Measuring Theoretic Probability
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Lecture 31
MCT and the Linearity of Measure Theoretic Integration

Welcome to this lecture. This is the first lecture of week 7. So before going forward with the discussions of this week, let us first quickly recall what we have done in the previous week that is in week 6. So, we have defined the integration of measurable functions with respect to the domain site measure.

We have also looked at some of the major properties of such an integration. And as an application, we have obtained the expectations of discrete random variables. So that is, in a nutshell, what we have discussed in the previous week. Let us go forward and look at the slides of this lecture.

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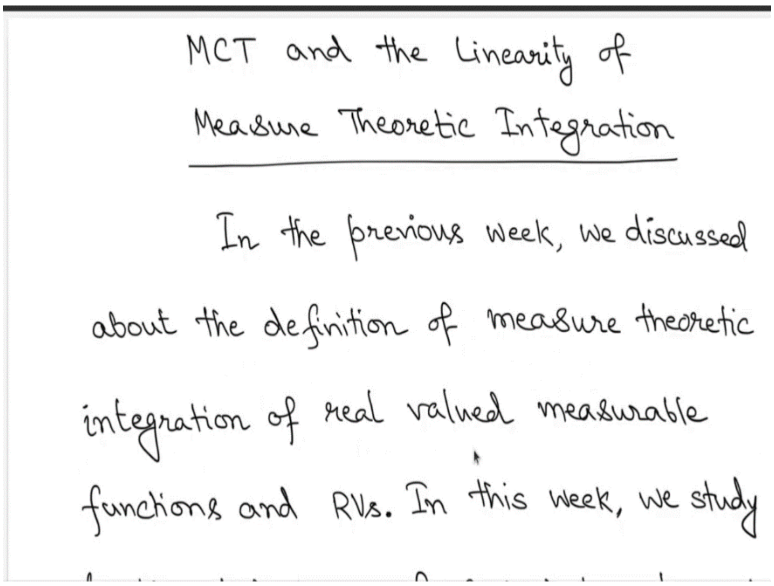

integration of real valued measurable functions and RVs. In this week, we study limiting behaviour of such integrals and expectations. These results shall be stated for $\mathbb{R}$-valued functions. However, these results may be extended/proved

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for $\overline{\mathbb{R}}$ valued functions in an identical fashion. we do not separately state these versions.

Note①: In the last lecture of week 6,

In the previous week, we defined the measure theoretic integration of real-valued measurable functions. And as a particular case, we also looked at the expectations of random variables. So, these are simply the integration of these measurable functions with respect to the underlying probability measures.

In this week, we are going to study the limiting behavior of such integrals. And of course, as a consequence, we are also going to get limiting behavior of expectations of random variables. These results are now being stated for real-valued measurable functions. However, if you can obtain the existence of the integrals, you can also consider these results in the extended real-valued functions case.

And using similar arguments you can extend or prove the results for these extended reroute functions. The arguments are identical and we are not going to separately state the origins for extended real-valued functions. As long as you can define the integrals for extended real-valued functions or you get the appropriate integrability conditions you can prove this.

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Note 1: In the last lecture of week 6, we computed expectation for discrete RVs. The fact that the law of these RVs are convex linear combination of Dirac measures, was an important result in these computations. However,

respect to different Dirac measures.

In many of the following arguments, similar situations shall arise - in that we have to consider the integrals of the same measurable function with respect to different measures.

To avoid any chance of a confusion, we shall use the following terminology to highlight the measures used.

let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

Definition ① (μ-integrable and μ-quasi-integrable functions)

But let us focus on one of the important steps that we did in week 6. We computed the expectation for discrete random variables. And what we had used there was that the fact that the law of these discrete random variables are convex linear combination of Dirac measures. So, this was used quite significantly, and we had to therefore consider the integrals of the underlying random variable with respect to these different, different probability measures or different, different Dirac measures.

So, in many of the following arguments, similar situations will arise. In that, we have to consider that the integrals are with respect to different measures or different probability measures, but the measurable function or the random variable that we are going to consider remains fixed. So, under this setup, it's important to keep track of the measure, and underlying measure with respect to which we are doing the integration.

So, to clarify this or to keep track of this, we are going to introduce this terminology to highlight, which measure we are using when we are doing the integration.

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Definition ① (μ-integrable and μ-quasi-integrable functions)

(i) Given $f:(\Omega,\mathcal{F}) \longrightarrow (\mathbb{R},\mathcal{B}_{\mathbb{R}})$ measurable, say that f is μ-integrable if

$$\int f^+ \, d\mu < \infty \quad \text{and} \quad \int f^- \, d\mu < \infty.$$

(ii) Given $f:(\Omega,\mathcal{F}) \longrightarrow (\mathbb{R},\mathcal{B}_{\mathbb{R}})$ measurable,

(ii) Given $f:(\Omega,\mathcal{F}) \longrightarrow (\mathbb{R},\mathcal{B}_{\mathbb{R}})$ measurable,

say that f is μ-quasi-integrable if one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite and the other is infinite.

Exercise ①: Restate the results of the previous week using the terms above.

So, in this lecture, we are going to continue with the standard notation that $(\Omega,\ \mathcal{F},\ \mu)$ will be a measure space. So that is μ being a measure on this measurable space $(\Omega,\ \mathcal{F})$. So, what are these terms? So, we are going to introduce these terms called μ integrable functions and μ quasi-integrable functions.

So, what are these? So, if you have a function measurable with respect to the appropriate σ-fields, you are going to say that this is μ integrable, if, $\int f^{+}\, d\mu$ is finite and $\int f^{-}\, d\mu$ is also finite. So, this is exactly the integrability setup that we had considered in the previous week. We are just adding the measure in front of the term integrable instead of just saying f is integrable we are now going to say if is μ integrable.

So, this is now highlighting the measure in question with respect to which you are doing the integration. And similar terminology will now going through be used for quasi integrability case where $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are going to be considered.

And if one of them is finite and the other is infinite, you are going to say, that f is quasi-integrable with respect to the measure μ or you can use this shorthand terminology that f is μ quasi-integrable. This is again, the same terminology that we had used in the previous week, but we are just highlighting the measure now, by saying f is μ integrable or f is μ quasi integrable

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Exercise ①: Restate the results of the previous week using the terms above.

We now consider the proof of MCT. We recall the statement.

Theorem ① of week 6 (MCT).

Let $\{h_n\}_n$ be a non-decreasing sequence of non-negative, Borel measurable functions defined on a measure space $(\Omega, \mathcal{F}, \mu)$. Consider the limit function $h: \Omega \to \mathbb{R}$ defined by $h(\omega) := \lim_{n\to\infty} h_n(\omega)$, $\forall \omega \in \Omega$. Then, $\int h_n \, d\mu \uparrow \int h \, d\mu$ as $n \to \infty$.

Note ②: All the functions $h, h_1, h_2, \cdots$ are non-negative and hence $\int h \, d\mu, \int h_1 \, d\mu,$
$\int h_2 \, d\mu$ exist and take values in

$h : \Omega \to \mathbb{R}$ defined by $h(\omega) := \lim_{n \to \infty} h_n(\omega)$, $\forall \omega \in \Omega$. Then, $\int h_n d\mu \uparrow \int h d\mu$ as $n \to \infty$.

Note ②: All the functions $h, h_1, h_2, \ldots$ are non-negative and hence $\int h d\mu$, $\int h_1 d\mu$, $\int h_2 d\mu, \ldots$ exist and take values in $[0, \infty]$.

Note ③: For $n = 1, 2, \ldots$ we have

So, with these terms in hand, you can restate the results of the previous week, and that is what you should do as an exercise. Try to write down all these statements by using the terms μ integrable or μ quasi-integrable. So, now, we are going to look at the proof of the Monotone Convergence Theorem that was left out in the previous week.

So, first recall the statement that you are going to consider a sequence of non-negative Borel measurable functions, such that, the function should be non-decreasing in n. And as soon as that happens, we had earlier explained that there will be a limit function, which you call as h and we had also mentioned that the limit function being a limit of measurable functions will become a measurable function.

So, therefore, you can consider $\int h \, d\mu$. Now, since you are given these functions h_n you can also consider $\int h_n \, d\mu$ and we are going to say that $\int h_n \, d\mu$ increases to the limit value which is given by $\int h \, d\mu$, so that was the statement of monotone convergence theorem.

So, before proving this a couple of comments that we should be aware of, is that the function that we are using here in the statement is non-negative and hence, you have the existence of all the integrals. So, these integrals will exist but can take values as ∞. So, the values possible values of these integrals are $[0, \infty]$. So, this is the range of possible values.

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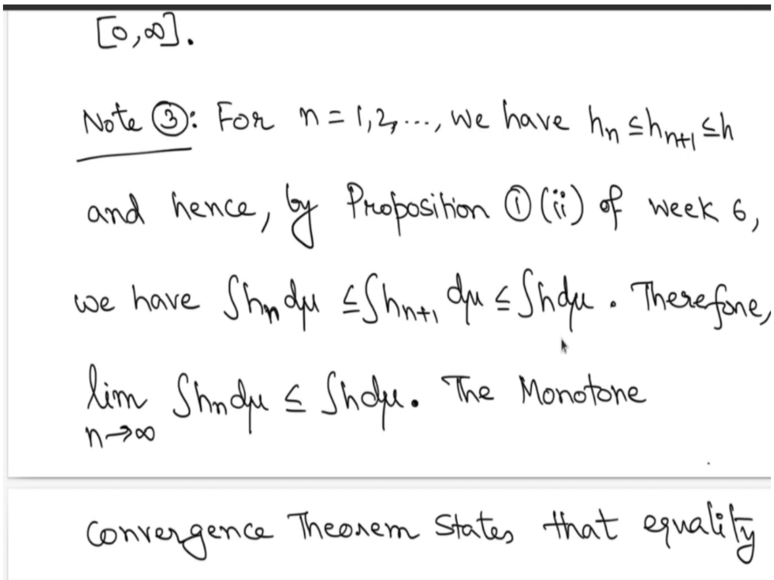

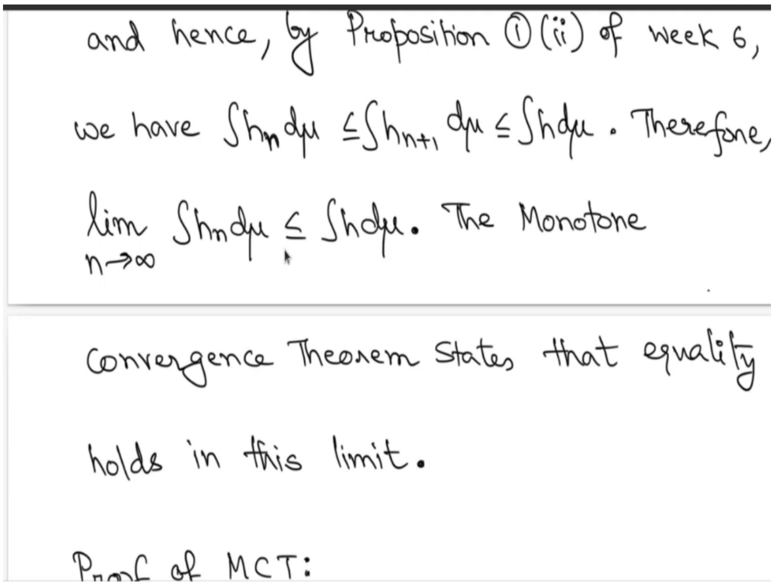

Next thing that we have to be aware of, is that, there are these comparison type inequalities between the functions. So, for every fixed n, $h_n \le h_{n+1}$, so therefore, corresponding integrals will also follow the same order. So, $\int h_n \, d\mu \le \int h_{n+1} \, d\mu$. But h being the limit of all these non-decreasing sequence of functions, you get that h is upper bound for all these functions.

So, therefore, you also get the upper bound for all these integrations as $\int h \, d\mu$. But since $\int h_n \, d\mu$ are increasing in n you therefore, get the limit value. And as per these inequalities you will immediately claim that $\lim_{n \to \infty} \int h_n \, d\mu \leq \int h \, d\mu$. So, that is a very simple inequality that is just going to follow from the original inequality that we just mentioned.

But what the modern convergence theorem is saying, is that, in these inequality the equality holds, as long as, you are dealing with these non-decreasing, non-negative functions. So, non-decreasing in n. So, that is the statement of the modern convergence channel.

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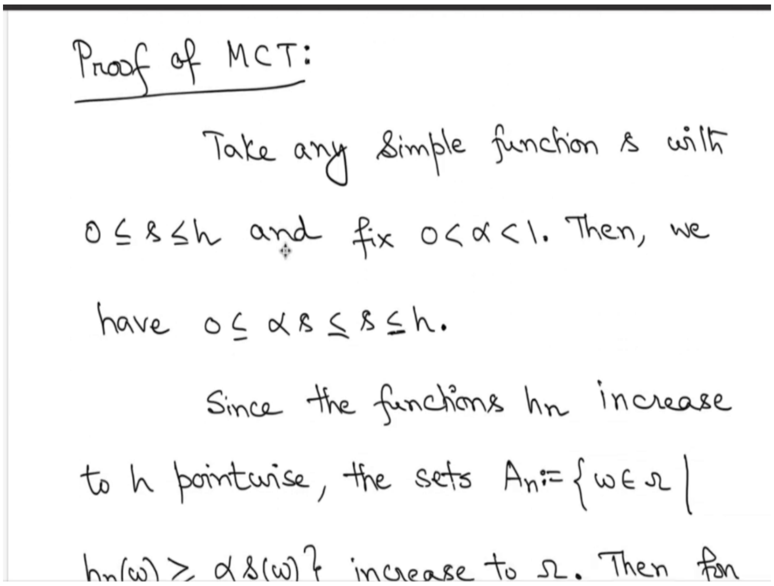

So, let us see how do you prove this statement. So, start with any simple function s, which is approximating the function h from below So, what you are going to do is to choose this simple function below h. So, since h is non-negative you are just going to choose these simple functions between 0 and h. And for the purpose of our argument we also need to fix a scalar α between 0 and 1. So, you will see the usage of the scalar α.

But use these two quantities, the function s and the scalar α to write down this inequality, that $\alpha s \leq s$, and so, of course, $s \leq h$. So, you have these inequalities. So, what we are going to do is to consider the integrations of this, but on specific type of sets.

What we are going to do is to now bring in the functions h_n. So, the functions h_n increased to h point wise, so that is the definition of h anyway so you can consider these sets A_n. So, for every fixed n consider the set of points on the domain such that $h_n(\omega) \geq \alpha\, s(\omega)$.

So, what is this? Look at the sequence of $h_n(\omega)$, they increase an increase to $h(\omega)$. So, therefore, in this inequality what is going to happen is that $h_n(\omega)$ is increasing, and increases to the value of h. So, therefore, it will close the value $\alpha\, s(\omega)$ at some certain stage, so we are going to consider that specific event. So, this is what these events or these sets are keeping track of.

We are going to look at $\{\omega \in \Omega / h_n(\omega) \geq \alpha\, s(\omega)\}$. But now, if you let n go up to ∞you will immediately say that $h_n(\omega)$, as they are approximating the value $h\ (\omega)$, so therefore, it is definitely going to cross this value $\alpha\, s(\omega)$. So therefore, these sets increase and increase to the whole set Ω, so this is the increasing ness in n.

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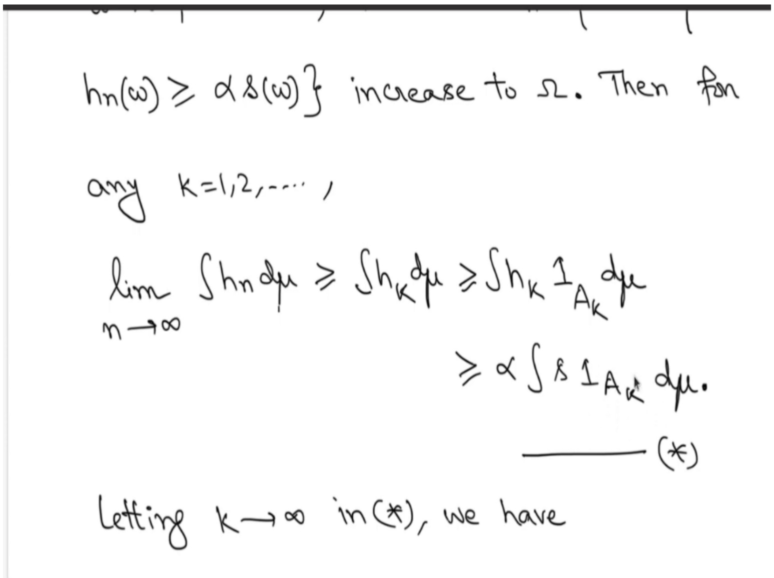

Now, you look at every fix at positive integer k, you look at

$$\lim_{n\to\infty} \int h_n\, d\mu \geq \int h_k\, d\mu \geq \int h_k\, 1_{A_k}\, d\mu \geq \alpha \int s\, 1_{A_k}\, d\mu$$

So, that is just by the increasing property of the functions h_n. But now look at these non-negative functions h_k and multiply by indicator of A_k. So, these functions h_k are, of course, greater or equals to $h_k\, 1_{A_k}$, so you get this inequality here.

But what is happening on these sets A_k, is that, $h_k \geq \alpha s$ there. So, on the sets A_k the function $h_k \geq \alpha s$. So, use that, and you get this relation star. So, therefore, limit of integrations $\int h_n d\mu$ has this interesting lower bound given by $\alpha \int s 1_{A_k} d\mu$. So, this is what we have observed now.

But in this inequality that we have obtained, the first term on the left-hand side that you see is independent of the natural number k. And the term that you see on the right most expression that is dependent on k. But here this A_k's increased to Ω as k increases to ∞. So, what we are going to do is to look at the limit in this relation star.

So, look at the first quantity on the left-hand side that is independent of our value k. So, therefore, you can just write this same value. So, I have just changed the indexing from n to k this is not a big change. So, therefore, this quantity of whatever this is, this is not dependent on k after you take the limit.

But what happens on the right-hand side? So, the right-hand side here will give you these limits here. So, you are taking limits in k for the functions h_k in between and then for the functions s here. So, that is all, you are writing down from the relation start.

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$$\lim_{k\to\infty} \int h_k d\mu \geq \lim_{k\to\infty} \int h_k 1_{A_k} d\mu$$

$$\geq \lim_{k\to\infty} \alpha \int s 1_{A_k} d\mu$$

$$= \alpha \int s d\mu \quad \text{---} (**)$$

The last equality follows from Note (15) of week 6. Now let $\alpha \uparrow 1$ in $(**)$.

Observe here, that the function s is kept fixed here only variation in k appears to the set A_k, and A_k is increased to the whole set Ω. Now, recall the fact that for non-negative measurable functions, and in particular, for non-negative simple functions the integrals work sets give you a measure. And in particular, this measure also is continuous from below. So, therefore, as soon as the integrals here that are being considered over the sets A_k these will increase to the whole integral, because A_k is increased to the whole set Ω.

So, therefore, here, you have $\int s\, 1_{A_k}\, d\mu$ and therefore, they will increase to $\int s\, d\mu$ here. And the scalar α just keeps intact here. So, therefore, you get the relation double star. This has been discussed in note 15 of week 6, about the continuity from below of this measure.

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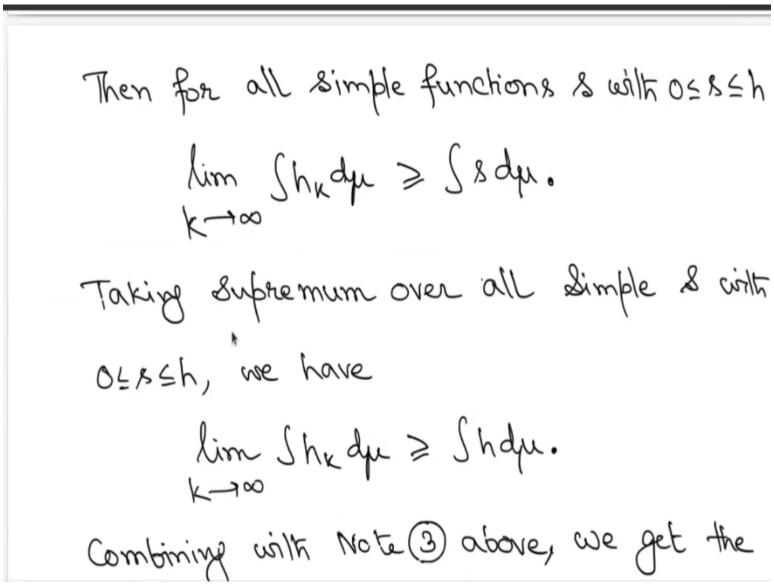

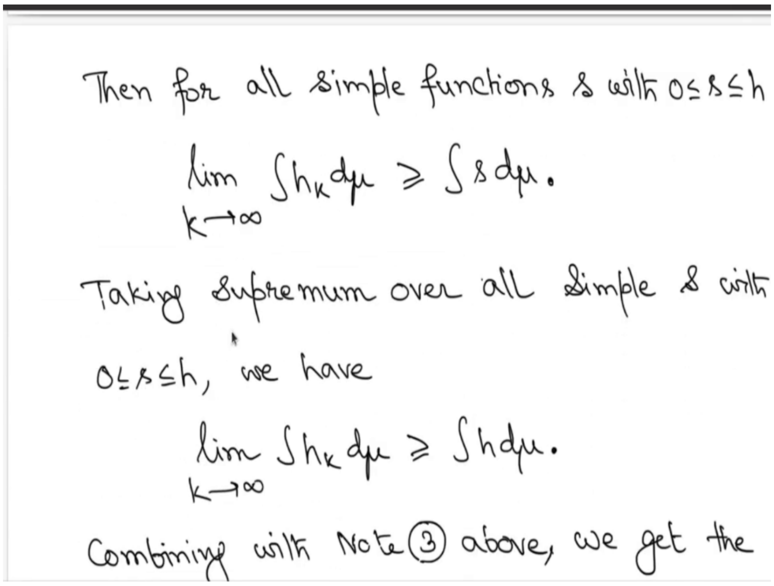

Now, observe that in double star, the left-hand side term is independent of α, and the right-hand side α is getting multiplied with the integral. So, here, the choice of α was between 0 and 1. If you will, now, let α increase to 1 in double star what do you get, you get that $\lim \int h_k \, d\mu$ is lower bounded by $\int s \, d\mu$, but this is true for all simple functions s, such that, $0 \leq s \leq h$.

Now, what you can do, is that, you can choose to take supremum over the right-hand side over all such simple functions. And therefore, you are immediately going to get the inequality

$$\lim_{k\to\infty} \int h_k \, d\mu \geq \int h \, d\mu.$$

But remember, that in note 3, we had mentioned earlier, that h_k's are dominated from above by the function h, and therefore, $\int h_k \, d\mu$, and therefore, the $\lim_{k\to\infty} \int h_k \, d\mu$ are dominated from above by $\int h \, d\mu$.

So therefore, we already had the other-sided inequality. Now, put it together with the inequality that we have just obtained, and we are going to get the required equality as stated in the modern convergence theorem. So therefore, we have proved the monotone convergence theorem. Now as an important consequence of the modern convergence theorem, in week 6, we had mentioned that linearity of measure-theoretic integration is true, and we had used that in our arguments. But now, it is a good time to see, the proof of the linearity or the additivity properties of measure theoretic integration.

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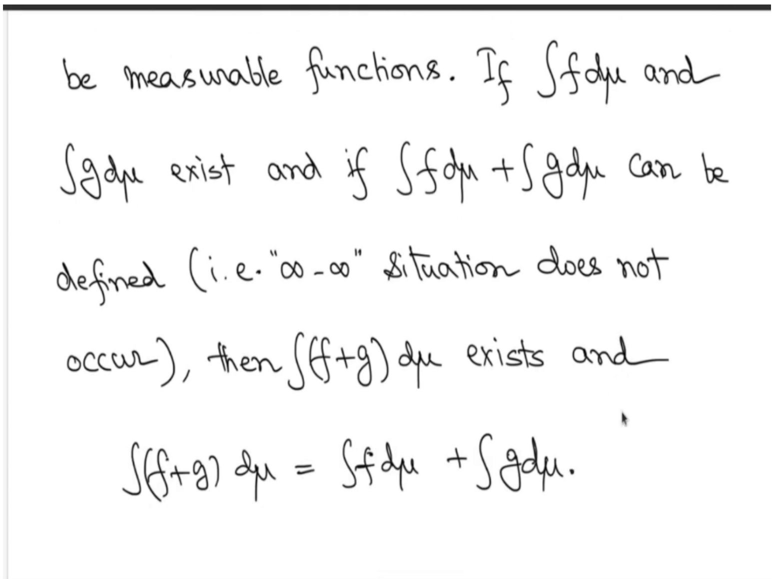

So, let us quickly recall the statement that you are considering two real valued measurable functions given to you. And if it so, happens that $\int f\, d\mu$ exists, $\int g\, d\mu$ also exists and then you also can make sense of $\int f + \int g$, then, the statement says $\int (f + g)\, d\mu$ exists and it is equal to the addition of the integrals $\int f d\mu + \int g\, d\mu$. Here, the $\int f\, d\mu$, addition with $\int g\, d\mu$ is defined only if $\infty - \infty$ does not occur. So, that was what we had discussed earlier.

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Proof: First, let f and g be non-negative and simple. Then $f+g$ is also non-negative and simple. The results follows from the definition of the integration.

Now, let f and g be non-negative and measurable. By Theorem 4 of

Let us see, how do we prove this. First, consider a very simple situation that f and g are non-negative and simple. In that case, we had already mentioned that $f + g$ is also non-negative and simple. Therefore, what you can now try to verify, is that, the relation holds and that will follow directly from the definition. This you can take it as a small exercise, but it is just following from the definition of the integration, as we had considered in the previous week. So, that takes care of the simple situation.

But then let us move ahead and try to consider the case when both m and g are non-negative and measurable. But recall, we had discussed earlier in week 3 that we can find sequences fn and gn of non-negative simple functions, which approximates the functions f and g from below. So, given any non-negative measurable functions, you can figure out such simple functions which will approximate the given functions from below.

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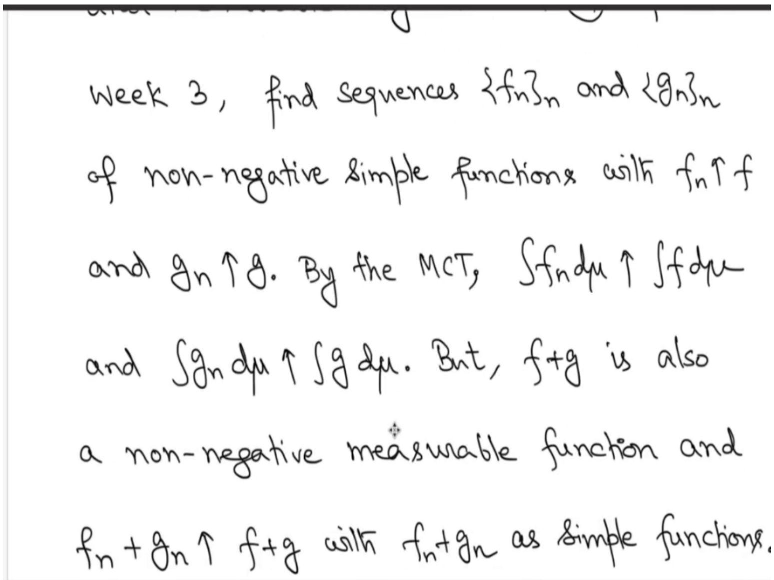

Here you are going to apply the modern convergence theorem. You will immediately say that

$\int f_n d\mu \uparrow \int f d\mu$ and $\int g_n d\mu \uparrow \int g d\mu$, so that is good. But observe that $f + g$ is also a non-negative measurable function, and $f_n + g_n$ are simple functions, which point wise increase to the function $f + g$ right. So, this is simply following the definitions of a f_n's and g_n's.

And therefore, you apply the monotone convergence theorem to these approximation now, you will immediately say that $f_n + g_n$. If you consider the integrations and then consider the limits of that, that will be exactly the $\int (f + g)\, d\mu$. So therefore, you get this equality.

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$f_n + g_n \uparrow f+g$ with f_n, g_n as simple functions.

By the MCT,

$$\int (f+g)\,d\mu = \lim_{n\to\infty} \int (f_n+g_n)\,d\mu$$

$$= \lim_{n\to\infty} \left[\int f_n\,d\mu + \int g_n\,d\mu\right]$$

$$= \int f\,d\mu + \int g\,d\mu.$$

If f and g are $\mathbb{R}$-valued measurable

$\int (f+g)^+\,d\mu$ and $\int (f+g)^-\,d\mu$. In general, $(f+g)^+ \neq f^+ + g^+$. The explicit argument is being skipped to reduce technicalities in the notes. The interested reader may refer to Theorem 1.6.3 from "Probability and Measure Theory" by in the notes. The interested reader may refer to Theorem 1.6.3 from "Probability and Measure Theory" by Robert B. Ash and Catherine A. Doléans-Dade, Second edition, Academic Press.

we now state an extended

But then you had already mentioned that you can split the integration, you can write it as an addition of the two individuals integrals when the functions are simple and non-negative, which is the case here. So, therefore, you split the integration here, as $\int f_n \, d\mu$ and $\int g_n \, d\mu$, therefore, you do this. And individually these terms approximate $\int f \, d\mu$ and $\int g \, d\mu$, just write that, and therefore, you get the required relation.

So, you have now proved it up to the case when both f and g are non-negative, and measurable. But the general case it will be slightly more complicated. You will need to look at the positive part of $f + g$, and negative part of $f + g$. But in general, there is a technical complication which is the fact that the positive part of $f + g$ is not necessarily equal to $f^+ + g^+$, it is not really the addition of the two positive parts.

So, what is going to happen is that we will have to be carefully do the accounting for the positive parts and negative parts and get the required equalities. What we are going to do is to skip the explicit arguments, we want to reduce the technicalities. We are going to assume that this can go through for measurable functions, which are now taking signs, but here you are assuming that the functions has integrals which exist.

So, for these cases, you can do this approximations and get the result. If you are interested, please refer to theorem 1.6.3 from this book by Robert Ash and Catherine A. Doleans from this book, titled, Probability And Measure Theory. So, this will be the second edition of the book published by Academic Press.

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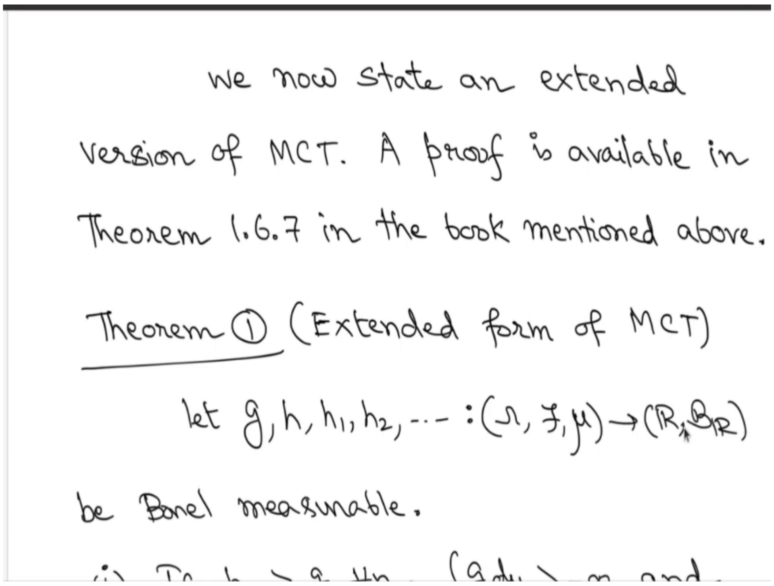

So, that takes care of the linearity or additivity of the integration, which was a proof remaining from the previous week. So, now, what we are going to do is to consider extended version of the monotone convergence theorem, which is also quite useful in practice. Again, we are going to skip the proof and we are just going to state this result, and we are going to use this. So, this is being stated without proof.

If you are interested you might also going to look at the book mentioned above, the book by Robert Ash and Catherine A. Doleans Dade, and you have to look at the theorem 1.6.7. So, what is this extended form? So, in the MCT monotone convergence theorem, we looked at the functions, which are non-negative So, the functions already were bounded below.

Here, what we are going to consider are some other function as a lower bound, and therefore, you will have to put some appropriate conditions on that lower bound function. So, you are going to consider this sequence of functions as in the previous case so, you will take h_1, h_2 and as shown that will give you the sequence, you are going to consider the point wise limit as h as considered earlier, but now, as a bound you are going to consider the function g.

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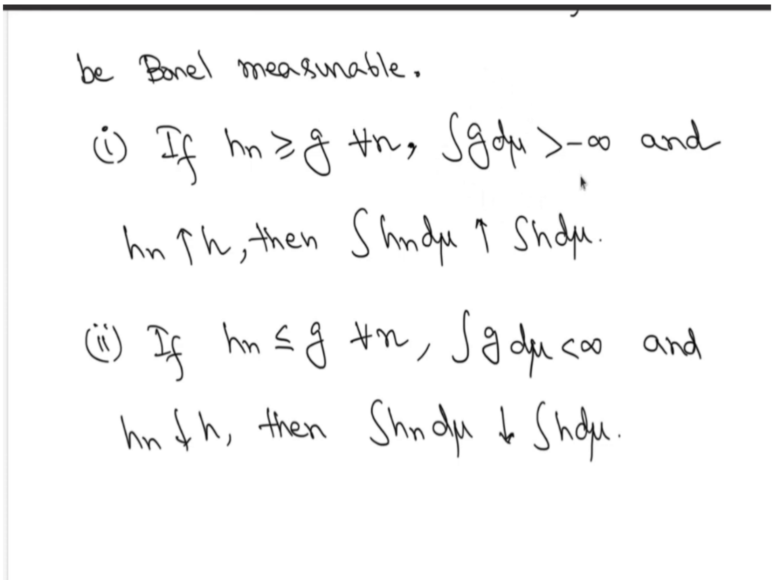

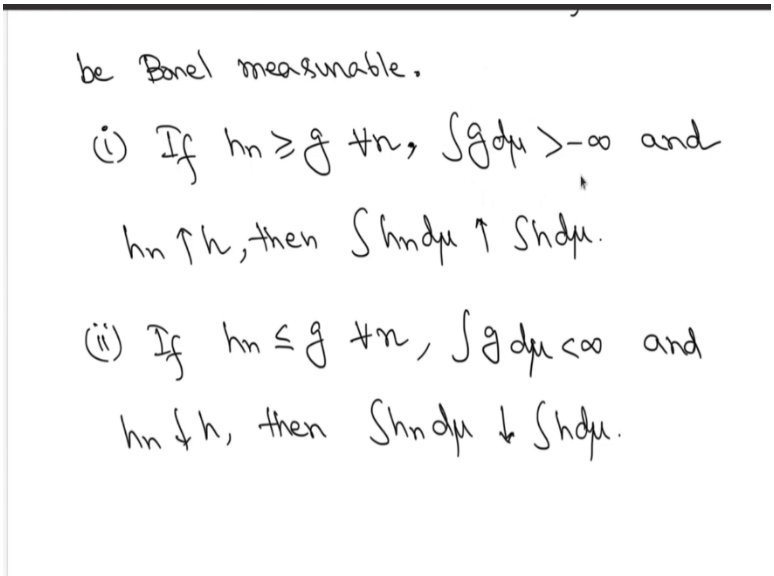
be Borel measurable.
(i) If $h_n \geq g$ $\forall n$, $\int g \, d\mu > -\infty$ and
$h_n \uparrow h$, then $\int h_n \, d\mu \uparrow \int h \, d\mu$.
(ii) If $h_n \leq g$ $\forall n$, $\int g \, d\mu < \infty$ and
$h_n \downarrow h$, then $\int h_n \, d\mu \downarrow \int h \, d\mu$.

So, there are two statements, here. So, if you have that h_n's are lower bounded by this function g, and if $\int g \, d\mu$ exists in the sense that it is greater than $-\infty$, so, you are allowing $\int g \, d\mu$ to exist and you need the integral value to be not equal to $-\infty$. So, you are allowing $+\infty$ as a possible.

So, in this case, if h_n's increase to the function h, so, it is approximating the function h from below then the integrals $\int h_n \, d\mu$ will increase and increase to the $\int h \, d\mu$, so that is the first statement. And the other statement, which you might expect is the exact opposite relation, is the case, when the functions are decreasing.

Here you will require an upper bound, again, we are writing it in terms of the function g. So, if all the functions are upper-bounded by this function g, and if $\int g \, d\mu$ is avoiding the value ∞, so, here $\int g \, d\mu$ can take the value $-\infty$, but you are not letting it be equal to $+\infty$.

In this case, if the functions h_n decrease to the function h then you are going to get that the $\int h_n \, d\mu$ will decrease, and is going to go to this value, which is $\int h \, d\mu$. So, we have now discussed the proof of modern convergence theorem, the additivity properties of the integrals, and we have finally seen the extended form of MCT where you can use instead of the 0 function you can use other functions as possible lower or upper bounds for the cases of increasing or decreasing sequences of functions. We are going to continue this discussion in the next lecture.