|f|$. So, look at this function $\tilde{g}$. So, on the situation on the points where mod of f is dominated by the g function there you retain the function value of g otherwise you assign the value ∞. So, that is how the new function $\tilde{g}$ is defined.

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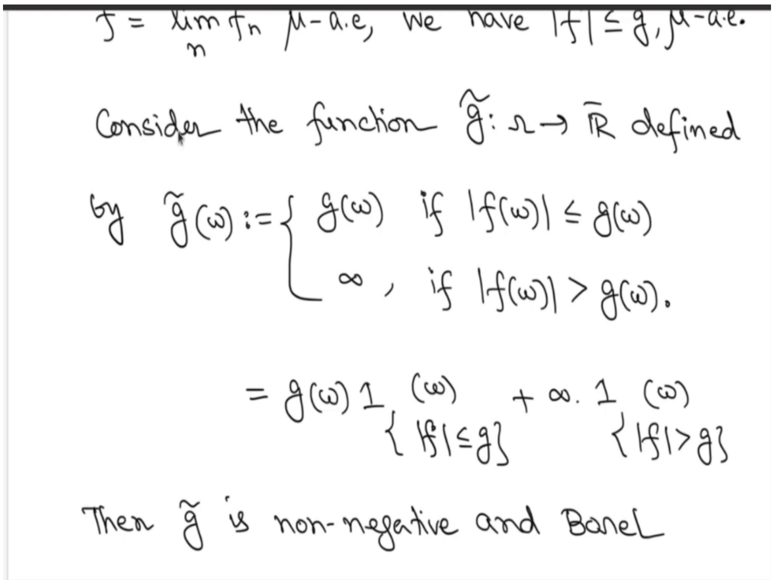

So, how does $\tilde{g}$ look like. So, in terms of this expressions involving indicators you can write it as this that $g(\omega)$ multiplied by the indicator of this set where $|f| \leq g$ otherwise it is ∞ on the set $|f| > g$. Now, here what is happening observe that you have already said that this set mod of is greater than f has 0 measure, so that is good.

And here if the function f is a limit of measurable function, so, that is measurable So, measurability here is not an issue. So, therefore, the function $\tilde{g}$ that you have such defined is a measurable function.

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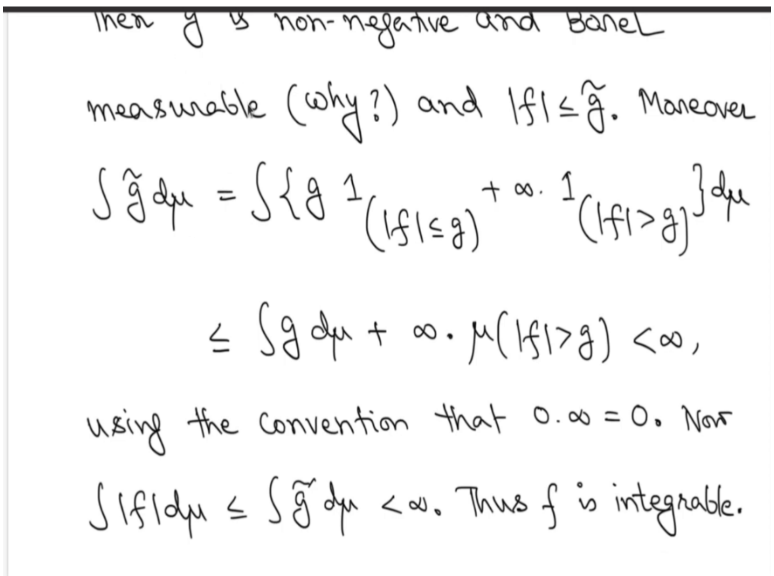

by $\tilde{g}(\omega) := \begin{cases} g(\omega) & \text{if } |f(\omega)| \le g(\omega) \\ \infty, & \text{if } |f(\omega)| > g(\omega). \end{cases}$

$$= g(\omega)\, 1_{\{|f| \le g\}}(\omega) + \infty \cdot 1_{\{|f| > g\}}(\omega)$$

Then $\tilde{g}$ is non-negative and Borel measurable (why?) and $|f| \le \tilde{g}$. Moreover

$$\int \tilde{g}\, d\mu = \int \{ g\, 1 \quad + \infty \cdot 1 \quad \} d\mu$$

Write down that argument that g function is Borel measurable. But now you observe that point wise the function $|f|$ is now dominated by $\tilde{g}$. So, on the points where $|f|$ was dominated by g there you have taken that value otherwise, of course, the function value is dominated by ∞. So, therefore, you get a $|f| \le \tilde{g}$.

But what happens to $\int \tilde{g}$ So, again, on a set of measure 0 the value ∞ does not matter, so therefore, you just get back the value of integration of g on the set in the $|f| \le g$. But this indicator is, of course, less equal to the indicator of the whole set, therefore, you can dominate the forced integration by integration of g itself, g is a non-negative function, so that is not a problem.

But g is given to be integrable so, therefore, that is finite. So, therefore, $\int \tilde{g}$, $\tilde{g}$ is non-negative that integration is finite.

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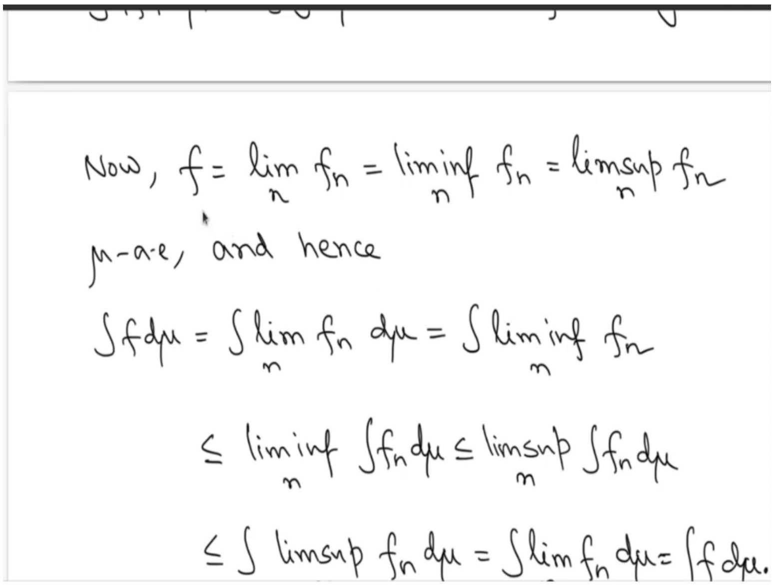

So, therefore, you observe the inequality. Now, that integration of $|f|$ is dominated by $\int \tilde{g}$.

So, this is true because $|f|$ is dominated from above by $\tilde{g}$. Therefore, f is integrable. So, you have proved the first part of the statement, but you would like to show that you can exchange the order of integration in the limits.

So, here, what is happening, look at the limit function. So, here, $f = \lim f_n$ almost everywhere, but if the limit exists then you can also identify it with $\lim\inf$ and $\lim\sup$.

So, here, again, we are interested in integrations, so therefore, in the following arguments whenever we are explicitly writing down the equalities or inequalities involving integrations, we are not going to care about this null set.

So, let us look at this equalities. So, $\int f\, d\mu = \int lim\, f_n\, d\mu = \int lim\, inf\, f_n\, d\mu$. So, as long as you expect that integration to exist, you have that.

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$$\leq \int \limsup_n f_n \, d\mu = \int \lim_n f_n \, d\mu = \int f \, d\mu. \quad \text{----}(*)$$

Here, we have used Fatou's lemma. Now, all the inequalities in $(*)$ must be equalities. Hence, $\liminf_n \int f_n \, d\mu = \limsup_n \int f_n \, d\mu$. Thus $\lim_n \int f_n \, d\mu$ exists. Then from $(*)$,

$$\int f \, d\mu = \int \lim_n f_n \, d\mu = \lim_n \int f_n \, d\mu.$$

Now, what you end up having, is that, $\liminf$ now comes out by an application of the Fatou's Lemma. So, $\liminf$ was inside if you bring it out, you get a less equal to inequality. But now, $lim\ inf \int f_n \, d\mu \leq lim\ sup \int f_n \, d\mu$, so that is a standard argument involving sequences of real numbers or extended real numbers, so that you already have.

Now, you push the limits superior inside. Again, you retain that inequality that is by the second part of the Fatou's Lemma. So, $\limsup$ now goes inside, but you still get

$lim\ sup \int f_n \, d\mu \leq \int lim\ sup\, f_n \, d\mu$. But now observe that pointwise limit superior of the functions is equal to limits of the functions, and therefore, you get the equality.

So therefore, you have originally started off with $\int f$, and you have ended up with $\int f$ itself.

But in between, there has been some inequalities, less equal to symbols. But you have obtained the left-hand side term and the right-hand side term which are equal. Hence, here we have used Fatou's lemma twice, but somehow you have obtained that the left-hand side and the right-hand side are equal.

Therefore, in these inequalities, you must have equalities because the left-hand side and the right-hand side match. So, therefore, you have immediately claim that $\liminf_n \int f_n \, d\mu = \limsup_n \int f_n \, d\mu$. As a consequence, you acclaim that limit of integrations exists, limit of integration of a f_ns exists, that is our first observation.

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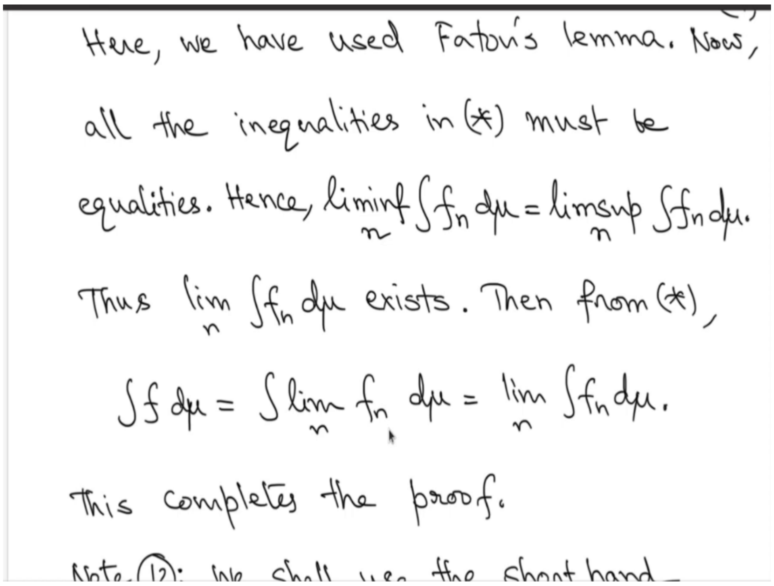

Moreover, if all the equalities hold, then you acclaim that $\int f = \int lim\, f_n$, it will be equal to the $lim \int f_n$. So, therefore, you have proved that $\int f = \int lim\, f_n$ and then that will be equal to $lim \int f_n$. So, here you are allowed to exchange the order of limits and integration.

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result in measure theoretic integration.

Theorem ③ (Dominated Convergence Theorem)

let $f, f_1, f_2, \ldots$ be Borel measurable functions such that $f_n \xrightarrow{n \to \infty} f$ μ- a.e. (i.e. there exists a μ-null set N such that for all $\omega \in N^c$, $\lim\limits_{n \to \infty} f_n(\omega) = f(\omega)$.

functions such that $f_n \xrightarrow{n \to \infty} f$ μ- a.e. (i.e. there exists a μ-null set N such that for all $\omega \in N^c$, $\lim\limits_{n \to \infty} f_n(\omega) = f(\omega)$.

If there exists an integrable function g, with $|f_n| \leq g \; \forall n$, then f is integrable and

So, this is possible by the boundedness assumption given by the function g. So, this is an important assumption that whenever you area considering general functions f_n, which can take arbitrary values now, but as long as the functions f_n are dominated by this integrable function g then you can exchange the limits and integrations. So, that is a statement of the dominated convergence theorem. And a particular case of that will also tell you that the limit function is integrable. Even if that the limit function is taken in almost everywhere sense. So, we have managed to prove the dominated convergence theorem.

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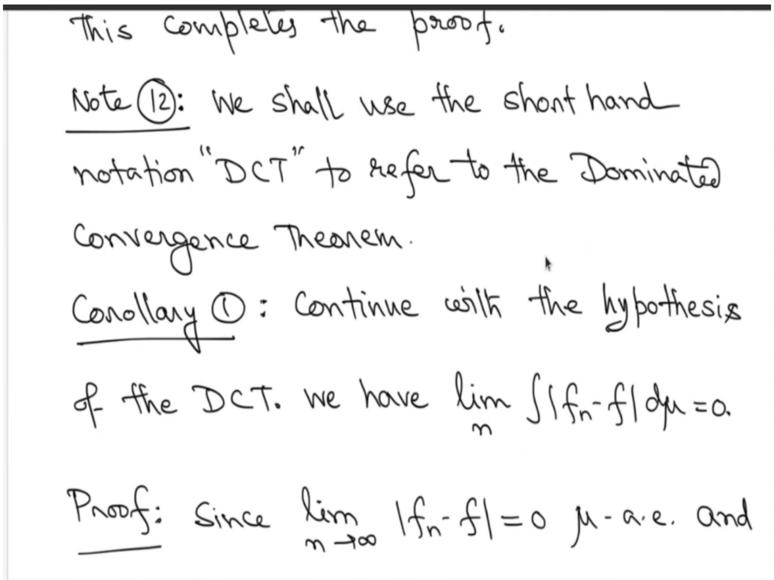

This completes the proof.

Note (12): We shall use the short hand notation "DCT" to refer to the Dominated Convergence Theorem.

Corollary (1): Continue with the hypothesis of the DCT. We have $\lim_{n} \int |f_n - f| \, d\mu = 0$.

Proof: Since $\lim_{n \to \infty} |f_n - f| = 0$ μ-a.e. and

Now, we are going to see that this dominated convergence theorem is a very general result and very useful in practice, we are going to use the shorthand notation DCT to refer to this dominated convergence theorem. Now, as a consequence of this dominated convergence theorem, we are now going to make some nice statements. So, continue with the setup of the dominated convergence theorem.

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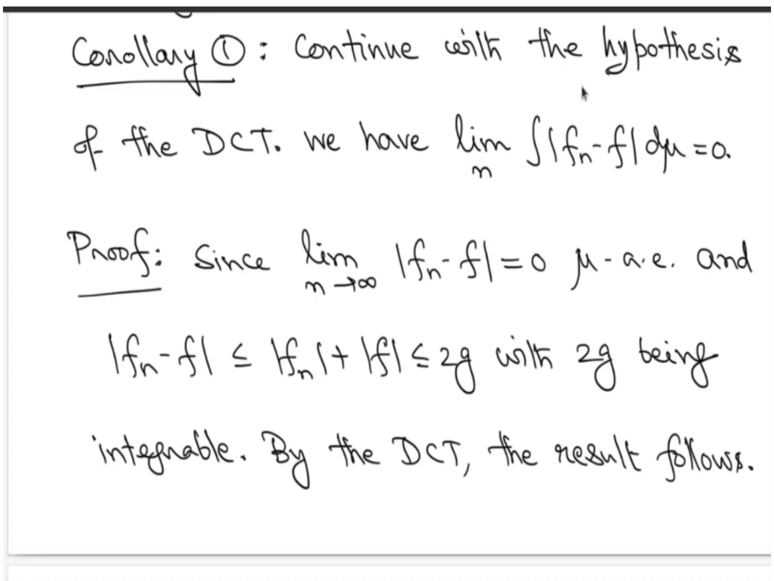

Corollary (1): Continue with the hypothesis of the DCT. We have $\lim_{n} \int |f_n - f| \, d\mu = 0$.

Proof: Since $\lim_{n \to \infty} |f_n - f| = 0$ μ-a.e. and

$|f_n - f| \leq |f_n| + |f| \leq 2g$ with $2g$ being integrable. By the DCT, the result follows.

So, here you assume that f_ns are converging to the function f almost everywhere, and the functions f_n are dominated by integrable function g so $|f_n|$ are dominated by integrable function g. Now, observe that if a f_ns converge to f almost everywhere then look at this sequence of functions $|f_n - f|$ that will converge pointwise to 0 μ almost everywhere.

Therefore, you do not care about the values that are occurring on that null set, outside the normal set, the convergence holds, and therefore, this limit will be the 0 from. But observe that the function $|f_n - f|$ can be dominated by $|f_n| + |f|$. However, you have again said that almost everywhere these things are dominated by g.

So, f_ns are dominated by g and $|f|$ is also dominated by g almost everywhere. So, therefore, in almost everywhere statement you get bound $2g$ for $|f_n - f|$, but $2g$, g is non-negative, $2g$ is also integrable. So, as long as g is integrable, $2g$ is also integrable.

Therefore, you apply the DCT to this sequence of functions now, which is $|f_n - f|$. And the result will follow because you are just considering the exchange of limit and integration. So, if you push the limit inside, as for the statement in this corollary, if the limit goes inside, you consider the limit function here, and the limit function is 0, so that will contribute integration to be 0. So, therefore, you get the statement. So, this is an important consequence of the dominated convergence theorem.

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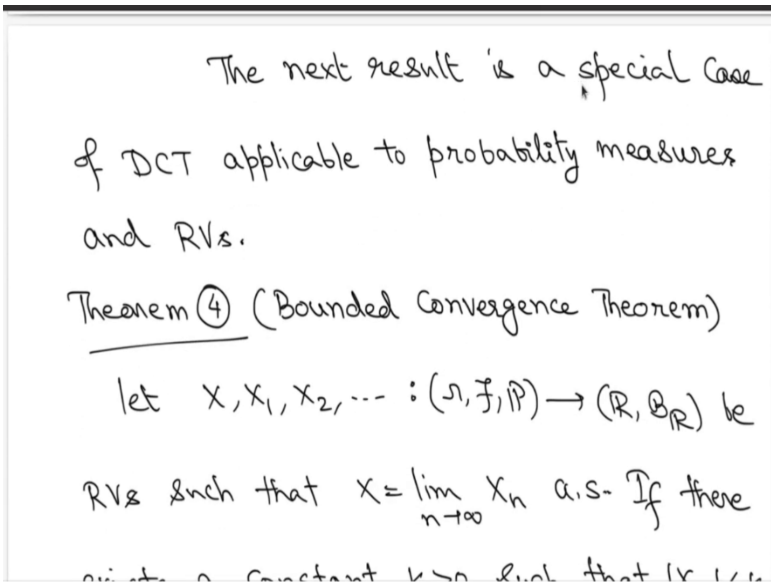

and RVs.

Theorem 4 (Bounded Convergence Theorem)

let $X, X_1, X_2, \dots : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be RVs such that $X = \lim_{n \to \infty} X_n$ a.s. If there exists a constant $K > 0$ such that $|X_n| \le K$, then the RVs $X, X_1, X_2, \dots$ are integrable and $\mathbb{E}X = \mathbb{E}(\lim X_n) = \lim \mathbb{E}X_n$

let $X, X_1, X_2, \dots : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be RVs such that $X = \lim_{n \to \infty} X_n$ a.s. If there exists a constant $K > 0$ such that $|X_n| \le K$, then the RVs $X, X_1, X_2, \dots$ are integrable and $\mathbb{E}X = \mathbb{E}(\lim_n X_n) = \lim_n \mathbb{E}X_n$.

Moreover, $\lim_{n \to \infty} \mathbb{E}|X_n - X| = 0$.

Now, look at further results, which are appearing as a special case of the dominated convergence theorem. So, consider the case of probability measures, and here, we are going to consider random variables. So, again, remember, random variables are nothing but measurable functions, as long as you have a probability measure on your domain side.

So, let us consider this theorem, which appears as a special case of the dominated convergence theorem. And it is known as the bounded convergence theorem. So, what do you do? You look at a sequence of random variables all defined on the same probability space. So, here are the sequences $X_1, X_2, \dots$ and so on.

Suppose, you get this random variable X such that X_n converges to X almost surely, so, by that, I mean that our outside the null set the convergence holds pointwise. Now, if there exists a constant key positive such that models of X_ns are dominated by k, then, you claim that all the random variables X_1, X_2 so on those random variables and limit random variable X, all of these are integrable, and you can exchange expectations and limits.

So, first of all $\mathbf{E}X$ will match with $\mathbf{E}\,(\lim_n X_n)$, but then, you allow the exchange to occur that limit can come out of the expectation, and that is what you will get as the statement. And, moreover, you can also claim, so that $\lim_n \mathbf{E}|X_n - X| = 0$. So, how do you prove this?

So, again, earlier we had mentioned that if you are going to bound a random variable by a constant, positive constant, you will immediately get the integrability. So, this was discussed in note 6 earlier, but now, the rest of the proof will follow from the DCT and the corollary that was discussed above. Let us make a few comments just to understand the steps.

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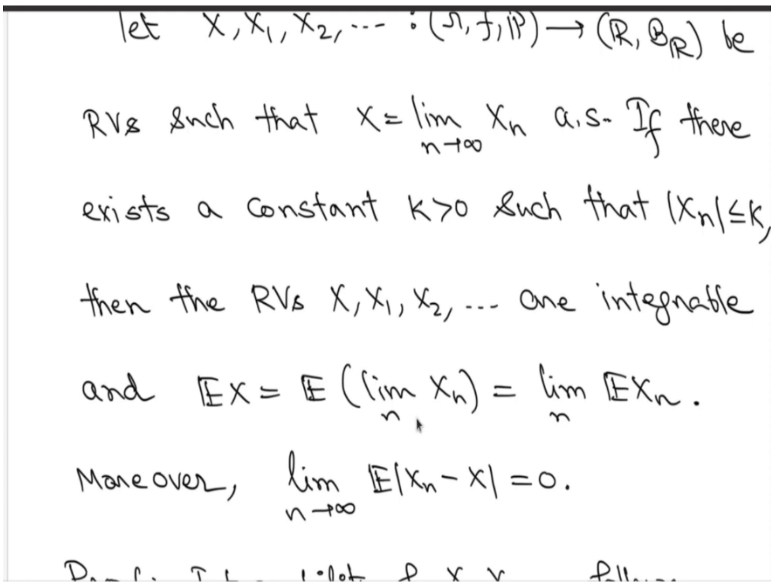

So, again as long as these X_ns are bounded by this constant and X_n's converge to X almost surely, you will get that X is bounded by k almost surely. Again, by note 6, as discussed earlier X must be integrable. In particular, this argument can also be discussed through the proof of the dominated convergence theorem.

So, X will be integrable. So, integrability is not an issue. But then if you are interested in only integrations or expectations as in this case, **E** X will be matching with **E** $(\lim_n X_n)$. So, that is fine because the equality holds almost surely for the X and $\lim_n X_n$. So, the expectations are matching.

But then, by the dominated convergence theorem, you are allowed to exchange the order of integrations and limits so expectations can be exchanged with limits in this situation. And finally, by the corollary, you get the expectation of the modulus of difference will converge to 0 that is by the corollary. So, that proves it.

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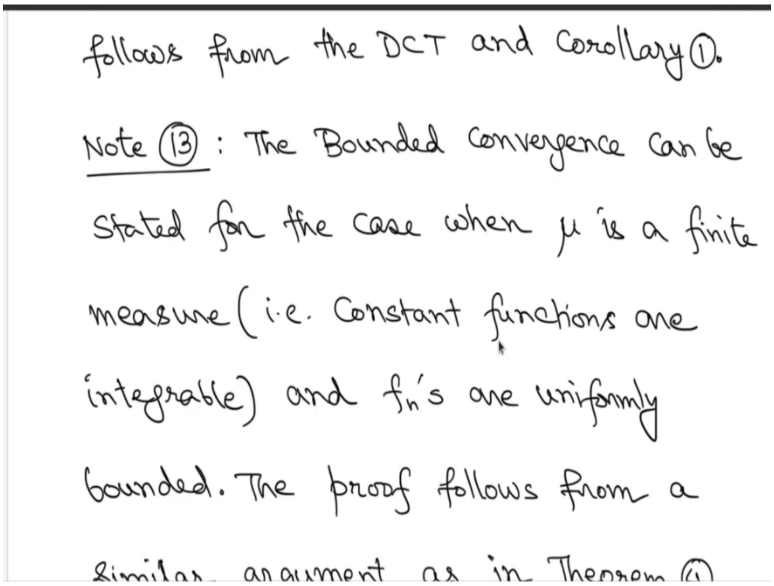

Now, here, we have used a probability measure for our discussion, but this bounded convergence this result can also be stated for the case when μ is a finite measure. So, here we had taken μ to be a probability measure, but if you take μ to be a finite measure there you cannot talk about random variables, but you can talk about measurable functions.

Here, if you can get that the measurable functions are dominated by some constant then you can argue the same that using the fact that constant functions are integrable you will claim that if all the measurable functions are dominated by a positive constant, then the functions will be integrable. Then if the functions converge point-wise to a limit function almost everywhere sense then, you will get that the limit function is also integrable.

And then finally, you are allowed to exchange the order of integration and limits. So, that appears as a special case of the dominated convergence theorem. But in this course, we are mostly interested in exchange of order of expectation and limits, which of course appears as a particular case when you consider the probability measures, and random variables.

So, with that observation, we have finished the discussion about the dominated convergence theorem, which is a pretty general result that allows us in computation of integration of the limit function. So, as long as you know the integrations of f_ns, you can consider the limit of integrations of f_n and obtain the integration of the limit function. So, that is quite useful in practice

Now, in later lectures, we are going to connect Reimann integration and Lebesgue integration and using that and using the nominator convergence theorem, we are finally ready to compute integrations over the real line. So, we have mentioned certain integrations with respect to the Lebesgue measure earlier, but once you have connected Reimann integration and Lebesgue integration, then by applications of DCT, we are going to see some very, very nice results. So, we stop the lecture here, and we are going to continue the discussion in the next lecture.