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Measure Theoretic Probability – 1
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Lecture 29
Monotone Convergence Theorem

Welcome to this lecture. In this week, we have been discussing about integration of measureable functions with respect to a given measure on the domain set. So, far we have concentrated our attention on obtaining properties for this integration procedure. In this lecture, we are going to look at a very important theorem involving this integration procedure that finally allows us to make comments about the linearity of the integration procedure. So, let us move ahead with the slides and discuss that result.

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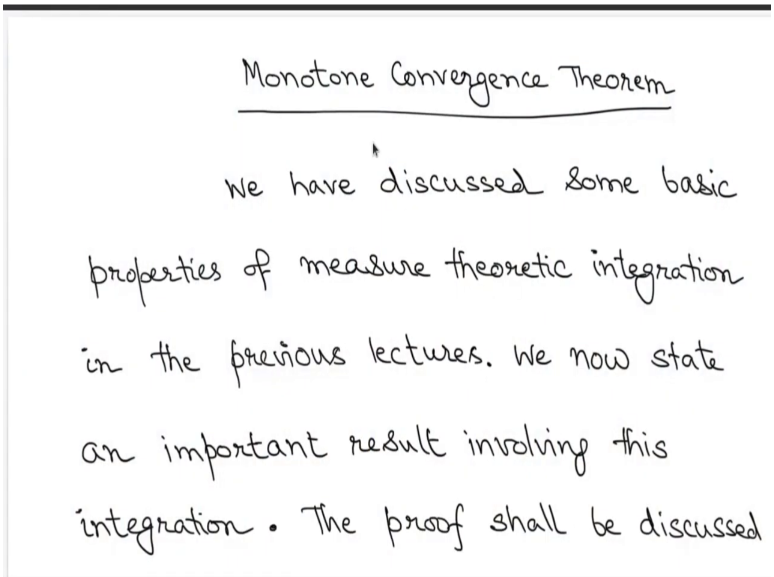

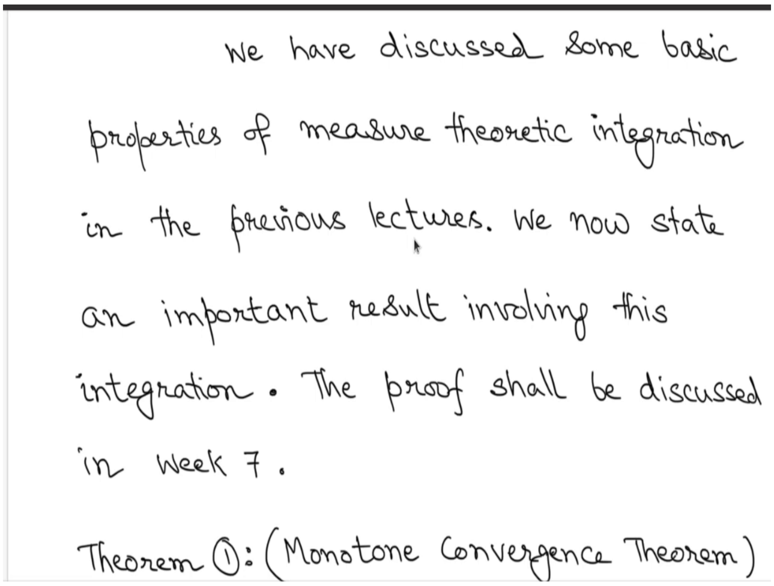

So, this result is called the monotone convergence theorem. So, so far, we have restricted our attention to basic algebraic properties, such as multiplication by scalars and certain comparison type inequalities. In the previous lecture, we have restricted our attention to a very specific type of integrals over sets. And we had seen that for non-negative measurable functions, by fixing that function and varying the set over which the function is getting integrated, we can get a measure.

So, that is some kind of a continuity behavior of this integration procedure. We are going to discuss that in more detail in later lectures. But for now, we are going to look at certain limiting behaviors for special type of sequences of measurable functions. And that is the content of this modern convergence theorem.

So, in this lecture, we are going to state it and look at certain applications of this monotone convergence theorem. But the proof will be discussed in the next week that is week 7. So, we are going to assume this and apply this the proof does not require any linearity property, and we are going to actually prove the linearity property as an application of the monotone convergence theorem. So, this is a very, very important theorem in the context of integration of measureable functions.

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Theorem 1: (Monotone Convergence Theorem)

Let $\{h_n\}_n$ be a non-decreasing sequence of non-negative, Borel measurable functions defined on a measure space $(\Omega, \mathcal{F}, \mu)$. Suppose that the limit function $h := \lim_{n\to\infty} h_n$ is real valued. Then

$$\int h_n \, d\mu \uparrow \int h \, d\mu \text{ as } n \to \infty.$$

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$$\int h_n \, d\mu \uparrow \int h \, d\mu \text{ as } n \to \infty.$$

So, what do we do we start off with non-decreasing the sequence of measurable functions. So, by that I mean if you take this sequence $\{h_n\}$, then $h_n \leq h_{n+1}$. And you also assume that these h_n's are bounded below in particular, you can assume them to be non-negative.

So, for such a sequence of measurable functions defined on the same measurable space with this given measure mu on the domain side, we are going to consider their integrals, but since this sequence of functions are non-decreasing, so, point wise you can consider the limit function. So, this limit function of course, will become measurable, so, we will comment about that in a minute, but let us look at the integration of this limit function h.

Now, since $h_n \leq h_{n+1}$, we have already mentioned earlier that integrals will also follow the same inequality that means, $\int h_n d\mu \leq \int h_{n+1} d\mu$. So, therefore, these quantities whatever they are they are also non decreasing in n. So, therefore, this has a limit. Now, since h_n's are going to this limit function h and h_n are non-decreasing each h_n is dominated from above by h.

So, therefore, you have $\int h_n \, d\mu$ dominated for above by $\int h \, d\mu$. What you are saying is that this limit is exactly the $\int h \, d\mu$ not the inequality that you get. So, this is the exact equality that the limit of this h_n is exactly $\int h \, d\mu$. So, a several comments before we go into applications as already mentioned, we are going to prove this in the next week not now.

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$h := \lim_{n \to \infty} h_n$ is real valued. Then

$$\int h_n \, d\mu \uparrow \int h \, d\mu \text{ as } n \to \infty.$$

Note 17: (i) We shall use the term MCT to refer to the Monotone Convergence Theorem.

(ii) In the MCT, h is measurable since it is the pointwise limit of since it is the pointwise limit of measurable functions.

(iii) In the case that h takes values in $\overline{\mathbb{R}}$, MCT continues to hold. We need to interpret $\int h \, d\mu$ as an integral of $\overline{\mathbb{R}}$-valued measurable function.

(iv) Given any non-negative and

So, this monotone convergence theorem is very, very important and it is used multiple times. So, for our reference, we are going to call it as MCT. So, this is for ease of notations, we are going to call it MCT to refer to monotone convergence theorem. So, in the MCT, you have this measurable function h because it is a point-wise limit of measurable functions. So, as long as you have the existence of the limit, so here you have the h_n to be non-decreasing. So, there therefore, you have the point-wise limits.

So, therefore h is measurable with respect to the given sigma fields. So, measurability is not an issue for the limit function. But it can happen that the limit function h takes values in the extended real line in particular, it might take values $+\infty$, it might take that value the MCT will continue to hold. So, even if the h_n's are real valued their limits could be $+\infty$ for some point. So, therefore, the limit function may take $+\infty$.

So, even then there is no issue because we have already mentioned that integration of extended real world functions goes in the similar way. So, again by this approximation by simple functions and all those procedure. Anyway we need to interpret $\int h \, d\mu$ in the case when h is taking values in the extended real line as an integration of extended real value measurable function. So, it is not a problem. So, you can define this, but the MCT is applicable even in this case.

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of $\mathbb{R}$-valued measurable function.

(iv) Given any non-negative and Borel measurable function, by Theorem 4 of week 3, we can construct a sequence $\{h_n\}_n$ of simple functions with $0 \leq h_n \uparrow h$. This sequence falls in the setup of MCT.

(v) Given any non-negative and Borel measurable function h, using (iv)

But important point to note is that given any non-negative and measurable function by this earlier theorem, in week 3, what we had done there is that we had approximated this non-negative and measurable function by simple functions. So, these simple functions approximated are general measurable function which is non-negative from below. So, then these simple functions, we had given explicit construction of that, such that these simple functions increase and increase to that given non-negative measurable function.

So, this falls in exactly in the setup of the MCT that these h_n's are certain nice functions simple of course, in particular, then they will be measurable and then h_n's will be dominated from below by 0. So, therefore, h_n's are non-negative and they increase to h. So, this is all falling in the exact setup of MCT.

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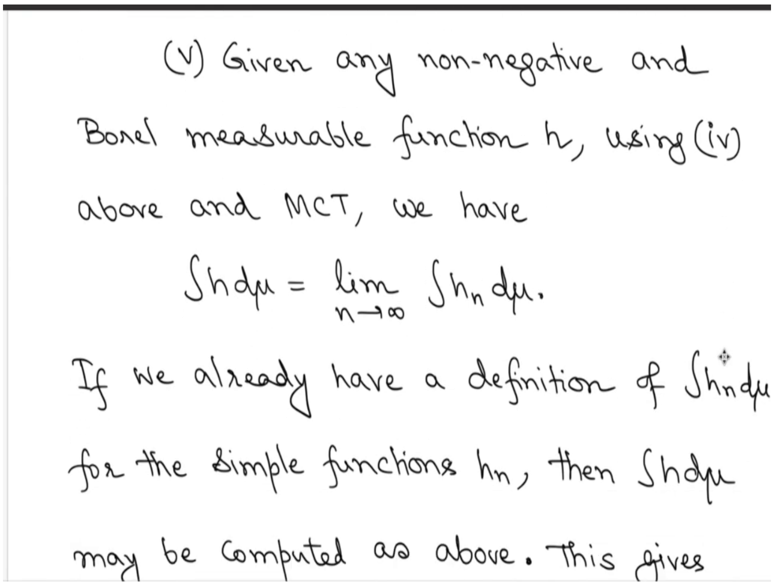

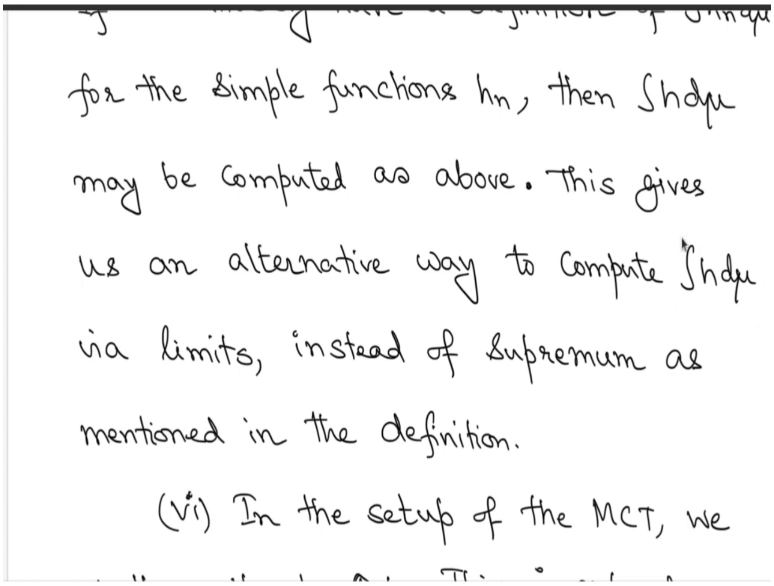

So, with this observation, you can now look at this point that given a non-negative and measurable function h choose the sequence of non-negative and simple functions h_n as mentioned in the previous point IV, continue with that sequence of simple functions, then as for MCT $\int h \, d\mu$ can be computed as $\lim \int h_n \, d\mu$.

So, if you can compute the integration of simple functions, which is appearing on the right hand side, consider the limit of that, then you have the value of $\int h \, d\mu$, but remember we had earlier defined the integration of non-negative measurable functions h as a supremum over a certain class of simple functions below h. Here, we have a alternative definition alternative expression in terms of a limit, not a supremum of course, these h_n increase so, therefore, this limit is nothing but a supremum over this countable set of values.

But, what is happening is that instead of using supremum says in the general case general definition limits are much easier to handle. So, this is a very important observation that by using MCT and using that approximation from below by simple functions, you can compute the integration of non-negative measurable functions by this limit of integration of the corresponding simple functions. Now, this will allow us to do computations in an easier fashion when we want to compute the $\int h\, d\mu$.

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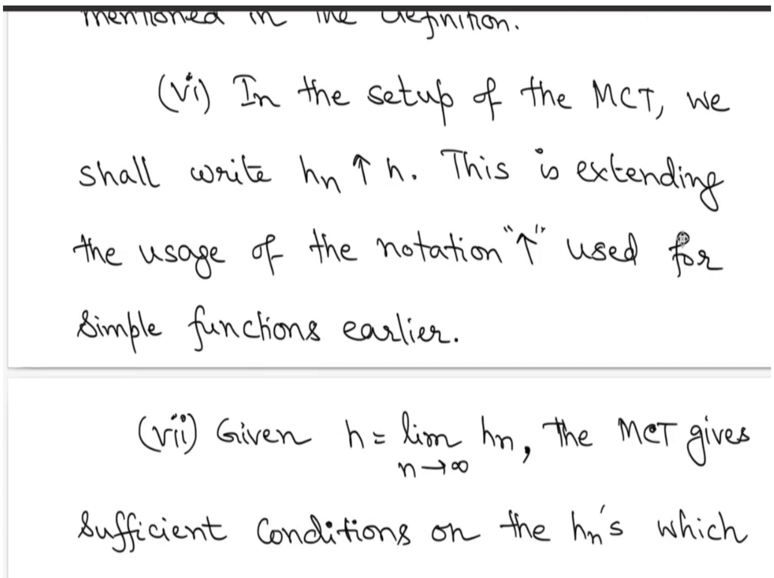

mentioned in the definition.

(vi) In the setup of the MCT, we shall write $h_n \uparrow h$. This is extending the usage of the notation "$\uparrow$" used for simple functions earlier.

(vii) Given $h = \lim_{n \to \infty} h_n$, the MCT gives sufficient conditions on the h_n's which simple functions earlier.

(vii) Given $h = \lim_{n \to \infty} h_n$, the MCT gives sufficient conditions on the h_n's which allows us to write

$$\int h\,d\mu = \int \left(\lim_{n \to \infty} h_n\right) d\mu = \lim_{n \to \infty} \int h_n\,d\mu.$$

In later lectures, we shall study more results of this type.

But in the setup of this monotone convergence theorem, we had earlier written that h_n's if they are simple functions and increase to h we had used this symbol. So, this was mentioned earlier in the construction of the simple functions that the simple function increased to h and that will denote by this upper arrow symbol.

But then in the setup of MCT, we can use any general sequence of non-negative measurable functions instead of a simple function, you can use a non-negative measurable function even then the limit holds. So, that is the statement of the MCT. And in particular of course, this applies to the case of simple functions approximation by simple functions.

Therefore, since this is a slightly more general statement in the MCT, where h_n's are now non-negative and measurable, then we will use the same upper arrow notation to denote that limiting behavior that these measurable functions are now approximating the function h from below. Now, given this structure that h is a point wise limit of this functions. So, MCT gives a sufficient condition for exchanging the order of limit and integration.

So, given a limit function h of the limit of these functions h_n, if you consider these sufficient conditions that h_n's are non-negative, and h_n's are non-decreasing. Under these sufficient conditions, you are allowed to exchange the order of limit and integration, so, that is what this is saying because $\lim \int h_n\, d\mu$ is nothing but the $\int h\, d\mu$ so that is as per the monotone convergence theorem, but observe that h inside is the point wise limit.

So, therefore, we were just exchanging the order of limit and integration. So, in later lectures, we are going to study more results of this type, which will allow us to exchange the order of limit and integration. So, this is a very important result, which allows us to exchange the order of limit and integration.

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We now look at various important applications of the MCT.

Theorem②: Let $f, g : (\Omega, \mathcal{F}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be measurable functions. If $\int f d\mu$ and $\int g d\mu$ exist and if $\int f d\mu + \int g d\mu$ can be defined (i.e. "$\infty - \infty$" situation does not

defined (i.e. "$\infty - \infty$" situation does not

occur), then $\int (f+g) d\mu$ exists and

$$\int (f+g) d\mu = \int f d\mu + \int g d\mu.$$

Note⑱: We shall discuss the proofs of Theorems ① and ② in Week 7.

The next result is a consequence

With that understanding of the monotone convergence theorem, we are now ready to discuss certain important applications of the monotone of convergence theorem. The first one being that linearity property that we have been after for a long time. So, take two measurable functions f and g if you know that $\int f \, d\mu$ exists, $\int g \, d\mu$ also exists then you consider the integration of $f + g$, but you would like to connect it with $\int (f + g)\, d\mu$. So, you have to make sense of $\int f\, d\mu + \int g\, d\mu$.

So, if it so, happens that $\infty - \infty$ does not occur that means, that $\int f\, d\mu = \infty$ and $\int g\, d\mu = -\infty$ if such a equation does not occur or vice versa meaning $\int f\, d\mu = -\infty$ and $\int g\, d\mu = \infty$. So, if this does not occur, then $\int f\, d\mu + \int g\, d\mu$ is well defined. Because $\infty - \infty$ is not occurring, in such a situation, you can now show that $\int (f + g)\, d\mu$ and is equal to the sum of the individual integrations.

So, this is the result that we have been after for a long time and this follows from the application of MCT. We are going to discuss this proof of this theorems, this theorem two, which is this linearity property and the MCT, which is theorem one in week 7. But we are going to use these facts in the discussions now.

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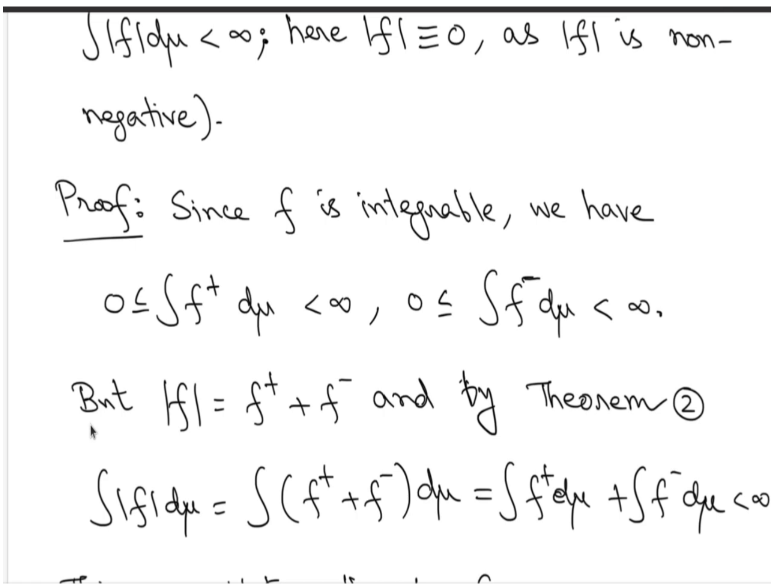

$0 \le \int f^+ \, d\mu < \infty$, $0 \le \int f^- \, d\mu < \infty$.

But $|f| = f^+ + f^-$ and by Theorem 2

$$\int |f| \, d\mu = \int (f^+ + f^-) \, d\mu = \int f^+ d\mu + \int f^- d\mu < \infty$$

This completes the proof.

<u>Note 19</u>: Corollary 1 states the converse of Note 9. Combining both results, we have

The next result is a consequence of Theorem 2.

<u>Corollary 1</u>: Let f be integrable. Then $|f|$ is also integrable (i.e. $\int |f|^+ d\mu = \int |f| \, d\mu < \infty$; here $|f|^- \equiv 0$, as $|f|$ is non-negative).

So, with this additive property at hand, we can now say that, if f is integrable, then $|f|$ integrable. So, why is this so, if f is integrable we are given by definition, that $\int f^+ \, d\mu$ is finite, $\int f^- \, d\mu$ is also finite, but then $|f|$ is nothing but $f^+ + f^-$ and therefore, by the additive property, we immediately get that $\int |f| \, d\mu$ which is $\int (f^+ + f^-) d\mu$ is equal to the separate integrations, but if both of these are given to be finite, then $\int |f| \, d\mu$ is also finite.

But, hence, what is happening is that $|f|$ is a non-negative measureable function whose integration is finite therefore, $|f|$ is a finite integrable function but this corollary which we have just proved states that f is integrable implies that $|f|$ is integrable.

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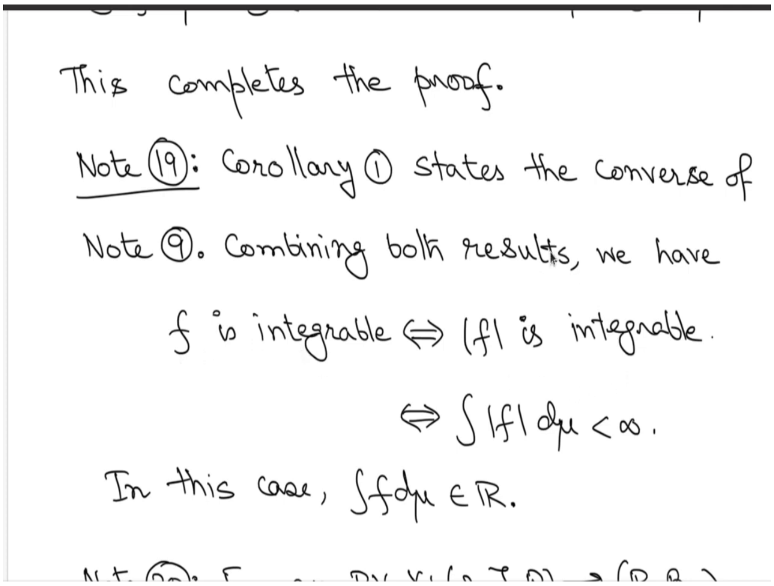

This is the converse of note 9 where we stated that $|f|$ is integrable implies f is integrable. Therefore, if you combine these results, you have the equivalence of these two statements that f is integrable if and only if $|f|$ is integrable.

And integrability is a $|f|$ is nothing but that $\int |f| \, d\mu$ that is finite. And just a quick observation once more that here we are looking at $\int f \, d\mu$, the $\int f \, d\mu$ is now computed as the difference of the integration of f^{+} and f^{-} if both are given to be finite in this case, the difference is a real number. So, that is an observation when f is integrable you have these things. But these results for measurable functions, you can now state it for random variables.

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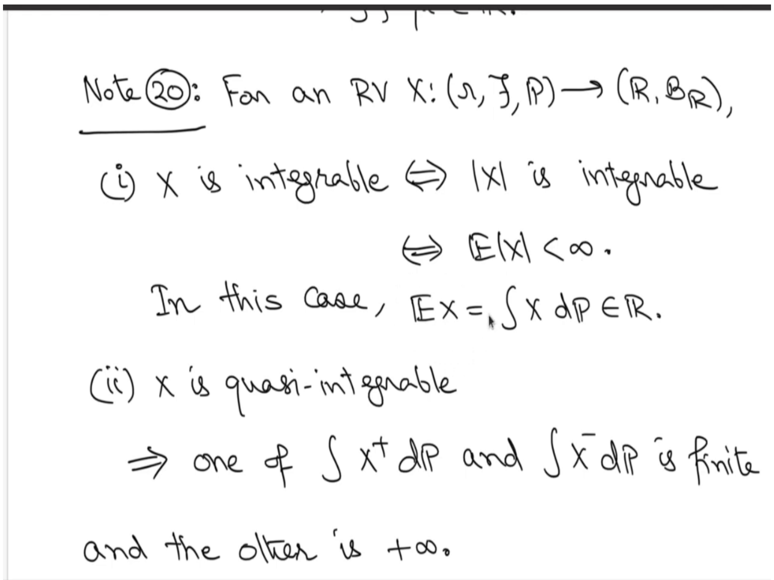

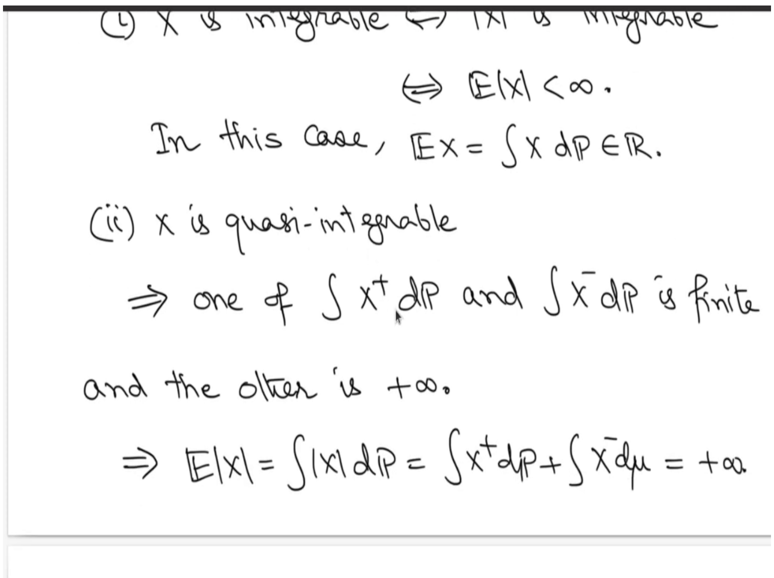

And you are now just restating these observations for random variables by this that X is integrable if and only if $|X|$ is integrable if and only if $\mathbf{E}\,|X|$ is finite. Remember, $\mathbf{E}\,|X|$ is nothing but the integration of $|X|$ with respect to the given probability measure. So, therefore, X is integrable if and only if $\int |X|$ is finite that is $\mathbf{E}\,|X|$ is finite.

And in this case of course, $\mathbf{E}\,X$ is a real number, but then if X is quasi integrable, you have to be careful in this case exactly one of X^+ and X^- has a finite integrable other has infinite integral. In this case, $\mathbf{E}\,|X|$ is the addition of these two quantities, one of them is finite, other is infinite. So, therefore, $\mathbf{E}\,|X|$ is ∞. So, in that case when X is quasi integrable $\mathbf{E}\,|X|$ is ∞.

But in this case the $\mathbf{E}\,X$ is $+\ \infty$ or $-\ \infty$. This thing we are considering now, when X is quasi integrable.

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In this case, $\mathbb{E}X = \pm\infty$.

Note (21): In basic probability courses, for the existence of $\mathbb{E}X$, we consider the condition $\mathbb{E}|X|<\infty$. In addition, to this case, from now onwards we may still talk about $\mathbb{E}X$ if X is quasi-integrable.

Note (22): For RVs $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$,

In the basic probability course, we had not considered the quasi-integrability case, but we had considered the existence of these expectations through this condition that $\mathbf{E}\,|X|$ is finite. But even if $\mathbf{E}\,|X|$ is ∞, if X is quasi integrable, we can still talk about $\mathbf{E}\,X$ that is the power of measure theory that we are using here. And since, we have defined the $\mathbf{E}\,X$ through these measure theoretic integrations, we are going to consider this in the cases when X is quasi integrable.

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talk about $\mathbb{E}X$ if X is quasi-integrable.

Note (22): For RVs $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$,
under appropriate hypothesis, Theorem (2) implies $\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$.

Note (23): Combining Theorem (2) and Proposition (1)(i), we have the linearity of the map $h \longmapsto \int h\,d\mu$.

Now, an important fact here is that the additive (())(15:51) of the integral that we have just discussed will allow us to prove this kind of statement that under appropriate hypothesis, if you have two random variables X and Y, then $\mathbf{E}(X + Y) = \mathbf{E}X + \mathbf{E}Y$. Now, remember in your basic probability theory, you had proved similar results using the fact when X and Y both are discrete or both are absolutely continuous here we are not stating them.

Here we are saying if X and Y are two random variables, be it continuous, be it discrete does not matter; X could be discrete, Y could be continuous, even then you can talk about their addition and their expectation and then you can split the expectation by the linearity of the integration and write this. So, here the appropriate hypothesis is simply that $\mathbf{E}X$ exists and $\mathbf{E}Y$ exists and their sum is defined.

That means $\infty - \infty$ situation does not arise on the right hand side, if that is the case, then you can talk about $\mathbf{E}(X + Y)$ being equal to the right hand side. Now, combining all of these results together, you can now say that this linearity of the integration holds. So, what we had proved earlier is that scalar multiplication was allowed.

Now, we are saying additivity is also allowed that if you take two functions, then under appropriate hypothesis, addition of the individual integrations will give you integration of the addition of the functions. So, you have the linearity of the integration procedure that we have discussed.

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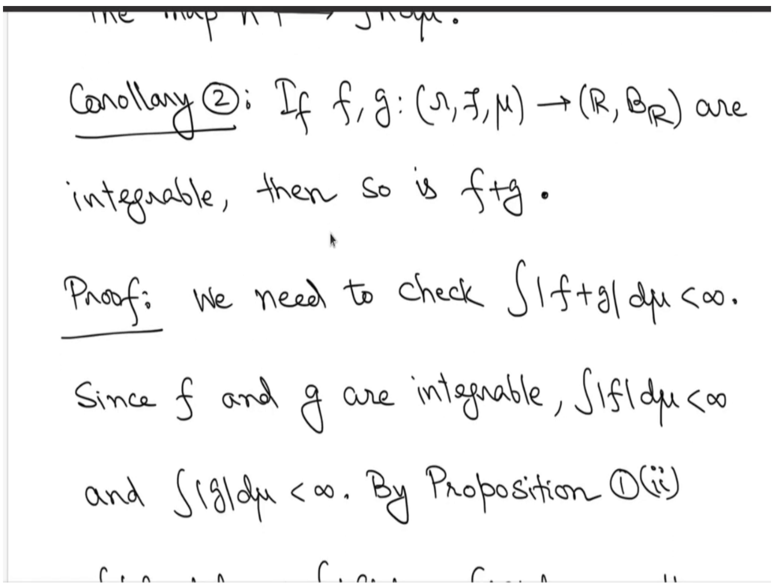

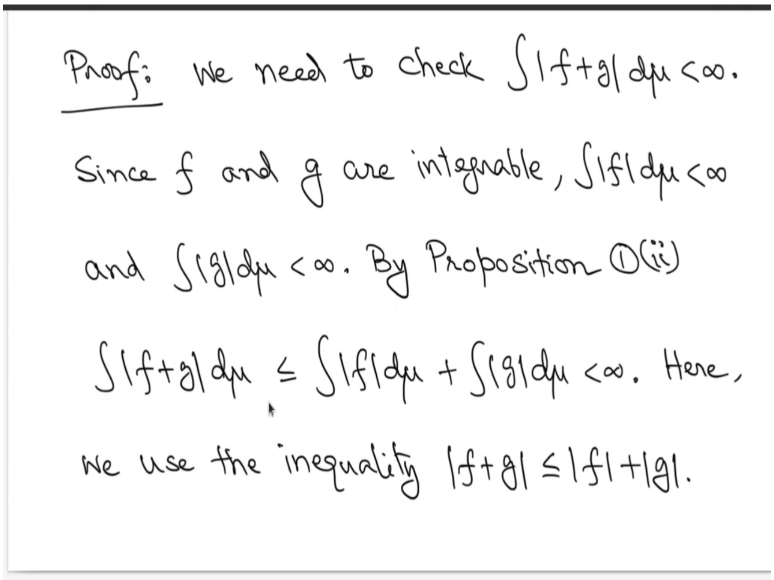

Now, with this observation at hand, you can now make this important comment that if both f and g are integrable, then $f + g$ is also integrable. Why, because observe that if f and g both are integrable, then $\int |f| \, d\mu$ is finite and $\int |g| d\mu$ is finite, this is by the equivalent condition that we just discussed in a few minutes back, but then $\int |f + g| \, d\mu$ will be dominated by the addition of the $\int |f| \, d\mu + \int |g| \, d\mu$ because $|f + g| \leq |f| + |g|$.

So, use that and use the additivity of the integrals, you get this inequality. But if $\int |f| \, d\mu$ is finite, and $\int |g| \, d\mu$ is finite, the addition is also finite. So, therefore, $\int |f + g| \, d\mu$ is also finite and hence, $f + g$ becomes integrable provided f and g both are integrable.

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Corollary ③: let $\{h_n\}_n$ be a sequence of non-negative Borel measurable functions.

Then $\int \left(\sum_{n=1}^{\infty} h_n \right) d\mu = \sum_{n=1}^{\infty} \int h_n \, d\mu$.

Proof: we have $\sum_{n=1}^{m} h_n \uparrow \sum_{n=1}^{\infty} h_n$ and by Theorem ②, $\int \left(\sum_{n=1}^{m} h_n \right) d\mu = \sum_{n=1}^{m} \int h_n \, d\mu$.

Then $\int \left(\sum_{n=1}^{\infty} h_n \right) d\mu = \sum_{n=1}^{\infty} \int h_n \, d\mu$.

Proof: we have $\sum_{n=1}^{m} h_n \uparrow \sum_{n=1}^{\infty} h_n$ and by Theorem ②, $\int \left(\sum_{n=1}^{m} h_n \right) d\mu = \sum_{n=1}^{m} \int h_n \, d\mu$.

Applying MCT, we have the result.

Nice, and we finish our discussion with this nice corollary that we can turn out talk about certain series of functions. So, take non-negative Borel measurable functions a sequence of that, a sequence of non-negative Borel measurable functions, then consider this infinite combination infinite summation. So, this linear combination infinite linear combination will give you a limit function, this are addition of non-negative quantities, so, you will get a limit function.

So, you are considering the integration of that on the left hand side; on the right hand side, you are saying look at the individual integrations and add them up. If you are looking at these integrations have non-negative measurable function, these quantities are non-negative. So, therefore, the summation is defined this summation could be infinity that is fine, we are saying this equality holds. So, you can exchange this infinite series and the integration here that is an important observation.

So, how do you prove this, observe that the finite sum $\sum_{n=1}^{m} h_n$ increase and increase to the limit function which is the complete summation, for this finite sum $\sum_{n=1}^{m} h_n$, then these finite sum increase, increase with a complete countable sum. But then what is happening is that for the finite sum you can exchange the integration and the sum that is by the additivity and apply limit in m as all these functions are increasing, increasing to this complete summation now.

And by the MCT you get the required result that you can exchange the series and the integration. So, with these properties, we are now ready to discuss computations involving expectations of random variables. That discussion we are going to have in the next lecture, we stop here.