Measure Theoretic Probability 1

Professor. Suprio Bhar
Department of Mathematics & Statistics
Indian Institute of Technology, Kanpur

Lecture No. 03

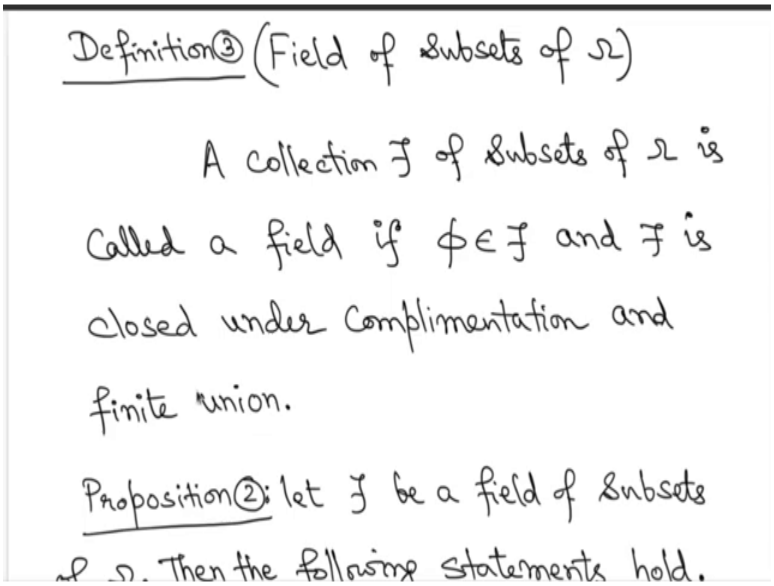
Fields and Generating sets for sigma-fields

Welcome to this lecture. In the previous lecture, we discussed random experiments and the events that occur in a random experiment. More generally, we looked at the structure of these events that appear and formulated these collections of events in terms of sigma fields. We had seen many examples of sigma fields and used this bottom-up approach to construct examples.

In this lecture, we will continue the discussions on the structures on top of the collections of events. So, more generally, as we have already talked about, we will take a non-empty set Ω for a random experiment that will be our sample space which is the collection of all outcomes. But in general, we will talk about a non-empty set and some collections of subsets of it.

We will now look at other structures that appear on top of these special collections of subsets. So, as before, I will move on to the slides that have already been prepared. So, I start on these discussions on the slides.

(Refer Slide Time: 1:33)

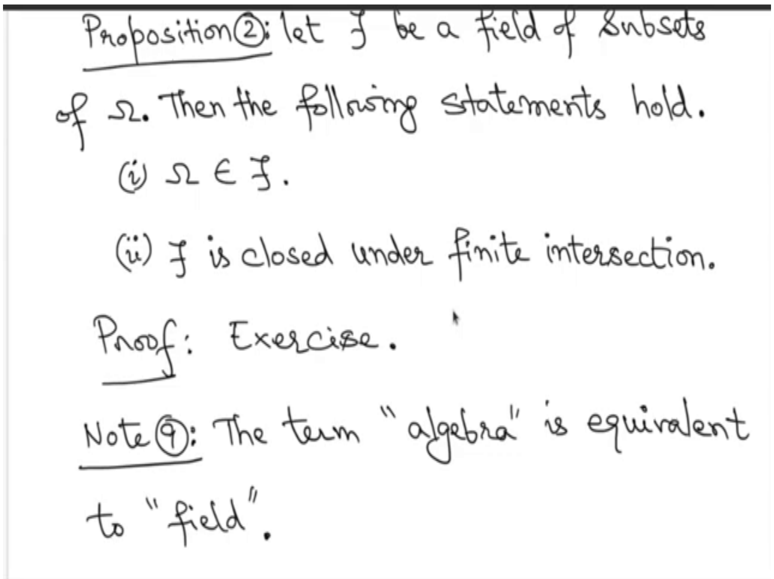


So, this is the definition.

Definition 3: A collection $\mathcal{F}$ of subsets of a non-empty set Ω is called a field if $\phi \in \mathcal{F}$, and $\mathcal{F}$ is closed under complementation as well as the finite union.

We already had an empty set in the collection to compare with the structure of sigma fields that we have already seen. Still, we had closure under complementation as well as countable unions.

(Refer Slide Time: 2:08)



But now, we are talking about closure under complementation. But we are dropping the countable operations on the union; we are only taking finite unions. So, as before for sigma fields, we will now look at structures or properties on top of fields. So, what are these structures, or what are these properties?

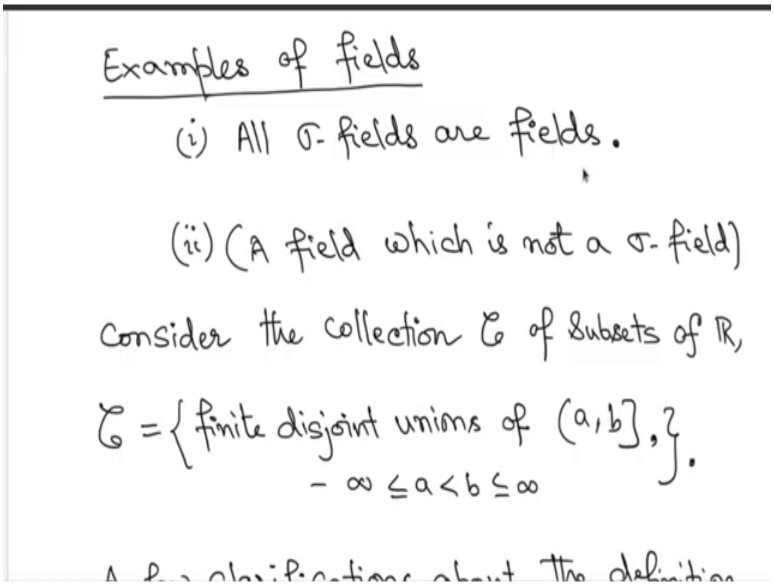
Proposition 2: If you take a field of subsets of Ω, then the following statements hold. Again, these arguments are similar to whatever you have seen for sigma fields.

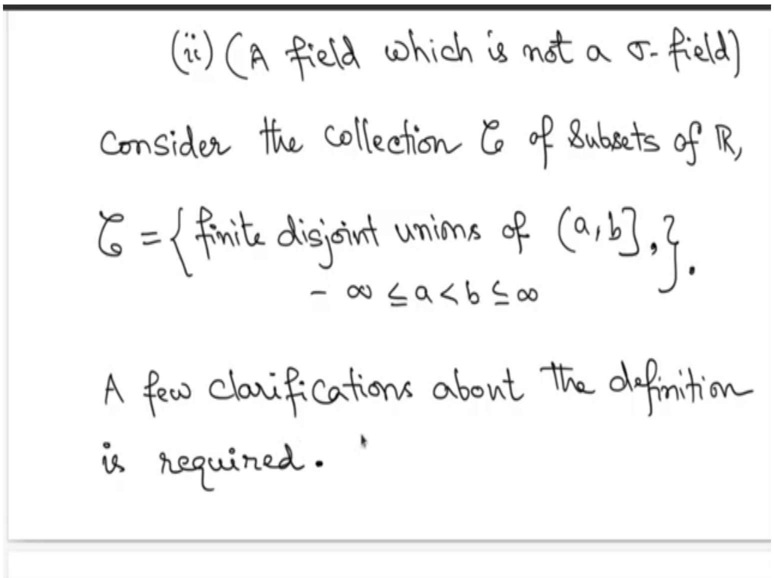
(i) So, the first property listed here is that the whole set must be in the list must be in the field. So, again, the idea is that you use the fact that the field is closed under complementation and the whole set Ω is nothing but the complement of the null set. The null set is already there, so, therefore, its complement must be there.

(ii) The second property that we have seen similar to the sigma field is that $\mathcal{F}$ closed under finite intersections. So again, as before, you have to use the operations of complementation and finite unions to prove this property. And the proof is extremely simple once more and is left as exercise again; it is similar to whatever you have done for sigma fields.

An important note is that some authors prefer to use the term algebra, and that is in our notion, algebra is the same as our field. So, the main idea is that we had the similar term sigma-algebra; now, for fields, we use the similar term algebra.

(Refer Slide Time: 3:48)





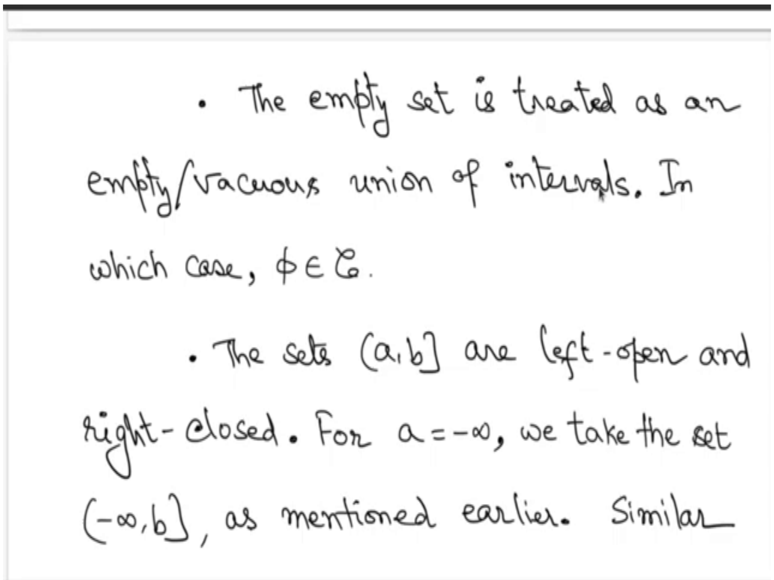
So, you would now ask what are now standard examples,

Examples:

(i) The standard example, first of all, turns out to be sigma fields. So remember, all sigma fields that you get are closed under complementation as well as countable unions. But then you would like to verify whether this is closed under finite union also. And that is easy to check because you can still take finitely many sets from the collection, let us say $A_1, A_2, \ldots, A_n$, you want to show that finite union $\cup_{i=1}^{n} A_i$ is there, but you can take this finite union $\cup_{i=1}^{n} A_i$ as the union $\cup_{i=1}^{n} A_i \ \cup \cup \phi$. Put empty sets and treat it as a sequence of empty sets throughout after the $n+1$ instead. And then what you will end up having is that the union must be there in the sigma field, which simply boils down to the finite union, this property we had discussed as a part of properties over sigma fields. So, therefore, sigma fields are closed under finite unions, and hence, you would satisfy the requirements of the fields. And there are, of course, closure under complementation is already there, ϕ belonging to the sigma field is already there as part of the definition of a sigma field. So, therefore, sigma fields are special structures that are automatically becoming a field.

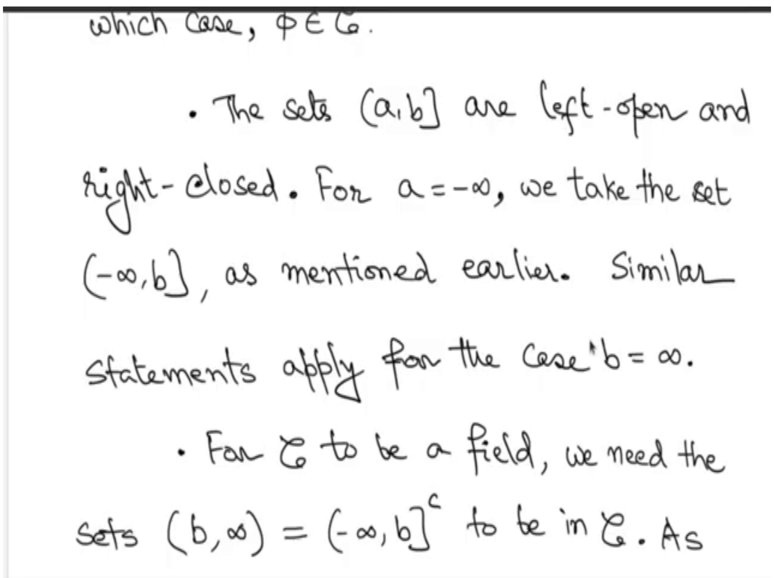
(ii) Now, an important example, that not all fields are sigma fields. So fields are somewhat less restricted. So, you would expect to have more examples. So, here is the first example of a field, which is not a sigma field. So, what we do, we first look at some specific collection of subsets of $\mathbb{R}$, so we denote it by $\mathcal{C}$. So, again $\mathcal{C}$ is not any standard notation just for the example here we are discussing this. So, let us look at intervals of this type $(a, b]$. So, on the left side, it is open; on the right-hand side, it is closed. What are a and b? a and b run between $-\infty$ to $+\infty$ but a < b. So, we are looking at such intervals, which will call us left open, right closed intervals. We would clarify what we mean when we put $a = -\infty$ or $b = +\infty$; we would clarify that. But let us say you understand what these left open, right closed intervals are, then what we consider are finite disjoint unions of such things. Meaning, if you have $(a_1 b_1], (a_2 b_2], \ldots, (a_n b_n], n$ many such things, let us say, then if they are pairwise disjoint take their union. So, that would be a typical element in this collection $\mathcal{C}$. So, let us say you have subsets $(a_1 b_1], (a_2 b_2], \ldots, (a_n b_n]$ left open, right closed, if they are pairwise disjoint then take their union whatever the set you get, that will be a set in our collection. Similarly, you can extend it to an end, many sets finitely many. So, let us now clarify what we mean by left open, right closed, and what we mean when $a = -\infty$ or $b = +\infty$. So, a few clarifications about this definition we are now going to discuss.

(Refer Slide Time: 7:00)



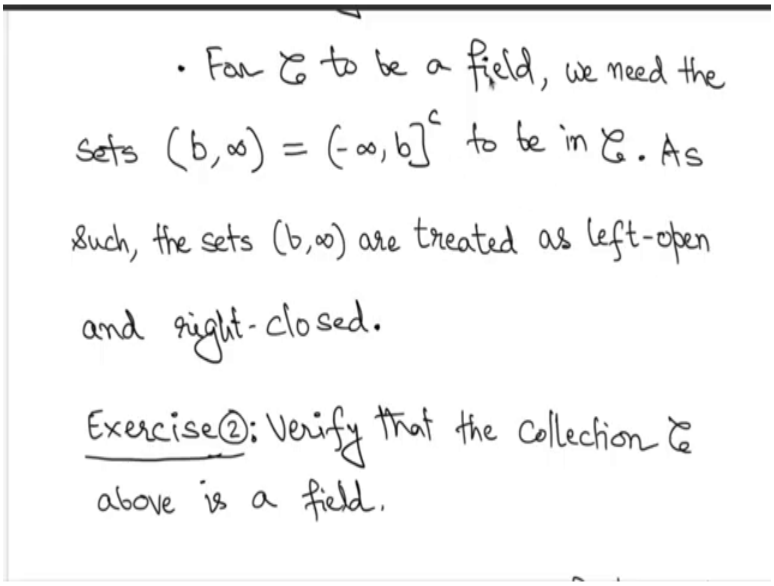
So, the clarification is like this. So, first of all, the empty set is treated as an empty or a vacuous union of intervals. So, the intervals that we are looking at are left open, right closed. We are saying that we are only going to consider finitely many unions of such sets. But for a finite union, we can also take an empty or a vacuous union. You do not take any sets, then automatically as part of this description, empty set belongs in this collection $\mathcal{C}$. So, whatever collections we will consider, we will allow empty set billing there, so this is the first thing.

(Refer Slide Time: 7:38)



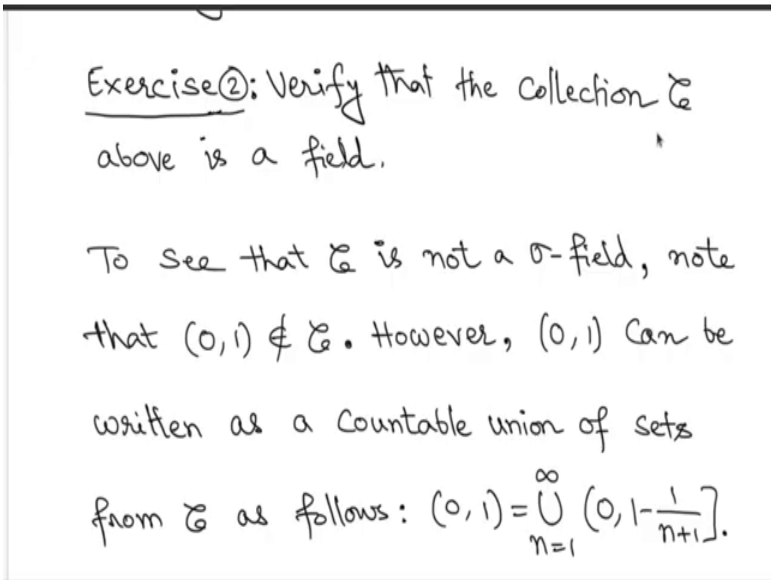
Next is the clarification about what happens for this left open right closed sets when $a=-\infty$ or $b = +\infty$. So, the idea is simple, suppose you are taking $a=-\infty$. Now observe that all these sets that you will consider $(a,b]$, must be subsets of the real line, and the real line does not have the points $-\infty$ or $+\infty$. So, therefore, whatever sets you are going to mean by $(-\infty, b]$ or whatever, you are going to mean that these are specific subsets. So for this case, when the left limit left boundary side, the a value is $-\infty$, we will take the set $(-\infty, b]$. So, that is easy to understand. So, for $(a, b]$, if $a = -\infty$., we simply look at the set $(-\infty, b]$. But then, similar statements would apply for the case $b = +\infty$. So, again, the idea will be that you are only going to consider (a, ∞). So we will not consider point $+\infty$ as part of this set because these sets are supposed to be within the real line, and the real line does not contain points + or $-\infty$. So, therefore, for left open, right closed subsets, we are going to consider such things.

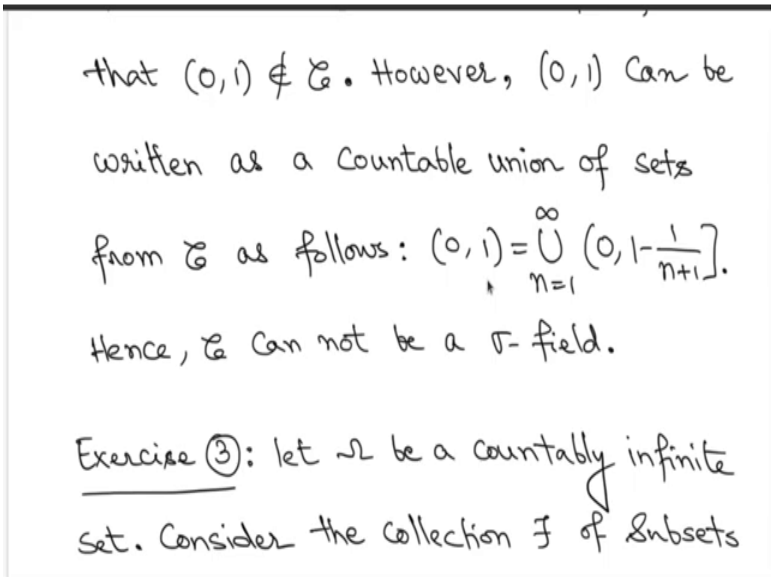
(Refer Slide Time: 9:13)



So, few more clarifications, why we want to do it this way is this because we want to make this C be a field, and for this, we want the closure under complementation. So, therefore, since we are aware that $(-\infty, b]$, that set is in the sets, that set is in the collection C, its complement must also be there. So, therefore, what we consider is that its complement is in there. Therefore we are asking that (b, ∞) is there. So, this complementation is being taken within the real line. So, we want these sets (b, ∞) to be inside the collection C. And for that to happen, we want these sets to be treated as left open and right closed intervals. So, this is the clarification. So, if your right boundary point is $+\infty$, in that case, what we incorporate is that the understanding is that we are going to write that specific interval with ∞ open, but that is still left open and right closed. So, all of this is happening within the real line. So, we are taking complementations within the real line.

(Refer Slide Time: 10:29)





Exercise 2: Now, what you can now try to verify is that with these structures and these notations at hand, this collection $\mathcal{C}$ above that we have described is a field.

So, we are going to consider this thing. So now, this is an easy verification. So, the setup for the complementation is already done. Still, you have to do many bits more verification because the collection $\mathcal{C}$ if we go back to the definition once more, the definition says you have to take finite disjoint unions of left open right closed intervals like this.

So then you will take finite many such left open right closed interval, you will look at their complementation So, how does it look like try to write it down you will immediately get the answer. Similarly, we have already described the empty set being in the collection script. The only thing left to verify is the finite union possibilities. Then you will also immediately get it because unions of left open right closed interval if you take unions of two such things, then either you get a finite disjoint union, or you simply get a left open right closed interval, just try to verify this.

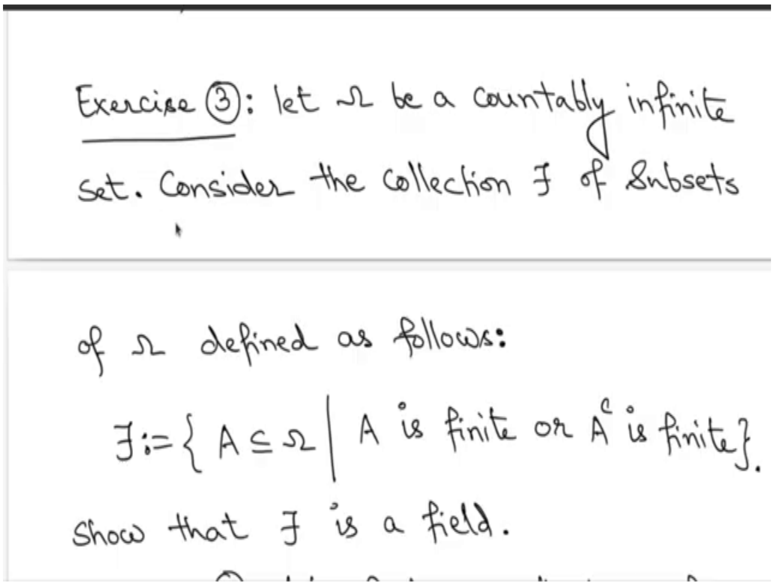
So, suppose you have$(a_1, b_1], (a_2, b_2]$. In that case, their union is either a disjoint union of these two intervals; otherwise, if they intersect, the union will simply become one big left open right closed interval. Try to check the calculation. So then, that is left as an exercise for you. Now, what we started this discussion about is that we want to claim that this collection $\mathcal{C}$ is a field but not a sigma field. So, therefore, this should be an example saying that fields are much more general collections than sigma fields.

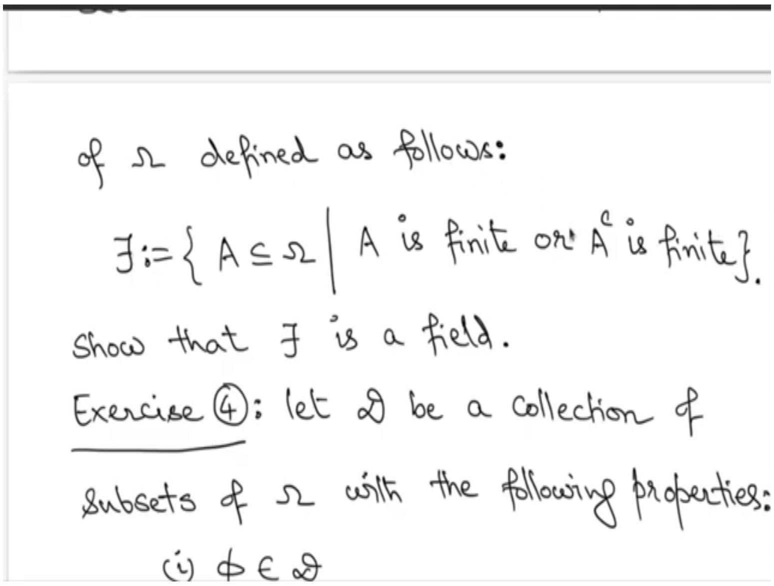
So, to see that $\mathcal{C}$ is not a sigma field, what do you observe if you look at this open interval, so now we are not considering left open right closed, but we are considering open intervals. So, what are these open intervals I will look at? It is this $(0,1)$. So, that interval I am going to look at. So, suppose I look at that interval. Now, what is happening is that since this is not right closed, this open interval is not in the collection $\mathcal{C}$. So, that is as per the description.

However, what you can do, you can still write $(0,1)$, that open interval as a countable union of sets coming from $\mathcal{C}$. How? So, you look at such intervals on the right-hand side, $(0,1 - \frac{1}{n+1})$ where n varies from 1 to ∞.

If you look at such left open right closed intervals, their countable union turns out to be the open interval $(0,1)$; try to check this. It is very easy to verify. But then each of these sets $(0,1 - \frac{1}{n+1})$. So, left open right closed, this is in the collection $\mathcal{C}$, but their countable union $(0,1)$ is not there in the $\mathcal{C}$. So, therefore, $\mathcal{C}$ is not closed under countable unions. So, therefore, it cannot be a sigma field. So, this gives an explicit example of a collection that is a field but not a sigma field.

(Refer Slide Time: 13:39)

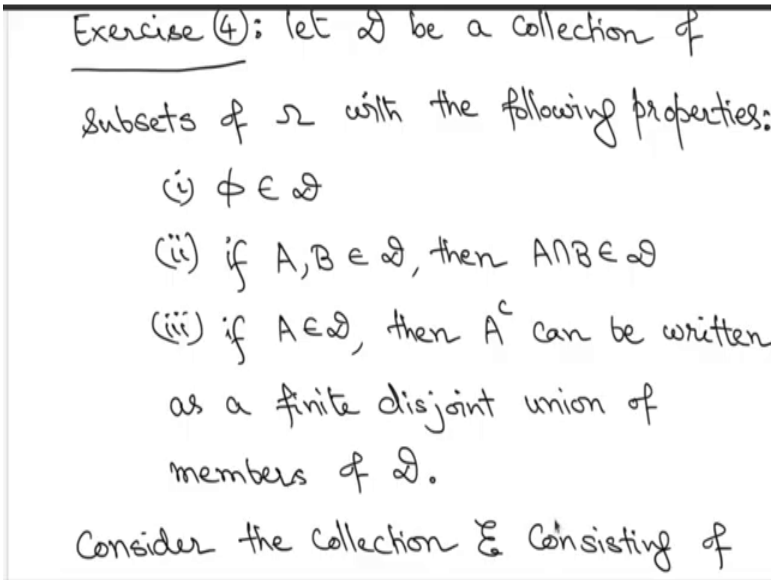




So, several exercises will help you understand these properties of a field much better. What are these exercises so? Please take your time working this out.

Exercise 3: If you take a countably infinite set, for example, it could be the set of integers or a set of natural numbers. So, for these subsets, consider the collection $\mathcal{F}$ of subsets we considered this way. So, you look at subsets of the original set Ω, but what are the subsets. So either A are finite, or its complement is finite. In either of these situations, if A is finite or its complement is finite, I will put A in the list, I will put A in this collection $\mathcal{F}$. If you take, $\mathcal{F}$ this way, you can now verify that it turns out to be a field. So try to work this out.

(Refer Slide Time: 14:33)



Now, here is some more structural properties involving fields.

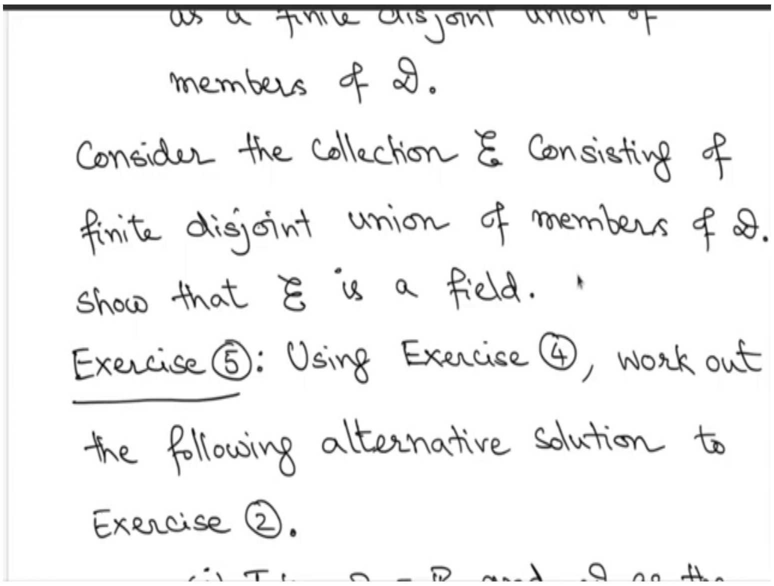
Exercise 4: Suppose you take a collection $\mathcal{D}$ of some subsets of non-empty sets Ω such that it has the following properties;

(i) the empty set is in this collection $\mathcal{D}$.

(ii) If A and B are two sets, then their intersection is also in $\mathcal{D}$.

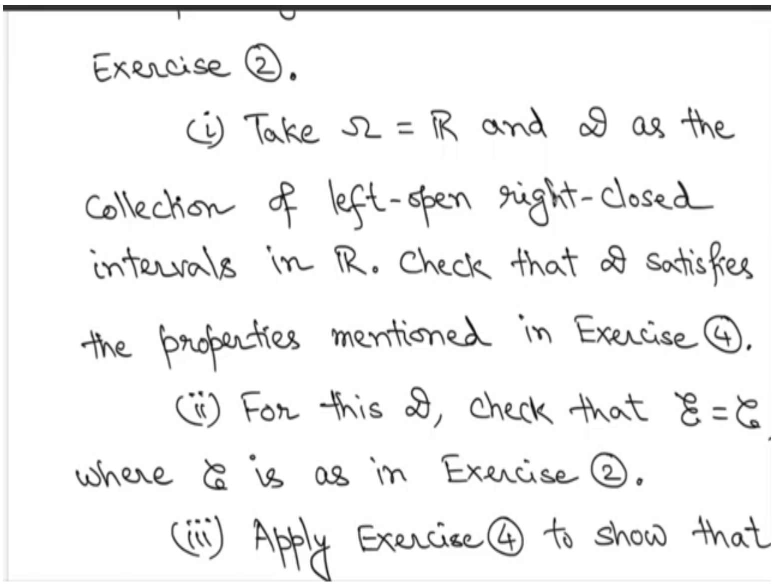
(iii) And the third property is interesting. It says that if you have a set in your collection, then the complement can be written as a finite disjoint union of members of $\mathcal{D}$. So, if A is there, you look at A^c, I am not claiming whether A^c is in the collection $\mathcal{D}$ or not, but I would be able to write it as a finite disjoint union of members of $\mathcal{D}$. So, there will be some sets, let us say $D_1, D_2, \dots, D_n$ whose disjoint union will turn out to be the A^c. So, for each A you can try to figure this out.

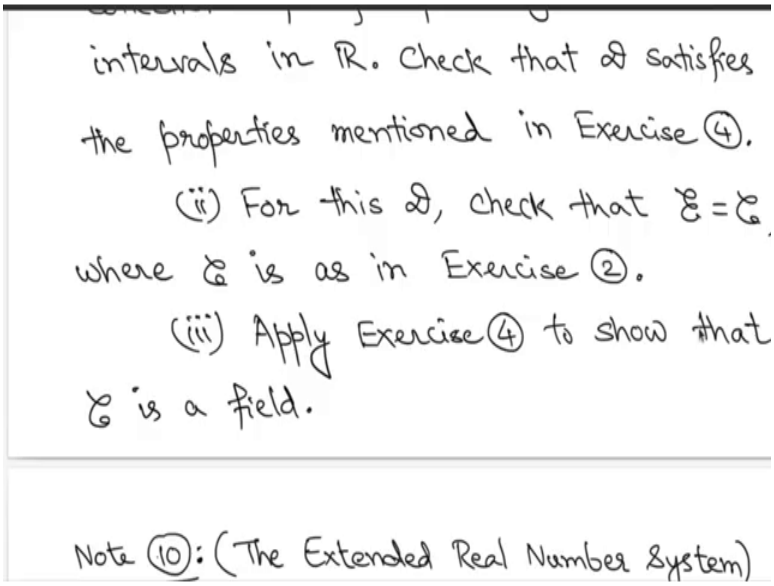
(Refer Slide Time: 15:36)



See, if you have these properties of $\mathcal{D}$ then what to consider is this collection $\mathcal{E}$ made out of properties of $\mathcal{D}$ this way that you now consider this finite disjoint union of members of $\mathcal{D}$. If you look at such sets, it so happens that $\mathcal{D}$ turns out to be a field. So, this is a general property that you can use to construct fields. But for this, you need a collection $\mathcal{D}$ satisfying these three properties. This may look abstract to you, but here is a more concrete example for you.

(Refer Slide Time: 16:06)





intervals in $\mathbb{R}$. Check that $\mathcal{D}$ satisfies the properties mentioned in Exercise (4).

(ii) For this $\mathcal{D}$, check that $\mathcal{E} = \mathcal{C}$, where $\mathcal{C}$ is as in Exercise (2).

(iii) Apply Exercise (4) to show that $\mathcal{C}$ is a field.

Note (10): (The Extended Real Number System)

So, we are going to work out exercise 5,

Exercise 5: We will discuss this exercise 5 using exercise 4. What you can do you can give the alternative solution to exercise 2.

Solution: What was exercise 2? Let us go back. So, in exercise 2, we had looked at this collection $\mathcal{C}$ of left open, right closed intervals and their finite disjoint unions. So, we said that that was a field on the real line.

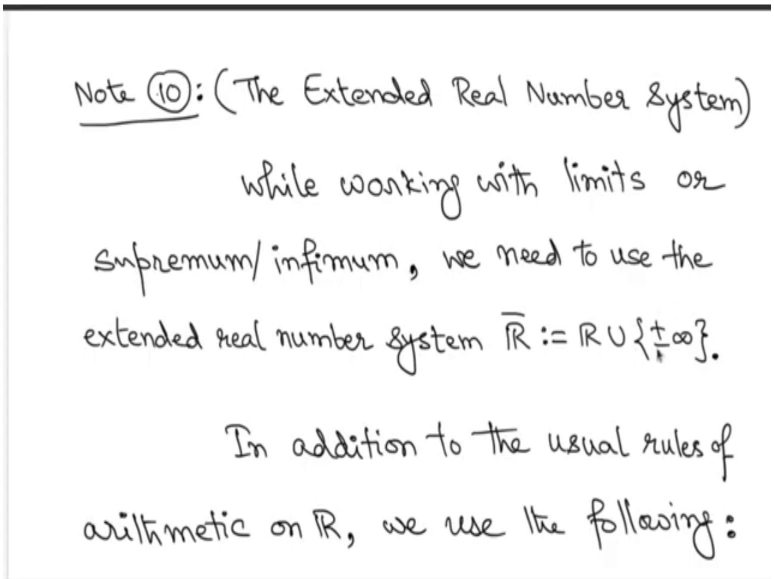
So, we will say this takes Ω the non-empty set to be the real line, and $\mathcal{D}$ as the collection of left open right closed interval in $\mathbb{R}$. You first try to check that $\mathcal{D}$ satisfies the properties mentioned above, meaning it should be closed under intersections and finite intersections and all these properties. Try to work this out.

Once you have a $\mathcal{D}$ for this specific case on the real line, check that this finite disjoint union some members of $\mathcal{E}$ exactly turns out to be the collection $\mathcal{C}$. And therefore, you apply exercise 4 to claim that since $\mathcal{E}$ in this specific example, of course, it must be a field. So, therefore, $\mathcal{C}$ which is equal to $\mathcal{E}$ in this case must be a field. So, this is an alternative way of proving that $\mathcal{C}$ is a field.

So, when you try to prove $\mathcal{C}$ is a field in exercise 2, we try to verify the actual properties of our field directly. Now, we are going via some actual construction behind in the background using some specific collection $\mathcal{D}$. So, that is what this exercise suggests.

But then, once you are looking at this collection $\mathcal{C}$, we understood that we had to understand some properties involving this $-\infty$ and $+\infty$, these points are not in the real line. So, we have to make sure that we understand this to do algebra of this.

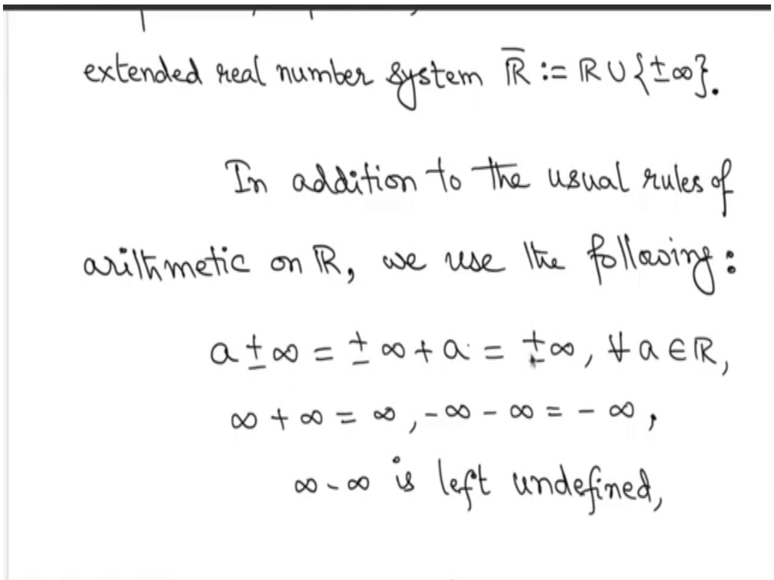
(Refer Slide Time: 17:59)

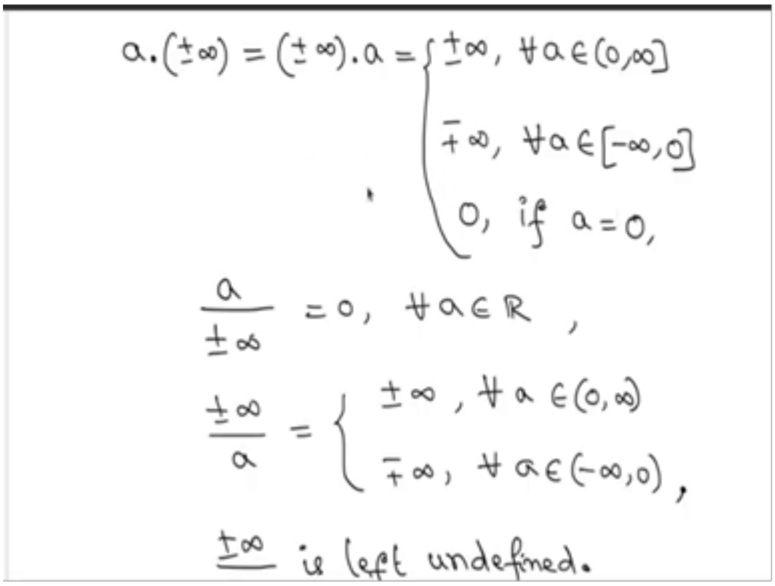


So, this is our following note on the extended real number system.

Note 10: This will appear when you work with limits or supremum, or infimum of sets. So, if you deal with limits of sequences or supremum or infimum of sets, all these operations sometimes will lead you to $-\infty$ and $+\infty$. But the problem is that you do not have these points $-\infty$ and $+\infty$ *on the real line*. So, what we do is that we use the extended real number system, which we write as $\overline{\mathbb{R}}$, which is nothing but the usual real line, and you do add these two extra points $-\infty$ or $+\infty$. So, now on this extended real number system $\overline{\mathbb{R}}$, there are still some nice operations that you can still do, standard algebraic operations.

(Refer Slide Time: 18:47)





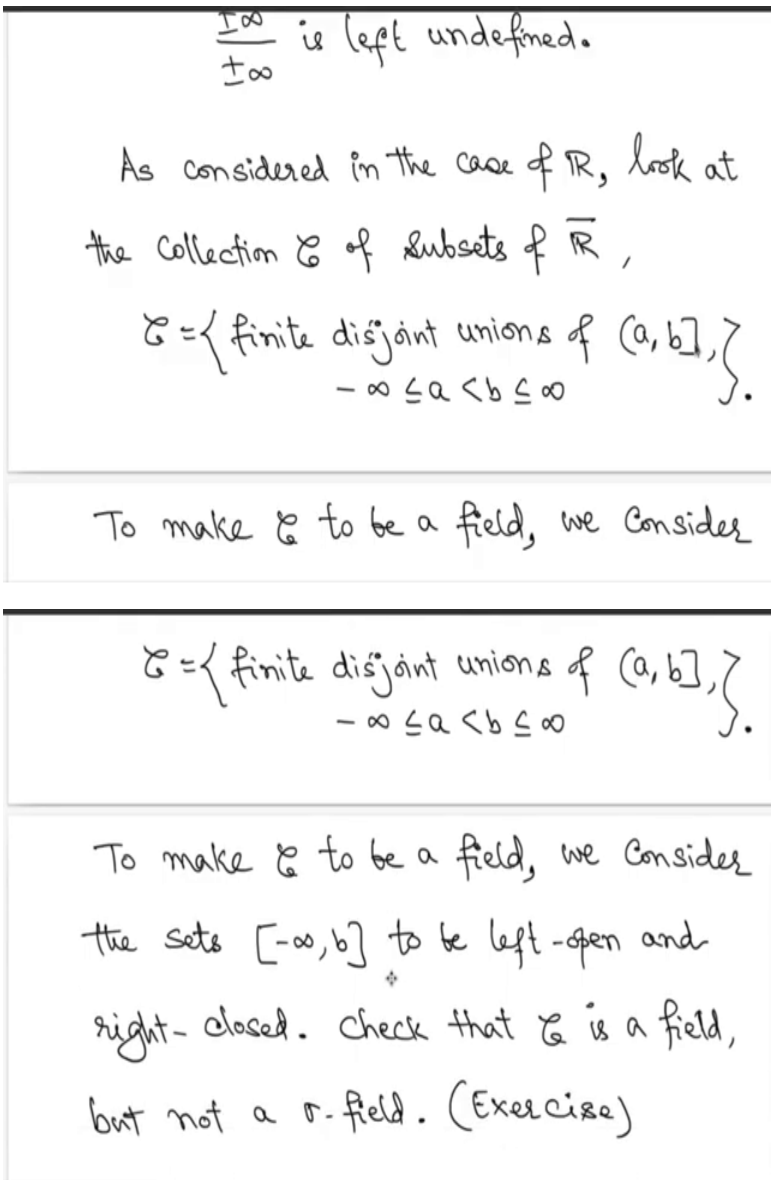
So what is this, so? We are looking at the rules of arithmetic on the extended real line. So on this, in addition to the usual arithmetic you already did on the real line, which you already know, you want to extend this to $-\infty$ or $+\infty$. So what you do is that if you add or subtract ∞ from a real number, you get $+\infty$ or $-\infty$. So that is what this plus-minus symbol represents.

It means that if you add $+\infty$, you will get $+\infty$. If you subtract ∞, you will get $-\infty$. Then the addition of two ∞ is nothing but ∞. And finally, you can also subtract $-\infty$ from $-\infty$ itself. So $-\infty$ and $-\infty$ that should represent $-\infty$. But it is an important point that $\infty - \infty$ is left undefined. So, these are the rules about addition and subtraction.

Now, this is about multiplication, that if you multiply a positive real number by $+\infty$ or $-\infty$, you $+\infty$ or $-\infty$. If you multiply your negative real number by $+\infty$ or $-\infty$, we end up exchanging the signs. So, if you multiply your negative real number with $+\infty$ you get $-\infty$ and so on. And the important thing is that if the real number a is 0, then this is the important convention that ∞ multiplied by 0, we will take it as 0, this is a very important convention.

Moreover, if you divide a number by + or $-\infty$, you get 0. And then finally, if you divide + or $-\infty$ by a real number, you, of course, get the expected results this way. And importantly, you do not divide ∞ by ∞, so these things are left undefined. So, again, two things are left as undefined, you do not subtract ∞ from ∞, and you do not divide ∞ by ∞.

(Refer Slide Time: 20:43)

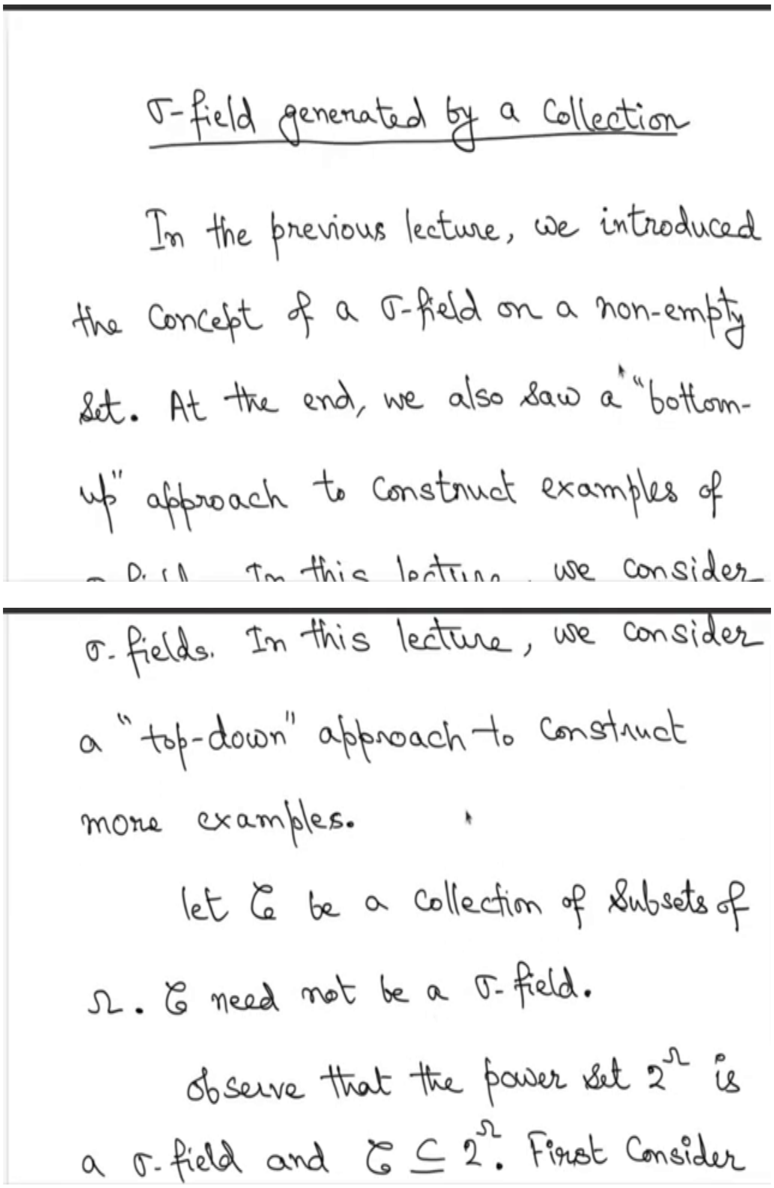


So, as considered in the real line, you can still look at the collection of subsets on the extended real line. You can still look at now left open right closed intervals, and you do not have to worry about whether these points a and b are equal to ∞ or not, because here you are now considering them as subsets of the extended real line, where $+\infty$ and $-\infty$ you are allowed to take.

So, if you allow this, then what happens is that you get these kinds of sets where a and b could be directly put as $-\infty$ or $+\infty$. So, if you look at subsets and the finite disjoint union of subsets on the extended real line, this will make it a field again. And then this is a clarification here, now $-\infty$ closed to be closed that should be taken as a left open right closed interval. So, these sets should also be included in the collection $\mathcal{C}$.

So, for all this setup, you should try to check that $\mathcal{C}$ must be a field, but it should not be a sigma field. So, this is a repeat of arguments that we had already done for the fields, but not a sigma fields case in the real line. With these ideas at hand, we are now going to consider going back to sigma fields, and this is a very important construction of the sigma field, general construction.

(Refer Slide Time: 22:10)



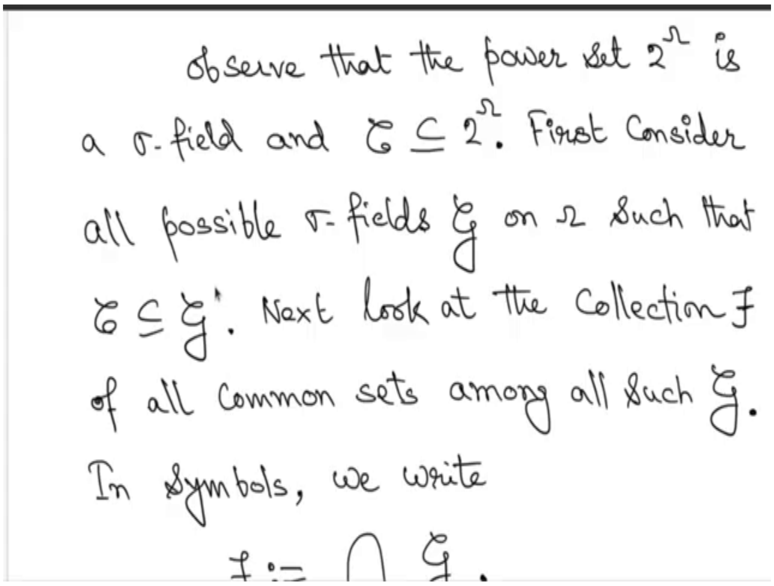
So, remember, we had already discussed in the previous lecture that we saw this bottom-up approach of constructing examples of sigma fields. But in this lecture, we will consider a top-down approach to construct more examples of sigma fields. So, what was this bottom-up approach? We had the trivial sigma field, the empty set, and the whole set. We added more sets and tried to make it a sigma field.

So, from the bottom, which is just a trivial sigma field, we added more sets to obtain bigger and bigger collections subsets, which turned out to be sigma fields. So, from the bottom to up, but now we are going from top to down. So, what do I mean by this? So, on the other end of the trivial sigma field, as we have discussed earlier, this is simply the power sigma, power sets sigma field. So, it consists of all possible subsets of the non-empty set Ω.

Now, if you look at the power set, and now so, that is on top of your hand, what we do is that we try to remove sets from the power sets collection. Remove sets do this way actually if you do it right, then what will happen you will get smaller and smaller collections, which in principle could give you examples of sigma fields. And we will now see some extract operations that will allow you to choose these subsets in a significant way.

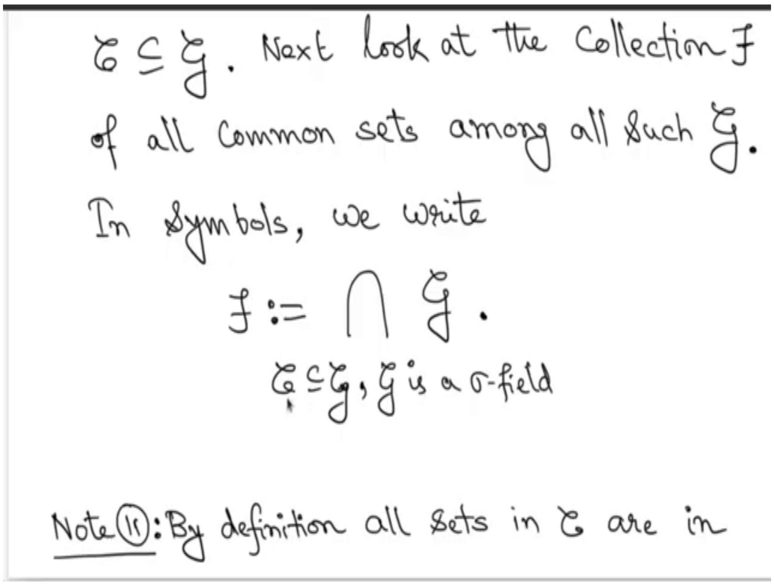
So, that this, once you choose them out in a certain way, if you cross them off from the list, from the power set list, if you cross them off, then it will turn out that you can get these smaller collections which also turn out to be sigma field. So, this is a top-down approach.

(Refer Slide Time: 23:51)



So, this idea is going through a certain parametric collection of subsets of this non-empty set Ω. What do you need to do? So, take a collection $\mathcal{C}$ of collections of subsets. So, this is just a collection of subsets. We do not assume any properties or any structure on $\mathcal{C}$. This is just a collection of subsets. Now, observe that the power set we denoted by 2^Ω, is a sigma field and contains this smaller collection $\mathcal{C}$. So, all the sets in the $\mathcal{C}$ are subsets of Ω. Therefore, it is contained in the big list, which is the power set.

(Refer Slide Time: 24:28)

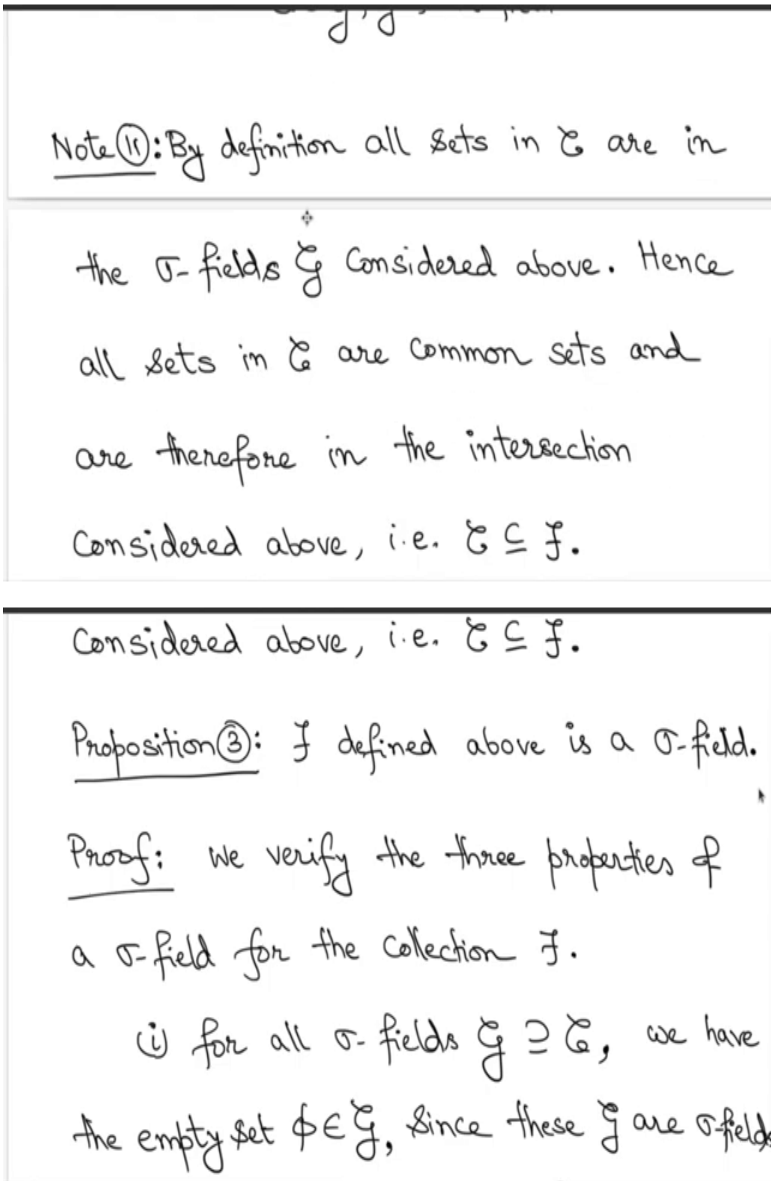


Now, consider all possible sigma fields $\mathcal{G}$, this is an arbitrary sigma on Ω such that our collection $\mathcal{C}$ is contained in $\mathcal{G}$. So, $\mathcal{G}$ are bigger lists that contain the list $\mathcal{C}$. So, these contain the collection $\mathcal{C}$. Of course, we already have such an example as the power set sigma field, but we would like to look at all possible such sigma fields, including the power set.

Now, what we do, we look at the collection $\mathcal{F}$ which we denote it this way, $\mathcal{F}$, but this should consist of all the common sets among also such $\mathcal{G}$, all such $\mathcal{G}$. So, what do we do? We first start with the collection $\mathcal{G}$, which contains our collection $\mathcal{C}$. $\mathcal{G}$ must be a sigma field. One example of this is the power set, but now what do we look at? We look at all the common sets among all such $\mathcal{G}$. So, we first list out all the sigma fields that contain our list $\mathcal{C}$.

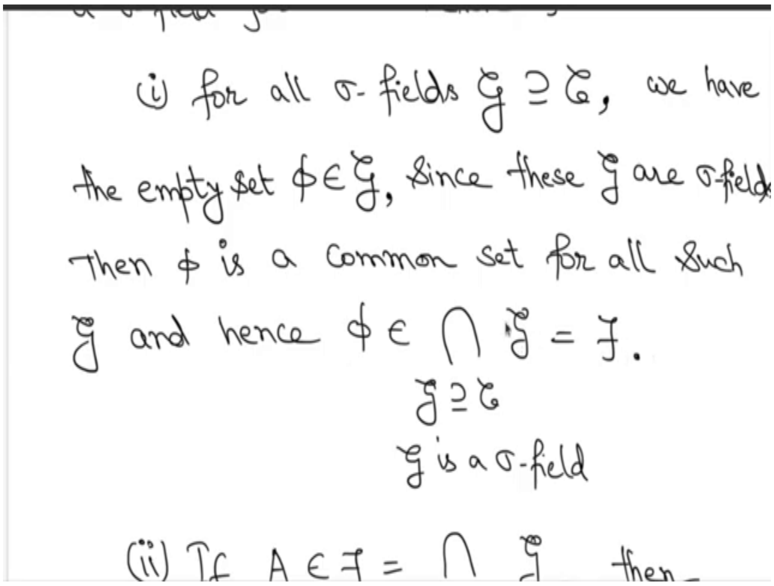
Look at all such $\mathcal{G}$, pick out the common sets in the lists. So, you might have two or three or more such lists of sigma fields. So, more such lists. So, look at all the common sets and create this collection $\mathcal{F}$. So, in symbols, we write this is the intersection of all these common sets, so intersections among all these $\mathcal{G}$. So,, what are the $\mathcal{G}$? These are must be sigma fields containing the $\mathcal{C}$. So, this is what the sigma field, this is what the $\mathcal{F}$ looks like.

(Refer Slide Time: 26:06)



By definition, all sets in $\mathcal{C}$ here are common sets. Because all sigma fields, all such sigma fields $\mathcal{G}$ by definition contain elements or the members of $\mathcal{C}$. So, therefore, this $\mathcal{F}$ also contains $\mathcal{C}$. So, therefore, all sets in the $\mathcal{C}$ our contained there, their common sets. So, therefore, they are in the intersection, so $\mathcal{C}$ by definition is contained in $\mathcal{F}$.

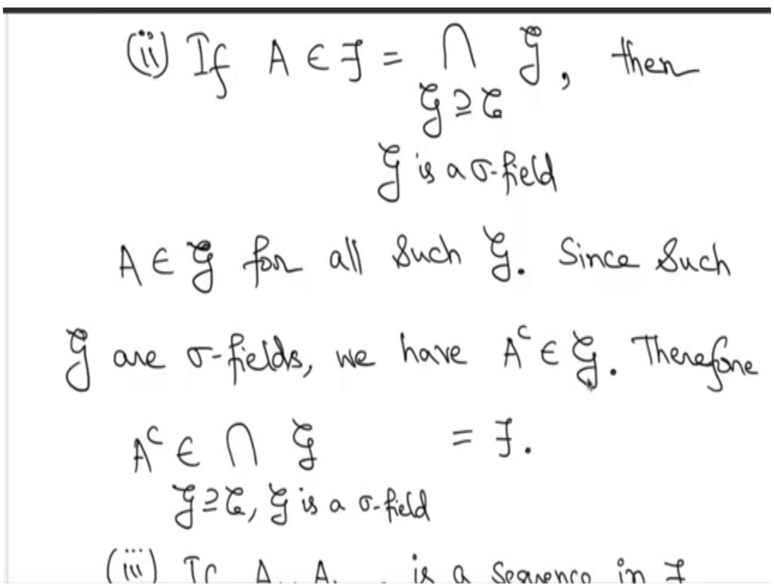
(Refer Slide Time: 26:33)



But then we now claim the $\mathcal{F}$ that thus defined is a sigma filter. How do you show this? So, we just verify the three properties of a sigma field for this collection $\mathcal{F}$. So, what do you need to show, first you need to show that the empty set is in this collection. For that, you need to show that an empty set is a common set among all the sigma fields. But since $\mathcal{G}$ are sigma fields themselves, they must contain an empty set in their list.

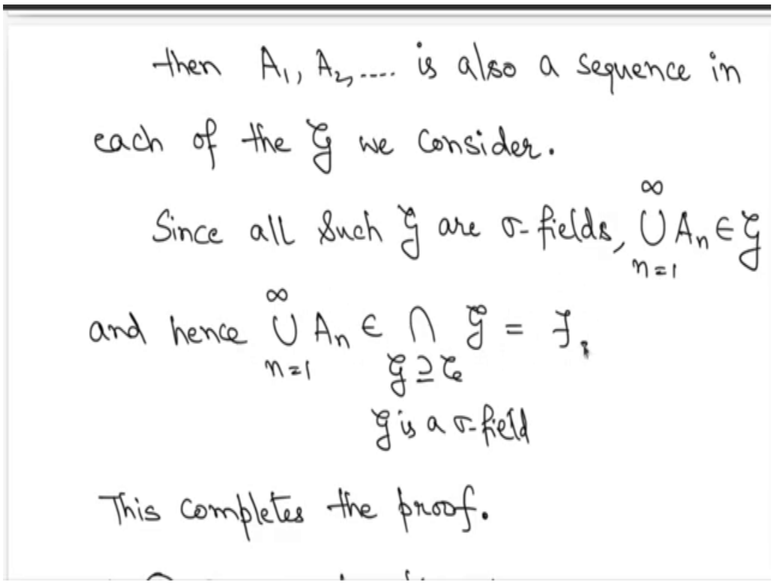
So, therefore, what will happen is that an empty set is in all of this $\mathcal{G}$ which are sigma fields containing $\mathcal{C}$., So, therefore, the empty set is a common set among all the sigma fields. Therefore it is in $\mathcal{F}$, that is by definition.

(Refer Slide Time: 27:16)



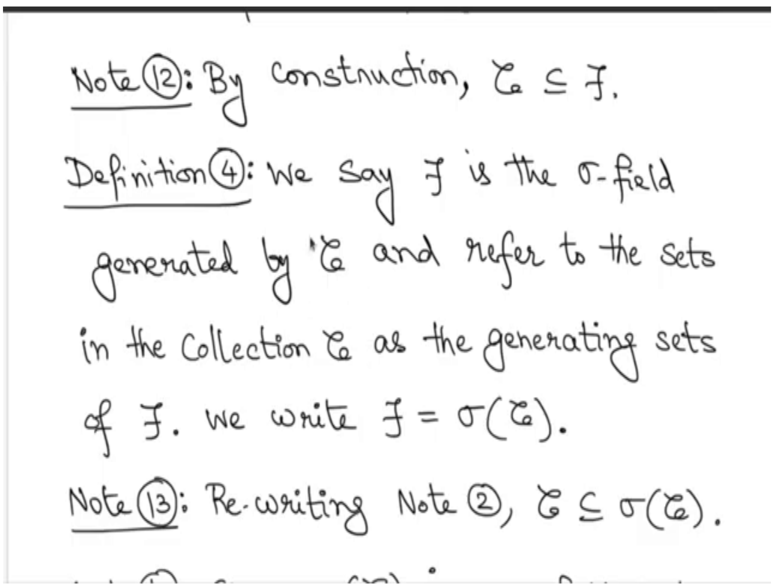
More generally, if you want to look at closure under complementation, you take a set in $\mathcal{F}$, but if A is a common set, then $A \in \mathcal{G}$ for all such sigma field $\mathcal{G}$. So, therefore, it is in each $\mathcal{G}$ is a sigma field. It is closed under complementation. Therefore, A^c is there in each of the $\mathcal{G}$. So, therefore, it is a common set. So, therefore, $A^c \in \mathcal{F}$.

(Refer Slide Time: 27:42)



A similar argument will show you that the union of these things countable union of such sets. Common sets must also be a common set in the collections. So, therefore, the countable union also must belong to $\mathcal{F}$. So, therefore, $\mathcal{F}$ thus defined is giving an example of a sigma field.

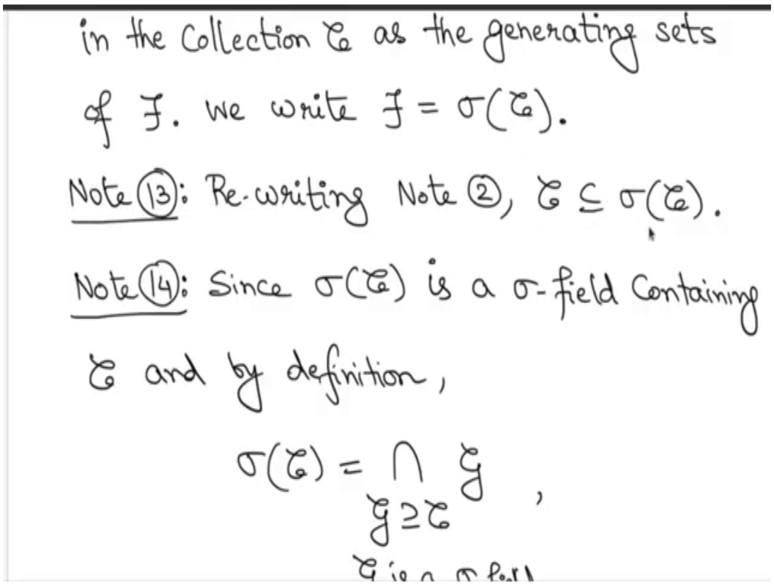
(Refer Slide Time: 28:04)



So by construction again, just to remind $\mathcal{C}$ is contained in $\mathcal{F}$. So, all these sets that you already started within $\mathcal{C}$, they must be in $\mathcal{F}$. Now, we will make the definition

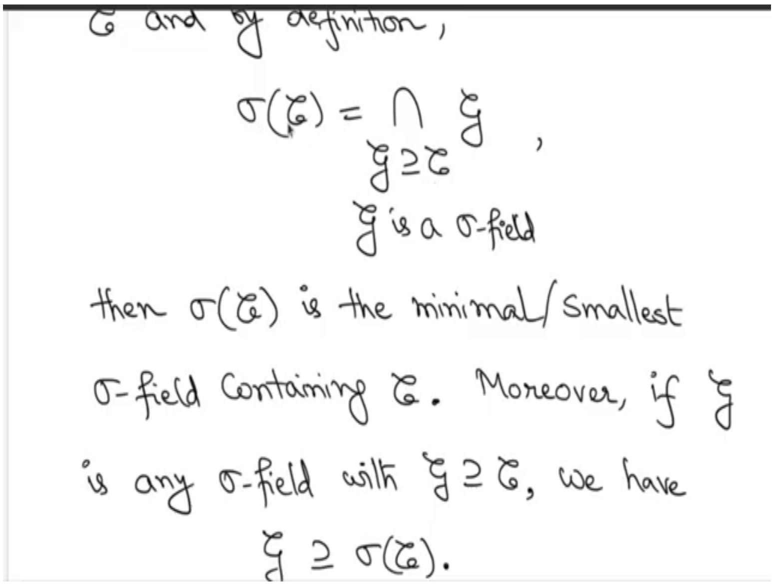
Definition 4: We say $\mathcal{F}$ is the sigma field generated by $\mathcal{C}$. And refer to the sets in the collection $\mathcal{C}$ as the generating sets of $\mathcal{F}$. So, you possibly get some extra sets that you started with. So, you had our list of sets in $\mathcal{C}$, but you added more sets by a certain way you are looking at certain common sets among all such common $\mathcal{G}$. So, that will give you the necessary extra sets to make $\mathcal{F}$ as the sigma field. So, as I said before, this sigma field will be called a sigma field generated by $\mathcal{C}$. And then elements or the members or the sets in $\mathcal{C}$ are said to be generating sets of $\mathcal{F}$.

(Refer Slide Time: 29:03)



And we will use this notation $\mathcal{F} = \sigma(\mathcal{C})$ to denote that $\mathcal{C}$ generate $\mathcal{F}$. But then remember that $\mathcal{C}$ is already contained in the sigma field $\mathcal{F}$. I am just rewriting this. We just observed that $\mathcal{C}$ is contained in the sigma field generated by $\mathcal{C}$.

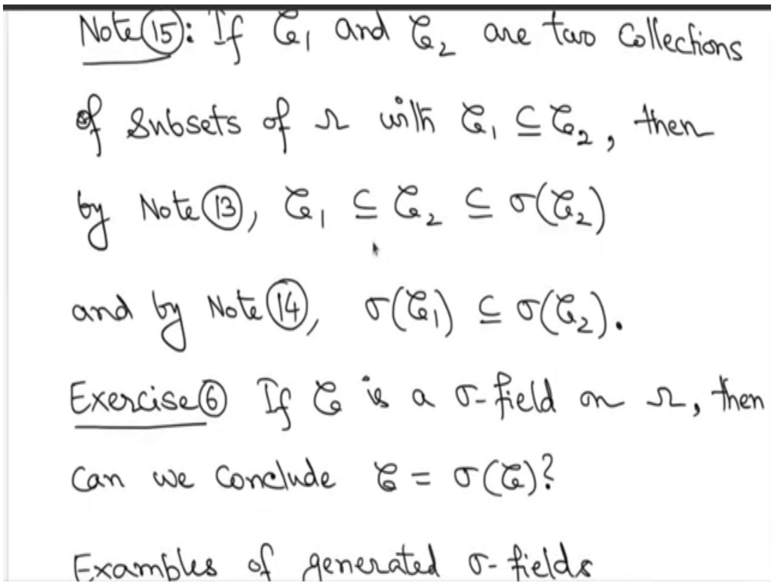
(Refer Slide Time: 29:30)



Now, the sigma field generated by $\mathcal{C}$ is a sigma field containing $\mathcal{C}$. So, therefore, what will happen is that the sigma field generated by $\mathcal{C}$ by definition is equal to this intersection. Therefore, the sigma field generated by $\mathcal{C}$ by definition is the minimal or the smallest sigma field containing $\mathcal{C}$. So, therefore, if you have any sigma field that contains $\mathcal{C}$, this sigma filters by $\mathcal{C}$, these are the common elements, and therefore sigma field generated by $\mathcal{C}$ is contained in any of the $\mathcal{G}$.

So, therefore, what you are saying is that if $\mathcal{G}$ is any sigma field such that $\mathcal{G}$ contains $\mathcal{C}$. So, therefore, $\mathcal{G}$ must contain the sigma field generated by $\mathcal{C}$. So, this is the minimal one containing $\mathcal{C}$. So, if you take any other sigma field, then $\mathcal{G}$ will contain the sigma field generated by $\mathcal{C}$.

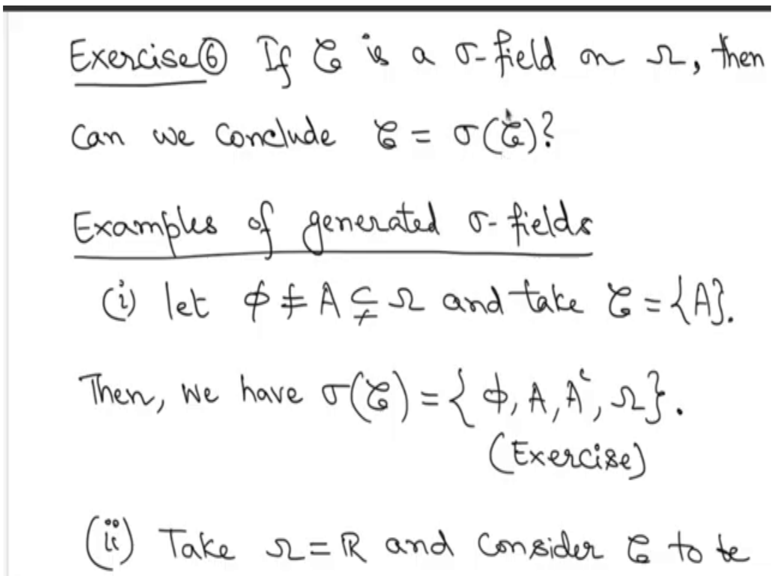
(Refer Slide Time: 30:20)



Now, we will look at the properties of generating sets.

Note 15: If you have two such collections with one collection contained in the other, you can easily show that sigma field generated by C_1 is contained in the sigma field generated by C. This is simply using the inclusions that we just discussed. This is just discussing the same properties that we have just mentioned. Take a minute to think about it; you will immediately get it.

(Refer Slide Time: 30:47)

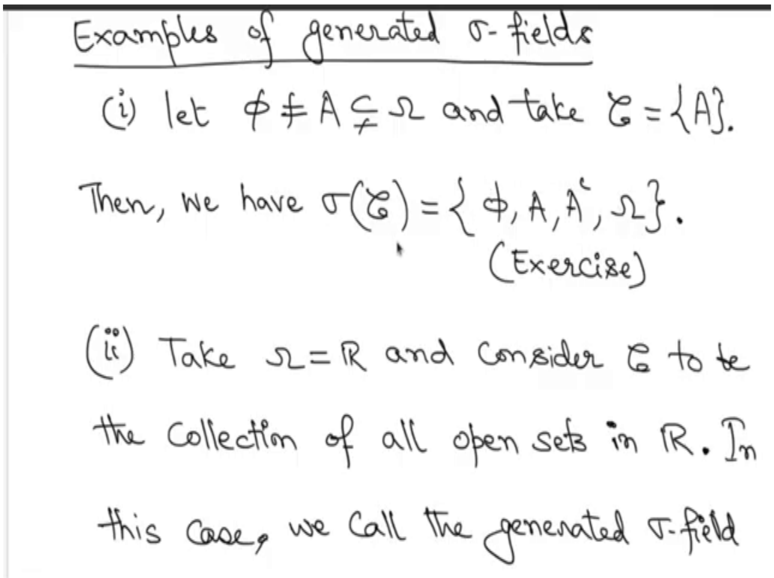


But then I leave this question to you,

Exercise 6: If $\mathcal{C}$ is a sigma field on Ω, can we conclude that $\mathcal{C}$ is equal to the sigma field generated by $\mathcal{C}$ itself?

So, originally $\mathcal{C}$ did not have any structure, but now, we are saying what about taking $\mathcal{C}$ and start with that try to generate sigma fields out of it. So, if you can generate this using the general technique we have just discussed, what happens? Does it equal the original $\mathcal{C}$ itself?

(Refer Slide Time: 31:21)

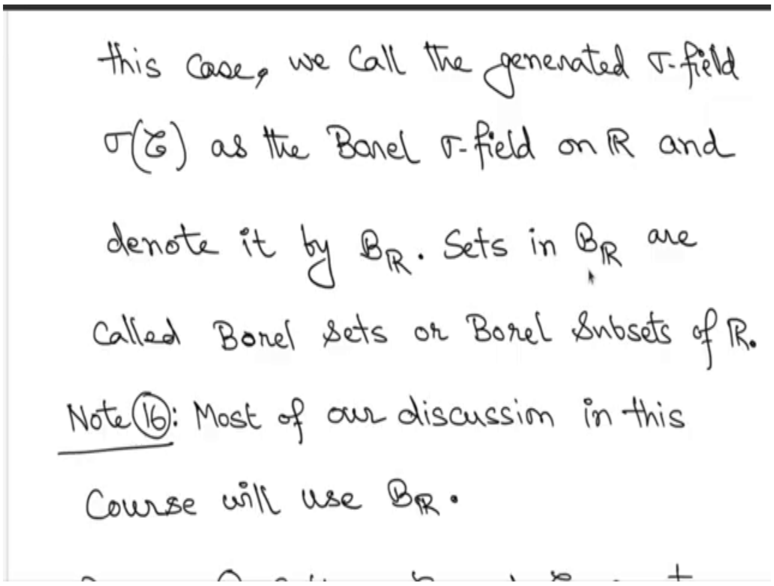


Now, examples of such generated sigma fields is that

Examples:

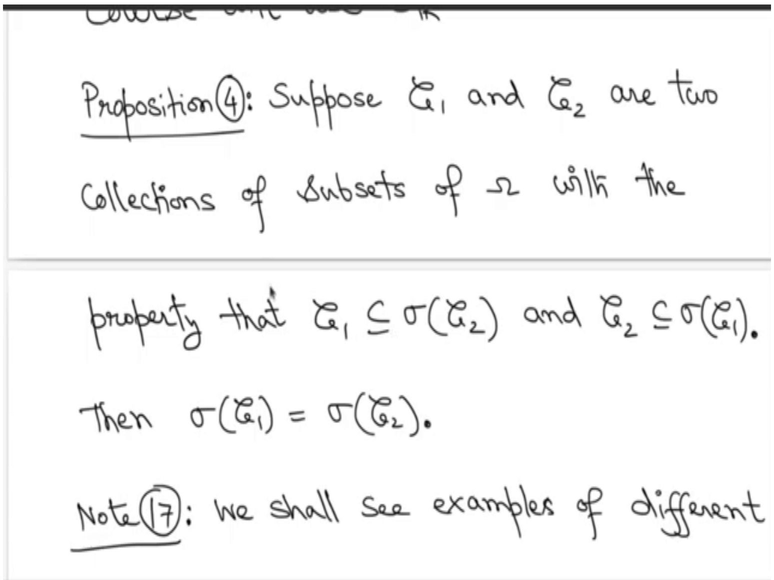
(i) If you take $\mathcal{C}$ to be just a collection of one single subset A, then you can show that the sigma field generated by $\mathcal{C}$ is nothing but this sigma field that we had already seen earlier. So, this is just the empty set, then A, $A^c$and Ω itself. So, you have this four sets collection, a sigma field that is exactly the sigma field generated by this subset A.

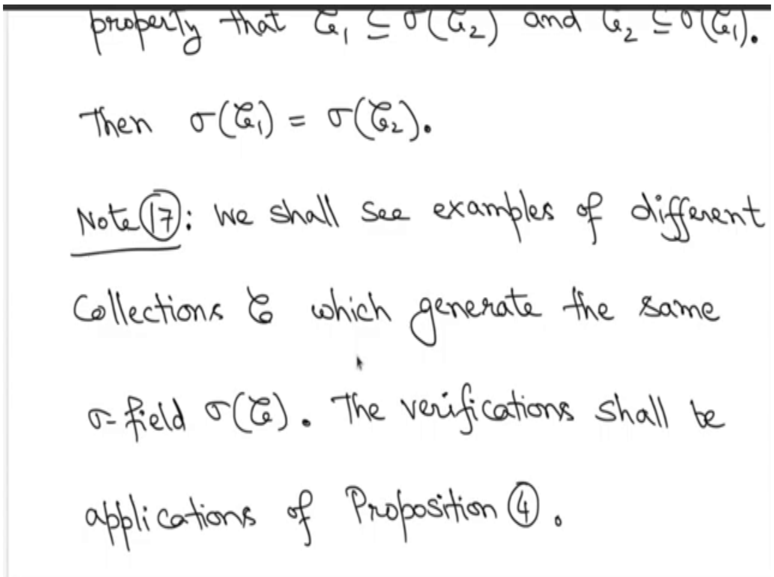
(Refer Slide Time: 31:49)



(ii) We will generally consider a very important collection of subsets, which are on the real line. So, if you take Ω to be the real line, consider the collection of all open sets. Now, important point, this collection of all open sets a priori does not satisfy the properties of a sigma field because you cannot have the properties of a sigma field here. Just try to check that. But then, you can still generate a sigma field out of these open sets, and what will happen is that you will end up with the Borel sigma field here, and then we denote it by $B_{\mathbb{R}}$. So, this is the notation that we are going to use. So, the generated sigma field out of all open sets in the real line, so that is the collection $\mathcal{C}$, you generate a sigma field, whatever you get, we refer to it as a Borel sigma field. And for this special sigma field, we introduce our notation, which is $B_{\mathbb{R}}$. So this is called the Borel sigma field on $\mathbb{R}$. Sets in the Borel sigma field that lists that you end up having at the end are called Borel sets or Borel subsets of the real line. So you start with open sets, generate a signal field, whatever extra sets or all the sets you get, including the open sets you had started with, all the sets you get are now called Borel subsets of the real line.

(Refer Slide Time: 33:14)





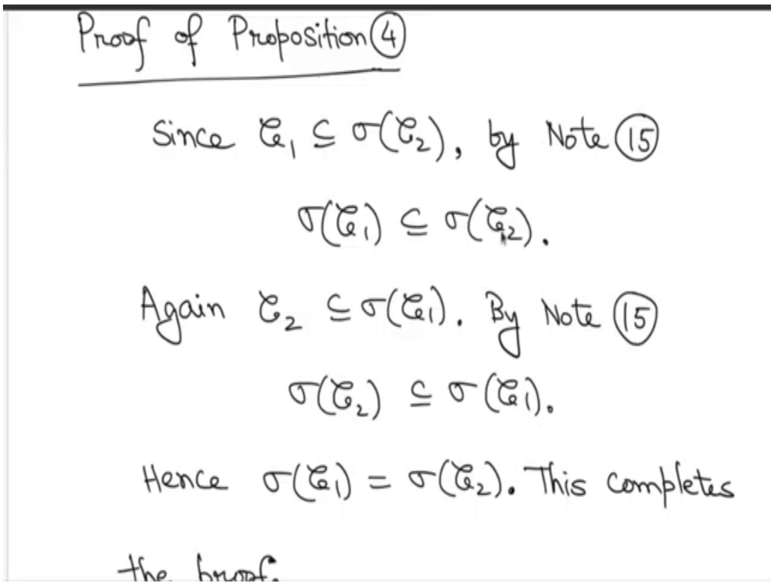
And most of our discussion in this course, we will use this Borel sigma field. And one final property involving these generating collections of sets that we will see the use of later on.

Proposition 4: So, if you have two such collections, and with this property, that$\mathcal{C}_1$ is contained in sigma field generated by $\mathcal{C}_2$, but sigma fields generated by $\mathcal{C}_1$ also contains $\mathcal{C}_2$ So, these are the reverse inclusion options. Then you claim that the sigma field is generated by $\mathcal{C}_1$ and sigma field generated by $\mathcal{C}_2$, these two must be the same as this proposition's claim.

So, how do you go about proving this is using the properties that we have already discussed. But we shall see examples of such different collections $\mathcal{C}$ which generate the same sigma field. This we will see in the next lecture, but the verification shall be done through these applications of the proposition that we have just discussed.

So, we are going to check that $\mathcal{C}_1$ is contained in the sigma field generated by $\mathcal{C}_2$ and vice versa. If you verify that, you will then claim that sigma field generated by $\mathcal{C}_1$ and $\mathcal{C}_2$ must agree. So, how do you prove this?

(Refer Slide Time: 34:25)



Proof: So, you already know that the sigma field is generated by $\mathcal{C}_1$ is the minimal one containing $\mathcal{C}_1$. But sigma field generated by $\mathcal{C}_2$ is a general sigma field, which contains $\mathcal{C}_1$. Therefore, by note 15 sigma field generated by $\mathcal{C}_2$ is a general sigma field and must contain the minimal sigma field generated by $\mathcal{C}_1$. So, you get this first inclusion that you want, but you again use the reverse inclusion exchange the roles of $\mathcal{C}_1$ and $\mathcal{C}_2$.

So, then again sigma field is generated by $\mathcal{C}_2$ is the smallest one containing $\mathcal{C}_2$, therefore, it must be contained in sigma field generated by $\mathcal{C}_1$. So, therefore, you have both ways of inclusion and immediately claim the equality you wanted to prove. This immediately follows from the relevant properties.

So, now, we have understood some structures of sigma fields and fields. We have used some examples and nice techniques, a bottom-up approach, and a top-down approach to construct many such examples. We will now look at the very concrete properties of one of these sigma fields, which we introduced a few minutes back, the Borel sigma field. So, we are going to discuss that in the next lecture.