Measure Theoretic Probability 1
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Lecture 26
Integration of Measurable Functions

Welcome to this lecture, this is the first lecture of week 6. In this week, we start the discussions about integrations of functions and in particular about expectations of random variables. So, before we move forward, it is a good time to have a recall of what we have already done. So, in the previous weeks, we have covered extensively about measures on measurable spaces, then measurable functions, then we moved on to random variables. So, we spent a lot of time in discussions involving their laws and distribution functions.

So, we obtained the correspondence between them. So, with all that knowledge at hand, we are now going to talk about expectations of random variables, which should be a special case of integrations of measurable functions with respect to a given measure on the domain side. So, without further delay, so let us move on to the slides and make the setting clear.

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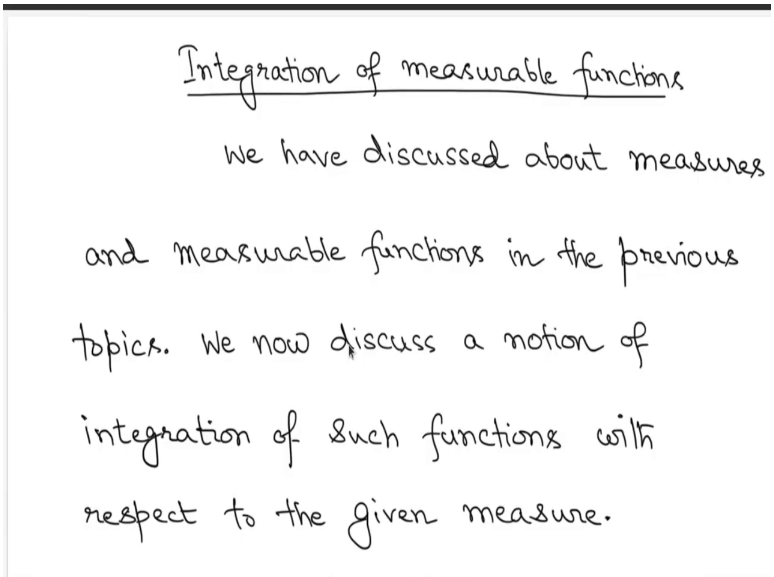

So, we have discussed about measures and measurable functions in the previous topics.

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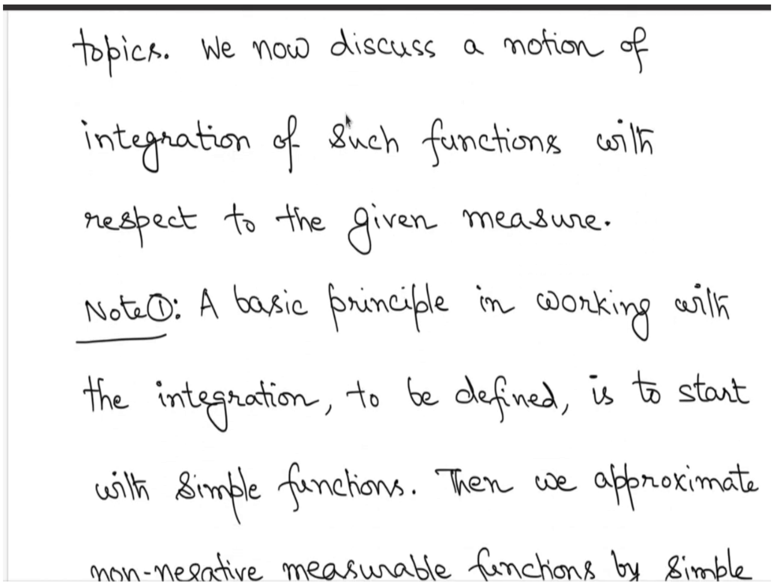

We are now going to discuss our notion of integration of such measurable functions with respect to this given measure. Now, before we even start, this is an important comment since, we are going to discuss integrations of measurable functions, before defining any kind of differentiation, it is a good idea to clearly understand the operation that we are trying to define. And a basic principle here in working with this integration procedure, which we want to define is to start with simple functions.

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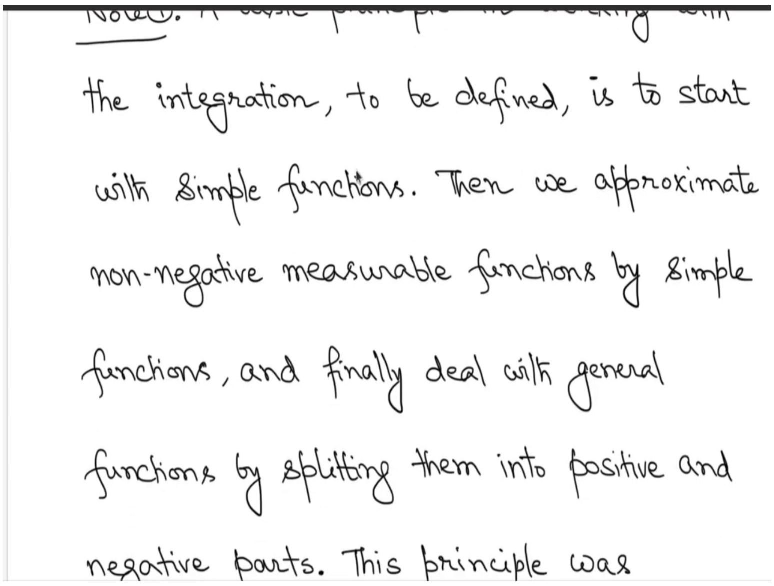

So, you first work with simple functions or maybe even go down to much simpler functions, which are indicator function. Then, after you understand the things for indicators and simple functions, we approximate non-negative measurable functions by simple function. So, this result was discussed earlier in relation with properties of measurable functions.

And finally, once you have understood all these operations in terms of non-negative measurable functions, you finally deal with general functions, what do you do you take a general function and split it into its positive part and negative part each of which is a non-negative and measurable function. And therefore, you try to put all that information from individual parts of the positive part and the negative part and put them together to get some idea about the general case.

So, this is basically the principle that we are going to follow So, in case we do not understand something, we will go back to these basics we will start with indicators and simple functions, then do limiting approximations to non-negative measurable functions. And then we shall split any general measurable function, signed measurable functions to be splitting into its positive part and negative part.

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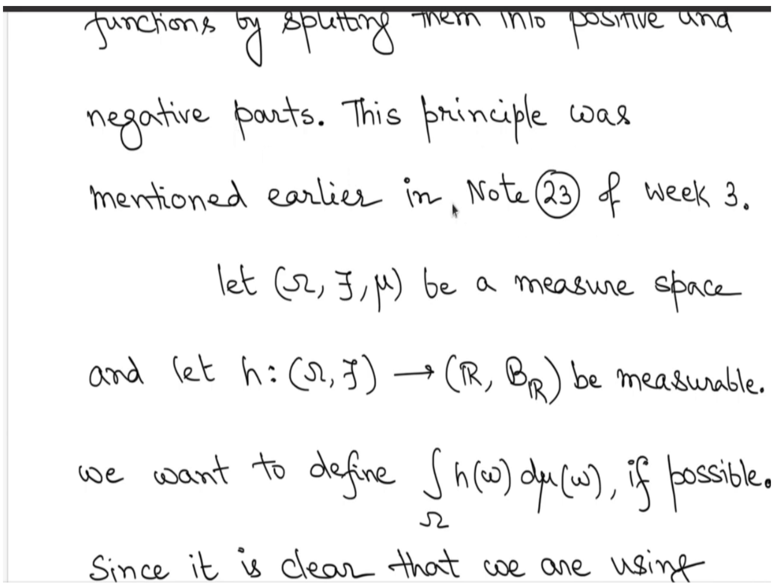

So, we have already mentioned this principle earlier in note 23 of week 3, but then again, we will come back to this issue again and again. So, let us move on and deal with the main setting.

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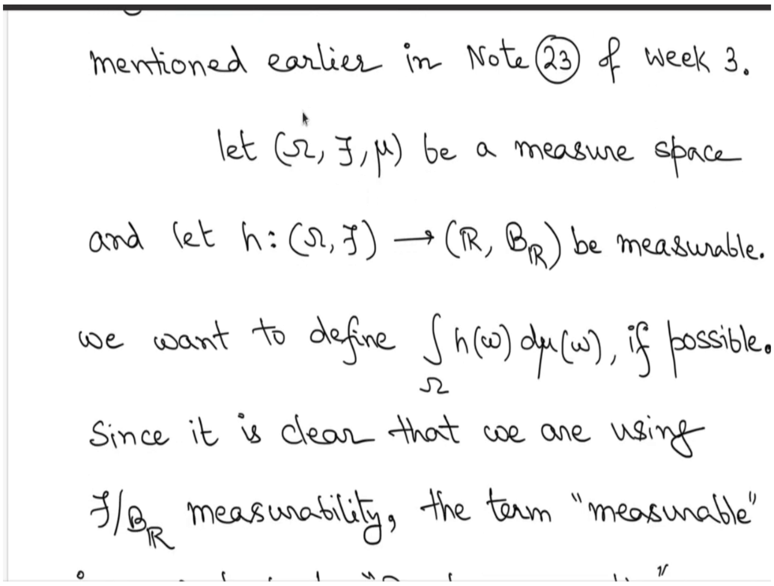

So, start with a measure space $(\Omega, \mathcal{F}, \mu)$ and take any measurable function h which is defined on this domain space together with this $\sigma -$ field and takes real values. So, this is the standard script F of R BR measurable functions. Remember that if your measure μ is a property measure, then what you can say immediately is that this function h is a measurable function which turns out to be a random variable. So, we are going to call it a random variable if μ is a probability measure.

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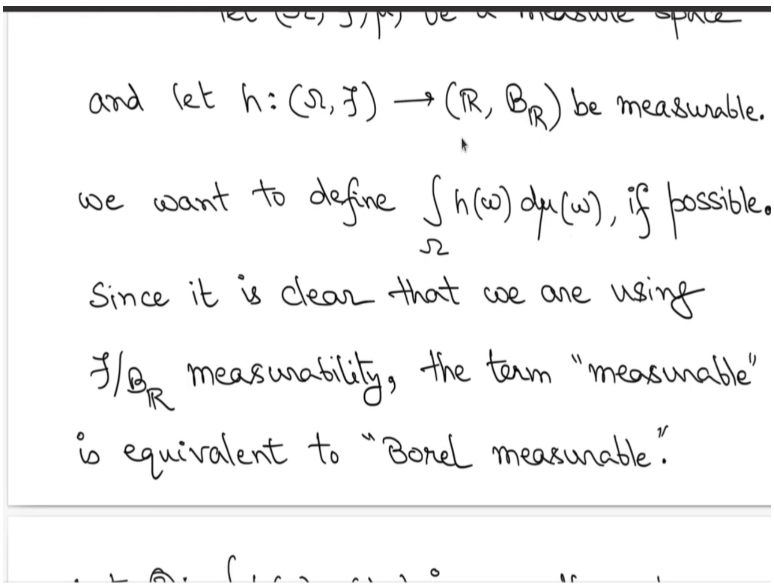

Now, what I would like to do is to define this value if possible. So, what is this, this is a formal notation now, integration over domain space Ω, we integration of the function h with respect to this measure μ which we write it as $\int_{\Omega} h(\omega) d\mu(\omega)$. So, ω's are points over the domain space.

Now, since it is clear that we are using the $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurability the term measurable in our discussion will be equivalent to Borel measurable. So, we are not going to make any distinction between them because we are first fixing all those σ −fields beforehand and then we are going to talk about all those nicer properties.

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Note ②: $\int_{\Omega} h(\omega)\, \mu(d\omega)$ is an alternative notation and represents the same integral. For the ease of writing, we shall use the short hand notation $\int h\, d\mu$.

As mentioned in Note ①, we take the following steps to define $\int h\, d\mu$.

we want to define $\int_{\Omega} h(\omega)\, d\mu(\omega)$, if possible.

Since it is clear that we are using $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurability, the term "measurable" is equivalent to "Borel measurable".

Note ②: $\int_{\Omega} h(\omega)\, \mu(d\omega)$ is an alternative

So, this next thing that we are going to mention is this, you can also rewrite this formal notation. So, what is the difference with the previous notation, so here we wrote $d\mu(\omega)$, here we are writing $\mu(d\omega)$. So, either notation is fine. And in fact, we are going to use a shorthand notation for simplicity, which will be $\int h \, d\mu$. So, we will suppress that variable ω and simply write it as $\int h \, d\mu$.

Another thing we are suppressing is the domain space Ω below this integration notation. So, again, as long as the setting is clear, as long as the domain space, measurable structures, measurable spaces, everything is clear, we can simplify the notation and use this. So, there should not be any chances of any miscommunication. As long as these things are fixed beforehand, we are going to use them in our analysis.

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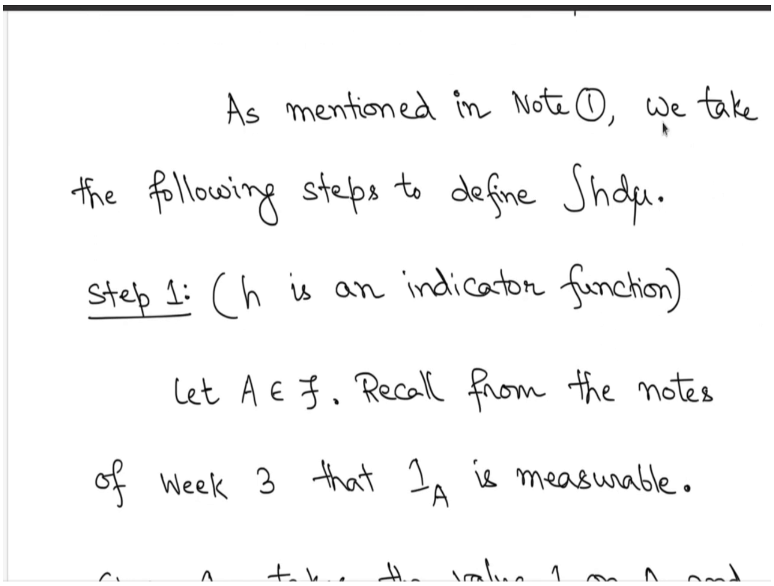

Now, as mentioned earlier our idea is that you want to define $\int h \, d\mu$, but you take first indicator functions or simple functions, try to do something there and try to generalize from there. So let us start with the case when h is the indicator function.

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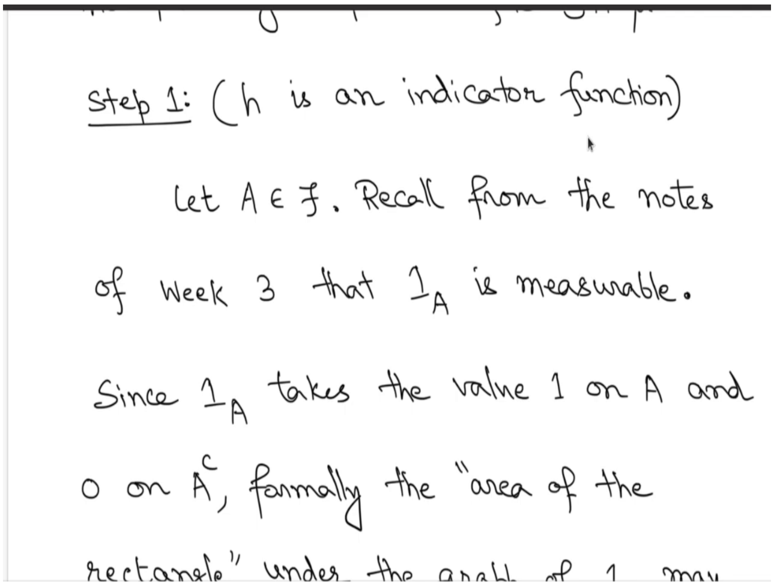

But then, remember, what you want is that h to be $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurable. And for that to happen if h is an indicator function, you need the set on the domain side σ −field. So, choose such as set of the domain size σ −field and recall from week 3's discussions that 1_A will be measurable in this case. Now, what do we want to do is to define the integration of such functions, which are of the form indicator of A. So now indicator of A takes the value 1 on the set A.

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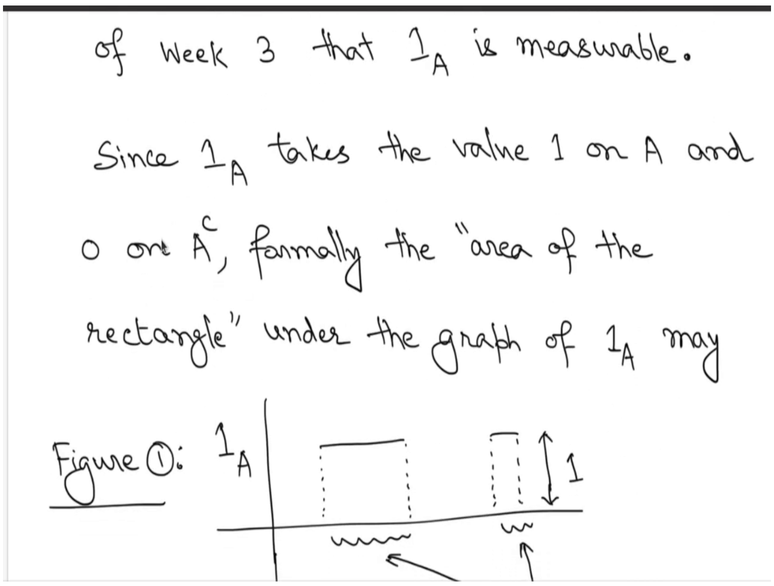

And takes them to 0 on the set A^c. Now, what I would like to think of integration as it is the area of the rectangle under the graph of 1_A. So that is what integration means. So, it simply means the area under the curve.

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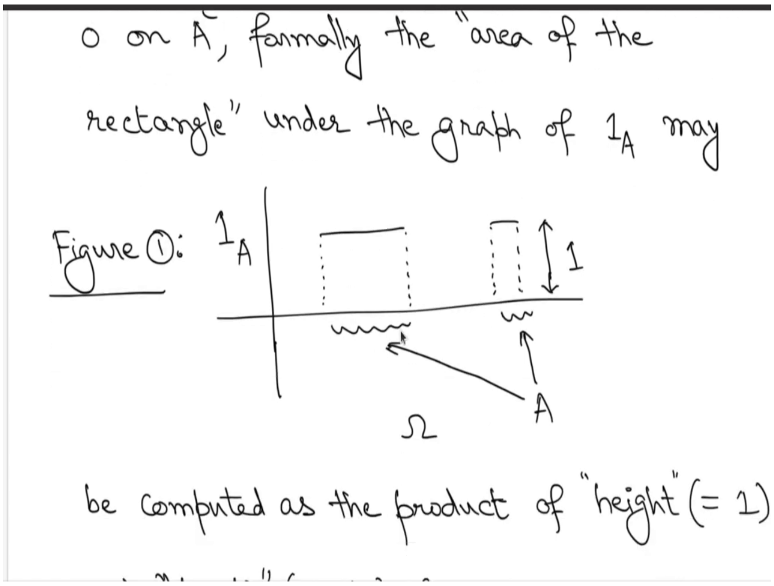

But then the idea is this, that if your set is roughly parts of the domain like this, so maybe split into chunks here and there. But then by 1_A, I mean a function which takes values 1 only on the set A, and otherwise it takes the value 0. So basically, the area under the curve should only be considered as the area of that all the rectangles that you see under the curve, everywhere else is 0 shields should not contribute to any area.

But then how would you compute the area of the rectangle that you see here. So, we follow the standard procedure, that we would like to look at the product of the length of the sides but what are the sides here.

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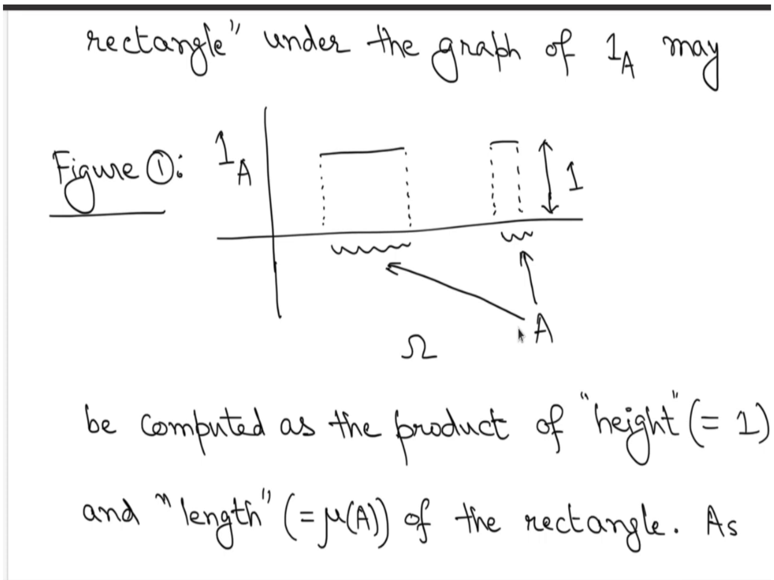

Just revisit this figure once more, the sides can be given as length of the rectangle as it goes along the domain side omega and the height as it takes the corresponding function value. So here, the function value remains constant, it is constant 1. So therefore, you can think of the height as 1 but what is the length, length here is the interesting part here you are using this measure, so you have this idea of length or size or measure of the set A. So, you use that on the domain side. So, on the value side, it is exactly equal to 1 on the rectangles.

So, it has a height 1, on the domain side you have the length or the size or the measure of that set to be given by $\mu(A)$. So, with these two quantities, you now can define the area as the product of this length and the height of these rectangles. So then, here one quick clarification is that by rectangle, we can mean all these things taken together.

So set A taken on the domain side, and value 1 taken on the other coordinate. So, the length will be the measure of A. And we are going to think of all of that together and call it a general rectangle. So, the term rectangle is not purely this rectangle being meant here, but it is putting together all the sets that you see, which constitute the set A. So, with that at hand.

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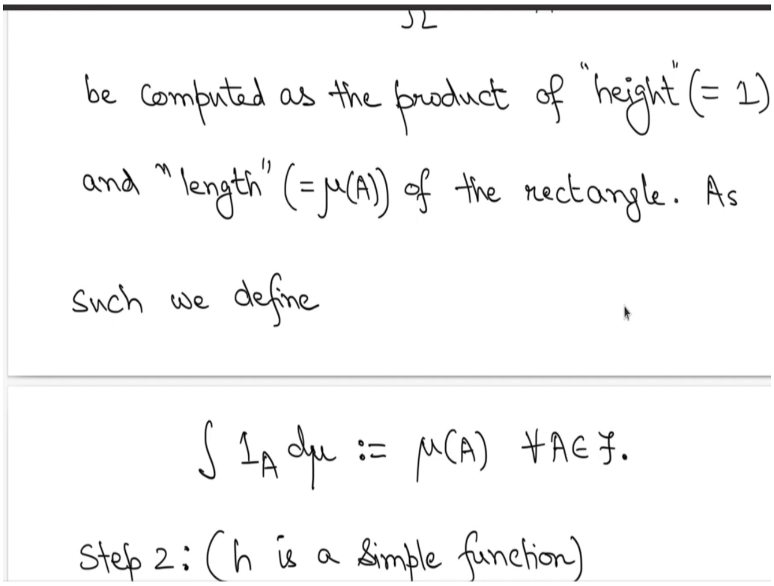

So now what is the integration. So again, as we just discussed, we would like to consider the product of height and length.

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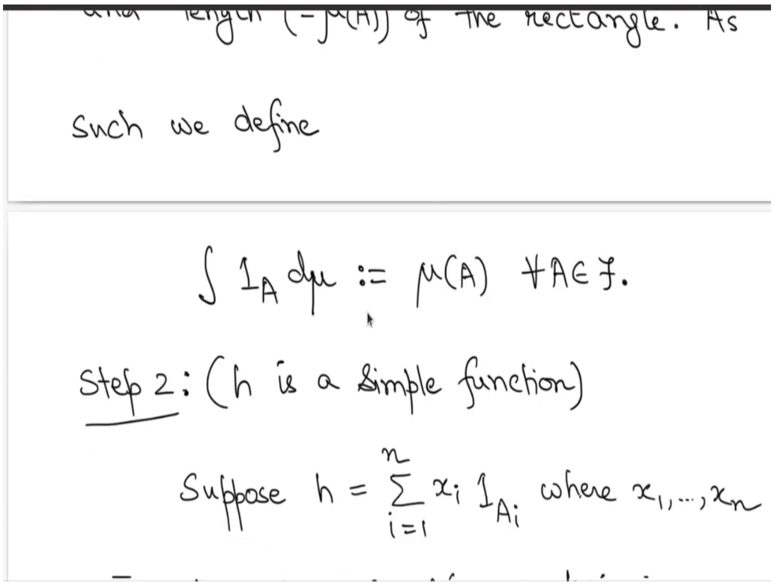

And that is what these values should be. So now, we define that integration of 1_A with respect to this measure μ should be taken as this quantity. So, this should happen for all the sets that come from your domain sides $\sigma -$ field. So, this is a very simple idea that you are just using area under the curve and here instead of the usual length that you might be putting on the domain side, here you have the domain Ω, so you have to use the given measure μ there, so that is all.

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$$\int 1_A \, d\mu := \mu(A) \quad \forall A \in \mathcal{F}.$$

Step 2: (h is a simple function)

Suppose $h = \sum_{i=1}^{n} x_i 1_{A_i}$ where $x_1, \ldots, x_n \in \mathbb{R}$, $A_1, \ldots, A_n \in \mathcal{F}$, A_i's are pairwise disjoint and $\bigcup_{i=1}^{n} A_i = \Omega$. If we are able

Once you have defined the integration of the indicator functions, you would like to extend these definition of the integration procedure, two simple functions. But remember, simple functions are nothing but certain linear combinations of indicators with certain conditions on the coefficients and the corresponding sets that appear in that representation. So let us not go into that technicality as of the moment, but let us just try to visualize how we can define the integration for simple functions, once we know for the indicator functions.

So now, if you have an integration procedure, it should be linear in the function h. What do I mean by that, I mean, that if you have two functions, h_1 and h_2, then $\int (h_1 + h_2)\, d\mu$, if you can define it, it should be the sum of $\int h_1 \, d\mu$ and $\int h_2 \, d\mu$ done separately. So, you add them up, you should be able to get back the integration of $h_1 + h_2$.

So, if you have that idea intuitively in hand, then you would like to say that corresponding to the simple function, that linear combination of indicators, if you integrate that, you should be getting back the same linear combination, but you have to just replace that indicator by the integration of the indicators and whatever that linear combination is, should give you the integration of the simple function.

So, if your integration procedure that you are going to define is linear in that function, linear in that integrand, then you should be able to describe it concretely. So let us put this idea in concrete mathematical terms. So let us start with h, which is $\sum_{i=1}^{n} x_i 1_{A_i}$, where the coefficients or the scalars x_i's are coming from real numbers, and the sets A_1 to A_n are coming from the domain side σ −field.

But remember, we would like to have that all these A_is to be pairwise disjoint. And as discussed earlier in week 3, that even if you consider any arbitrary linear combination of indicators, you can always rewrite it in terms of indicators, where the sets are pairwise disjoint. So, this restriction is not too much. So, you can just start off with some appropriate finite linear combinations of some indicators, where the sets need not be pairwise disjoint.

But you can further decompose the sets and rewrite in another possible linear combination, where the sets are pairwise disjoint. So, this is not too restrictive. Moreover, you could also associate one more condition that is quite useful is that the union of the sets covers the whole domain. So again, we have discussed this in week 3, that if the union of the sets is not the whole domain, you can always add the remaining part and multiply the scalar 0 to it.

So, therefore, what you can expect here is that this combination here is in some sense, a complete description with this Ais are pairwise disjoint and their unions covers the whole domain.

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Step 2: (h is a simple function)

Suppose $h = \sum_{i=1}^{n} x_i 1_{A_i}$ where $x_1, \dots, x_n \in \mathbb{R}$, $A_1, \dots, A_n \in \mathcal{F}$, A_i's are pairwise disjoint and $\bigcup_{i=1}^{n} A_i = \Omega$. If we are able to define an integration procedure, it should be linear in h. As such, keeping to define an integration procedure, it should be linear in h. As such, keeping the definition in step 1 in mind, we define

$$\int h \, d\mu := \sum_{i=1}^{n} x_i \mu(A_i),$$

provided the sum on the right hand side makes sense. The sum may also be

So, with this description in hand and keeping that idea in mind that you would like to convert integrations of the indicators to the corresponding measures of the sets, you would now like to look at such a linear combination. So here, what we are doing we are looking at the linear combination that was given to us, linear combination of the indicators, then replace the indicator Ais by the corresponding integral values.

So the integral values will be measures of the sets and you would like to look at when this summation makes sense. So, again this is simply following the idea is that if you can define an integration this should be linear in that integrand h. So, let us now concentrate our attention to this linear combination, but one thing we have to be careful about here note that the measures of the sets A_i can be ∞.

And the x_is that we are choosing could be positive or negative. In that case, the products that you look at $x_i \mu(A_i)$ could be $+\infty$, could be $-\infty$ or in nice cases when measures of A_i's are finite then the product will be some real number. But then again, for some sets A_i, $\mu(A_i)$ might turn out to be $+\infty$.

And in that case, if you are multiplying by some positive scalar x_i it will be $+\infty$, if you are multiplying by a negative scalar it will be $-\infty$. So, you will have to be careful when you want to talk about these kind of a summation. So, as long as this summation makes sense, you can go forward.

So, therefore, what you want is that this summation should make sense. So, if this summation makes sense only then you define the integration of such a simple function. So, you are not defining it for all simple functions taking values in the extended real life, you are only defining this notation to take this value only if this summation makes sense. So, this is important.

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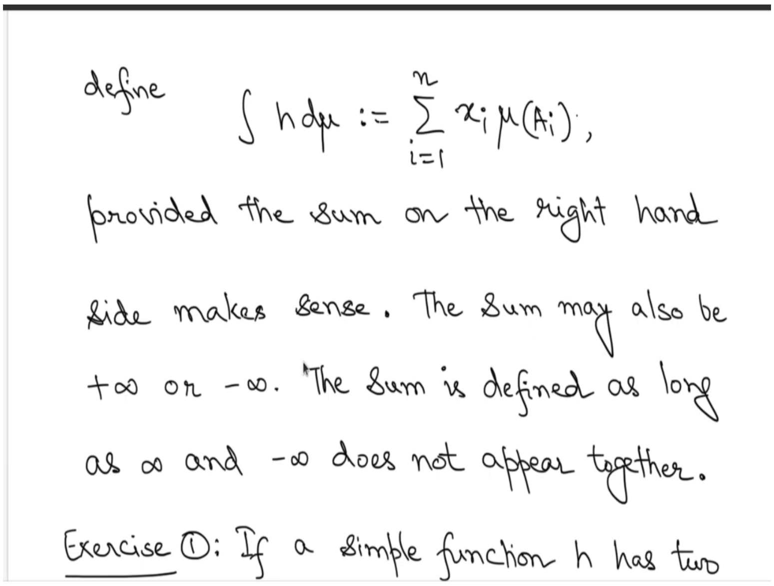

But you are allowing this sum to be $+\infty$ or $-\infty$ that is allowed. So, as long as this summation makes sense, this could take values in the extended real line. But as long as $+\infty$, $-\infty$ appears, you have to be careful, you cannot define it, but if the summation turns out to be $+\infty$, that is perfectly justified. So, then the sum is defined as long as $+\infty$ and $-\infty$ does not appear together, so that is all it is. So, therefore, you have defined the integration for simple functions with some appropriate justifications.

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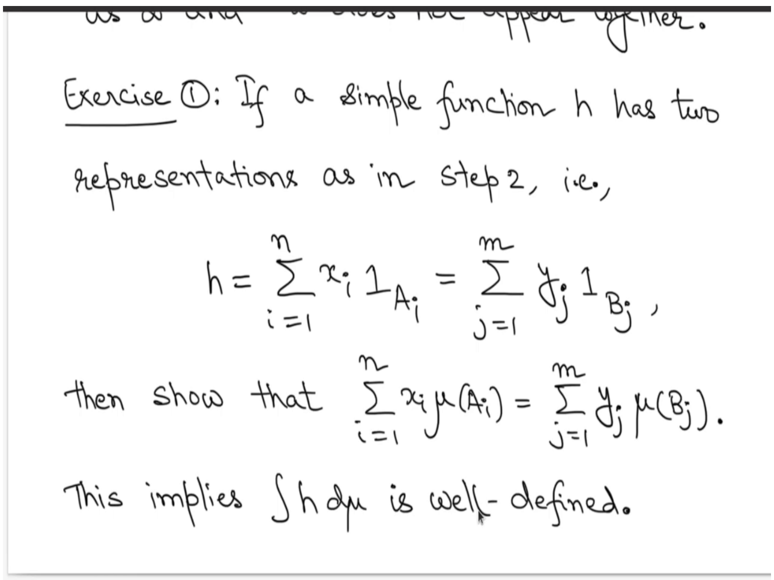

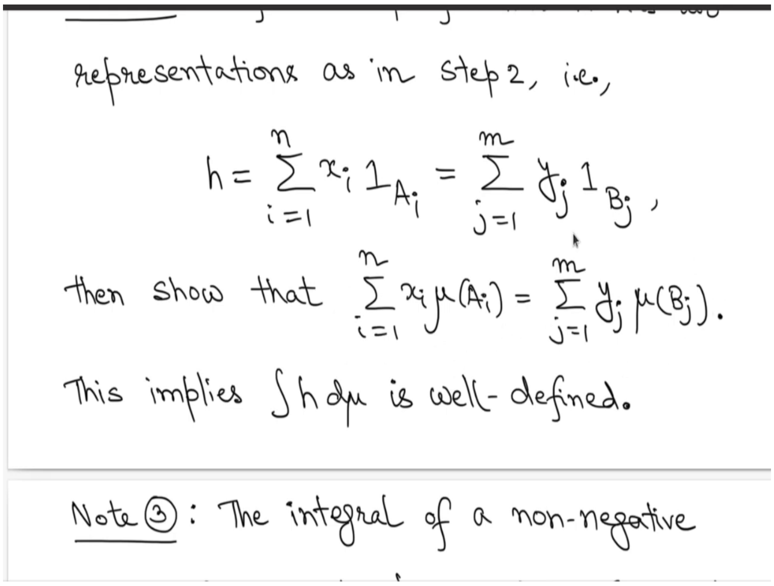

But then there is an important step here is this. So that if you have a simple function, it might have two representations in terms of sets and scalars. So, indicators used here may be different, the scalars used here may be different. But what you need is that the sets A_i should be coming from the domain side σ −field they should be pairwise disjoint and their union should cover the whole domain. So, the same thing should be true for these B_j's.

So, as long as those things are satisfied, if your function h has these two presentations, and you can define at least one of these summations corresponding to the integration then what you can say is that this equality should hold. So, as long as one side exists, the other side will also exist and will be equal to that. And this will imply that no matter what kind of representation of h you take in terms of the all these different indicators, $\int h \, d\mu$ is well defined. So, it does not depend on the representation that you will take and this is going to help us in future. So, this is a very important point. So, please try to work this out. So, this is being left as an exercise.

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Note ③: The integral of a non-negative simple function h always exists, since all x_i's are non-negative. The value of the integral may be ∞ in this case. We use this existence in the next step below.

Step 3: (h is a non-negative measurable function)

So, with that at hand, let us now move forward and try to see what is the case when we can take more general measurable functions, but one clarification here is that the integration of a non-negative simple function will always exist. Why, because you are taking this values x_i's to be non-negative then indicators are always taking values 0s and 1. So, signs can only come from the scalar x_i. So, whenever you are considering this linear combination of indicators scalars must be non-negative if the function is taking non-negative values.

Now, in this case, the value of the integration will be non-negative, including the value $+\infty$, so this will always exist now. So, this is a very important observation. So, you do not have to worry about $+\infty$ and $-\infty$ appearing together. If the function is non-negative simple function, then you will immediately get that its integration exists, it could be some 0 value or positive real number or ∞.

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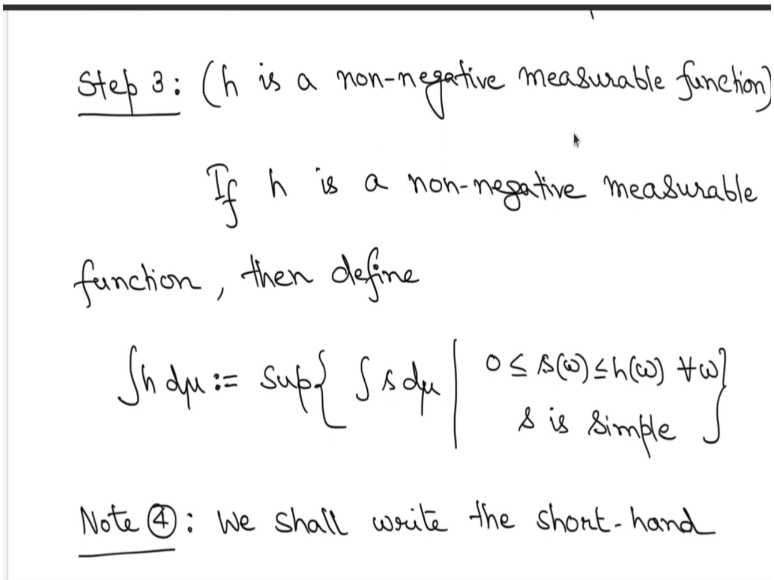

So, we are going to use this fact in the next step when we now discuss the non-negative measurable functions case. So, what we do is that we take a non-negative measurable function now.

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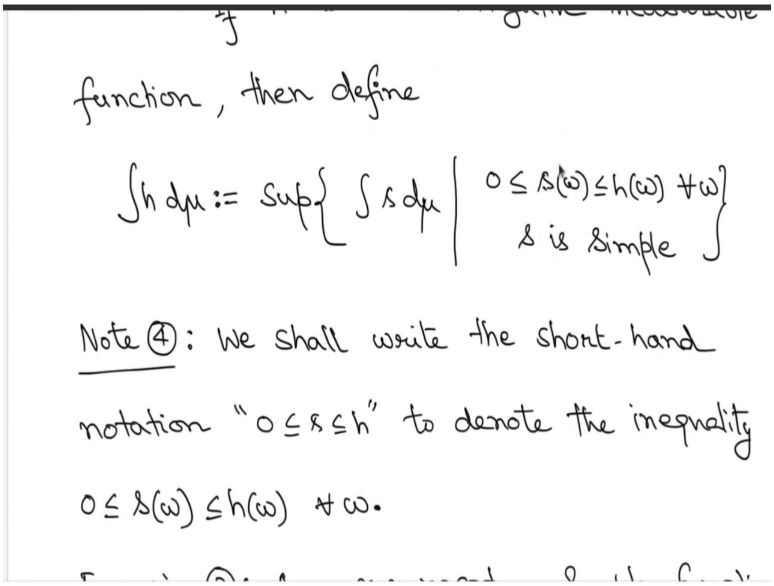

And look for all the simple functions that are dominated by h. So, take all points in the domain and find out simple functions s, such that $s(\omega)$ is dominated from above by h. So, if the values of the smallest falls between 0 and h. So, take such simple functions. Then, what you can look at is the corresponding integration for simple functions. This is already defined because you are taking the simple function to be non-negative. So, these values whatever they are they are non-negative including the value $+\infty$, this could happen.

Now, for all such simple functions you look at this values $\int s\, d\mu$ and collect these values consider the supremum. Whatever that supremum is you assign that value to $\int h\, d\mu$, again you have to be careful that these values whatever they are could be $+\ \infty$.

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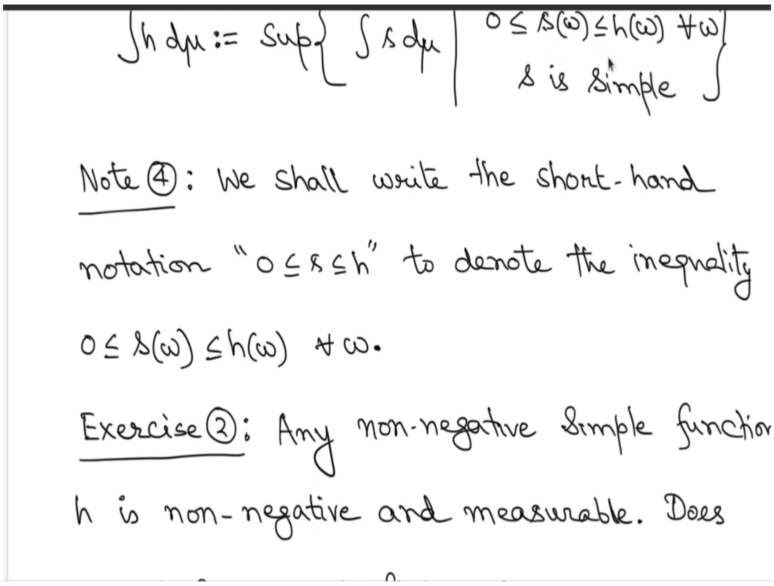

Now, as we shall see is that we are going to extensively use such inequalities between functions and to ease the notation what we are going to write is this thing. So, here s and h are functions. So, by this notation $0 \le s \le h$, we simply mean that this inequality holds for all points in the domain Ω. So, for any points $\omega \in \Omega$ function value $0 \le s(\omega) \le h(\omega)\ \ \forall\omega$. so that is it. So, you follow this notation.

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$0 \leq s(\omega) \leq h(\omega) \ \forall \omega$.

Exercise ②: Any non-negative simple function h is non-negative and measurable. Does the definition of $\int h \, d\mu$ in step 2 and step 3 match?

Exercise ③: Check that the collections of simple functions appearing on the right

function, then define

$$\int h \, d\mu := \sup \left\{ \int s \, d\mu \ \middle| \ \begin{array}{c} 0 \leq s(\omega) \leq h(\omega) \ \forall \omega \\ s \text{ is simple} \end{array} \right\}$$

Note ④: We shall write the short-hand notation "$0 \leq s \leq h$" to denote the inequality
$0 \leq s(\omega) \leq h(\omega) \ \forall \omega$.

Exercise ②: Any non-negative simple function h is non-negative and measurable. Does the definition of $\int h \, d\mu$ in step 2 and step 3 match?

Exercise ③: Check that the collections of simple functions appearing on the right
hand side of the definition in step 3

Now, with that at hand. So, we have now talked about the integration of non-negative measurable functions, but you have to be careful here. So, there are a couple of points that needs clarification. So, the first thing, in step 2 you have defined integration of non-negative simple functions. So that was defined as some kind of a linear combinations. So that is one definition that you have, but then you could also consider the definition that is given in step 3.

So, assume that there are some ways of defining integrations for simple functions that are below a given simple function h. So, if h is given to be simple, and you consider all simple functions below small h and suppose you can define this integration, then does this match? So that is the interesting question. So, take h to be a non-negative simple function, then $\int h \, d\mu$ can be defined as, as defined in step 2.

But then there is another way you look for all the simple functions that are below this given simple function h, look for all those integrations and consider the supremum, that is another definition of h that is being considered in step 3.

So, try to see that these two definitions match, these two definitions must match for all the things to make sense. So, for step 2 and step 3 to fit together, you need this kind of a consistency condition. So, please check this.

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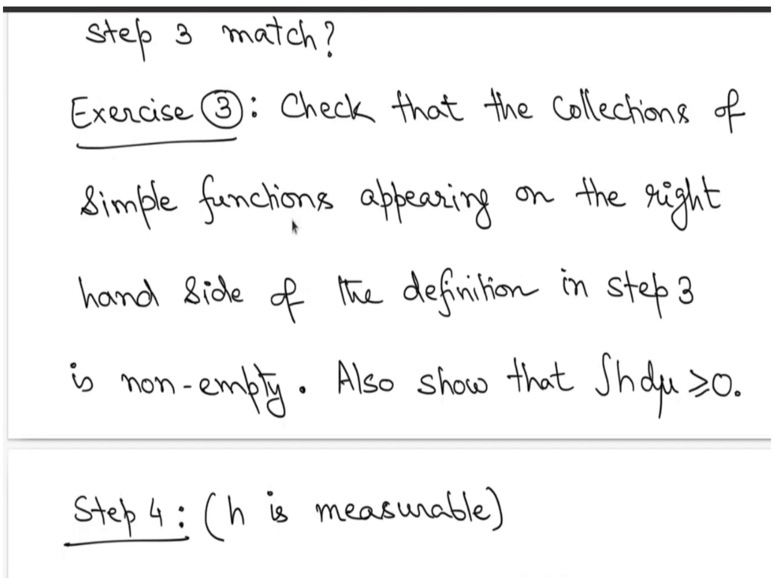

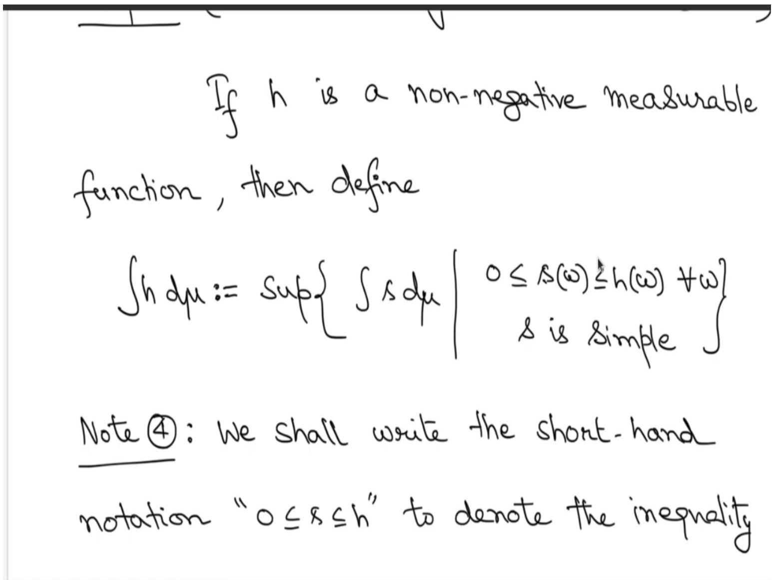

But in addition, another important thing happens is that the collection of simple functions that you see in this definition, so let us go up. So, we are considering all simple functions that are below the given function h. So, here h is a non-negative measurable function and you are considering all simple functions that fall below h.

What we are saying is that there is at least one simple function that is there. And therefore, you can consider its integrals values. And therefore, that supremum that you are considering that supremum is not over any empty set. So, there are certain values of integrations of non-negative simple functions. So, those values will appear there and you are considering supremum of those quantities.

So, in particular, this supremum is well defined, because it is not a supremum over our empty set. And this will give you some interesting values. So, in particular, you can also show that $\int h \, d\mu$ is non-negative, as long as h is non-negative and measurable.

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is non-empty. Also show that $\int h \, d\mu \geq 0$.

Step 4: (h is measurable)

Given a measurable function h, first write $h = h^+ - h^-$ (refer to the notes of Week 3 for notations). Since h^+ and h^- are non-negative measurable, the

So, with that at hand, so we have now discussed integrations up to the case when h is measurable function, but taking only non-negative values. But what do you do for the case when h is measurable, when h takes signed values. So, again, you go back to the notations for week 3, there we have discussed this h^+ and h^- which are the positive part and negative part of the given function h. h is now taking values with signs then you split it and then h^+ and h^- the positive parts and negative parts will now be individually non-negative and measurable.

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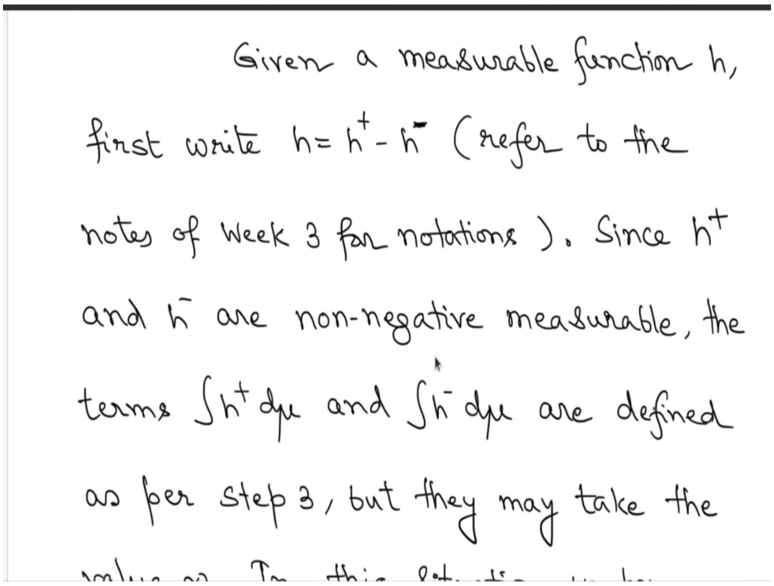

And for h plus and h minus you can separately talk about their integrations.

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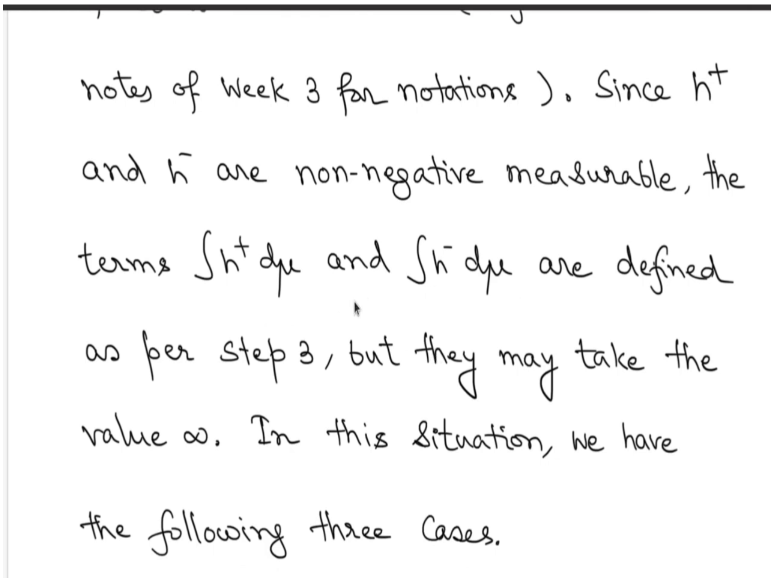

So, then these things are defined. So, these are certain values which are non-negative. So, integration of h^+ is non-negative integration of h^- is non-negative by the exercise which we just mentioned.

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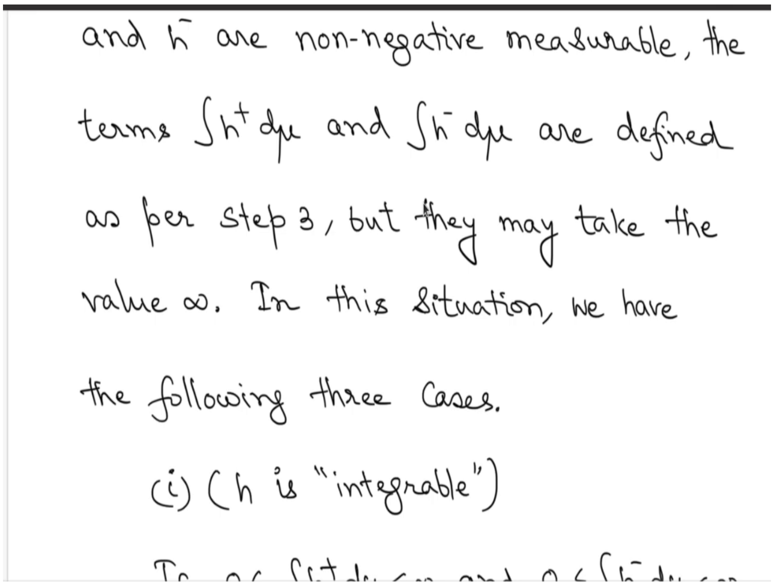

But then with these values in hand, we are now going to define the integration of h. So, you have following 3 cases. So, you have to be careful here.

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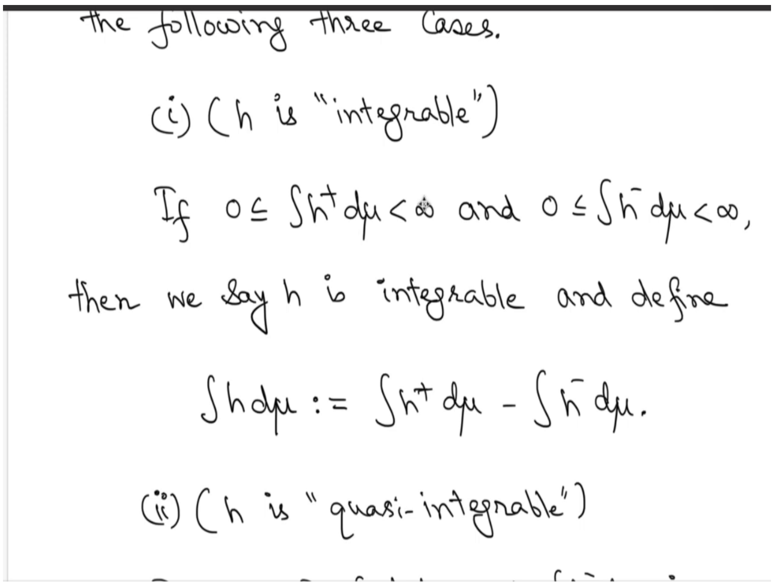

So, the first case is when we shall refer to the case as h is integrable. So, what is this case, so in this case $\int h^+ d\mu$ and $\int h^- d\mu$ both are finite quantities. So, again they are known to be non-negative, but I am saying that in this case when both of them takes finite values you say that the given function h is integrable and define the integration of h as the difference of these two integrations. So, that is exactly the idea again, you are keeping that linearity structure at hand.

So, linearity is being built into the definition. So, first linearity was built in when you went from integrated functions to simple functions, but then for non-negative measurable functions, we use some supremum. So, you have to be careful, we have to discuss about the linear structure there, but then when you come to the general case of measurable functions and split it into positive part and negative parts, then you are building in again certain linear structures here. So, in this case, you say that h is integrable and assign them value of $\int h\, d\mu$ as the difference of these two integrals.

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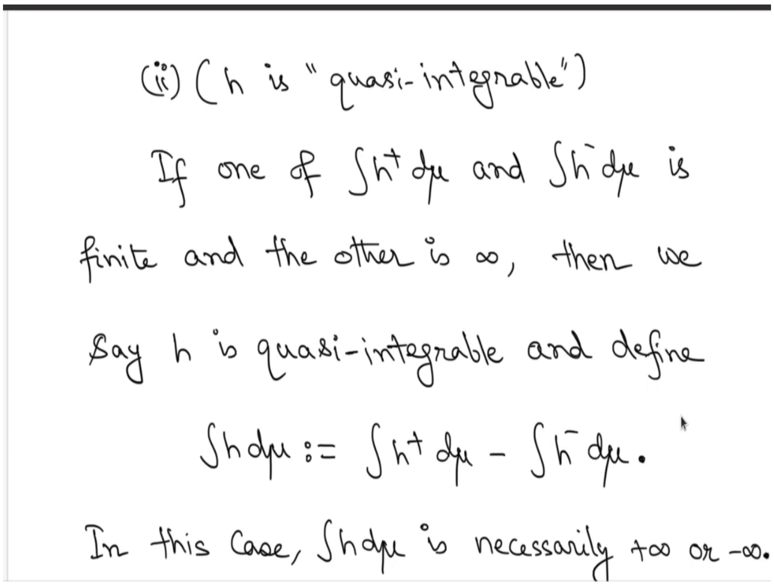

But then there is another interesting case when integration of h can still be defined. So, if it so happens that $\int h^+ d\mu$ and $\int h^- d\mu$, 1 of them is finite, $\int h^+ d\mu$ and $\int h^- d\mu$ are non-negative, but may take their value $+\infty$. So, I am saying that let one of them be finite and the other be infinite.

So, in this case, we can still define the difference between these two values because $\infty - \infty$ the situation is not appearing, one of them is finite, and the other is infinite. So, in this case the difference is still defined and therefore, you take that value and assign it to $\int h\, d\mu$. In this case you say that h is quasi-integrable, you do not say h is integrable, you say h is quasi integrable. In this case $\int h\, d\mu$ is still defined it will take values $+\ \infty$ or $-\ \infty$ depending on the situation.

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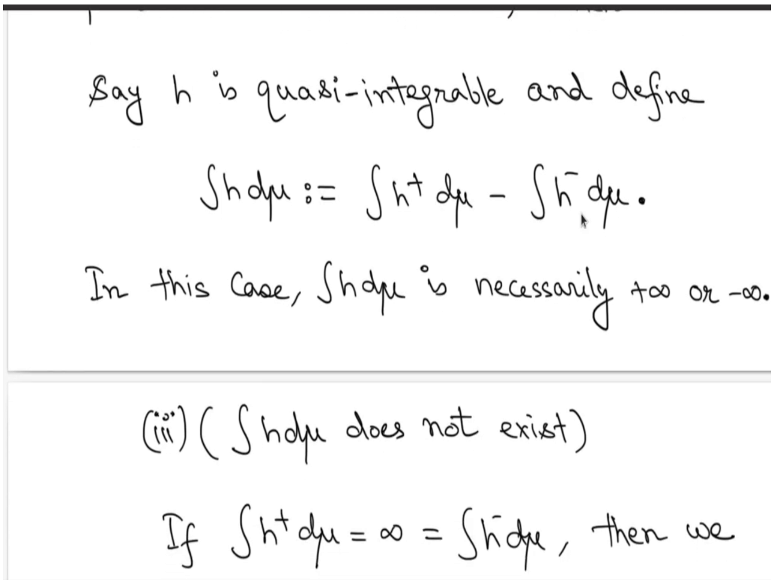

When h^{-} gives you an infinite integral and h^{+} gives you a finite integral. So, then $\int h\, d\mu$ will be defined and it will take the value $-\ \infty$. So, there is this quasi integrability case that you should be aware of.

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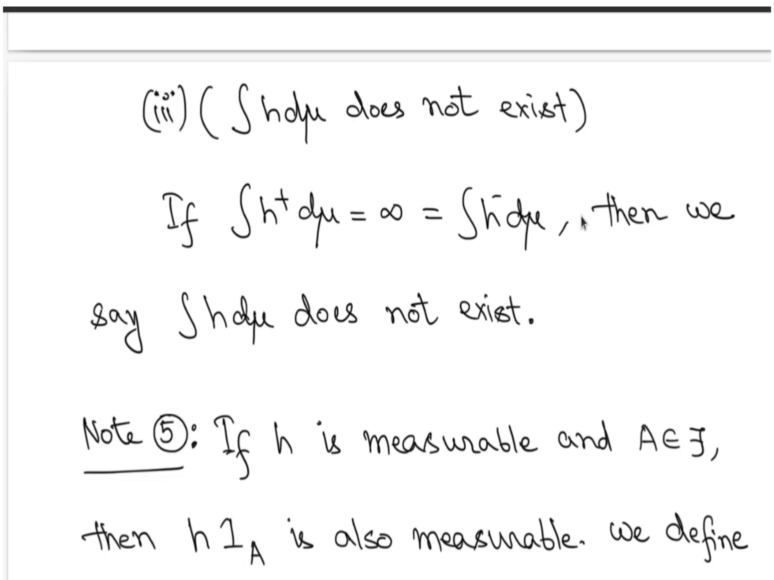

But then there is another case when we say we cannot define the integral, $\int h\, d\mu$ does not exist. What is this case? So this is the case when both $\int h^{+}\, d\mu$ and $\int h^{-}\, d\mu$ are ∞ . So, in this case both h^{+} and h^{-} has infinite area under their curve. So, in this case we say that $\int h\, d\mu$ does not exist and we are not going to define it. So, this is the case we are not going to consider in practice.

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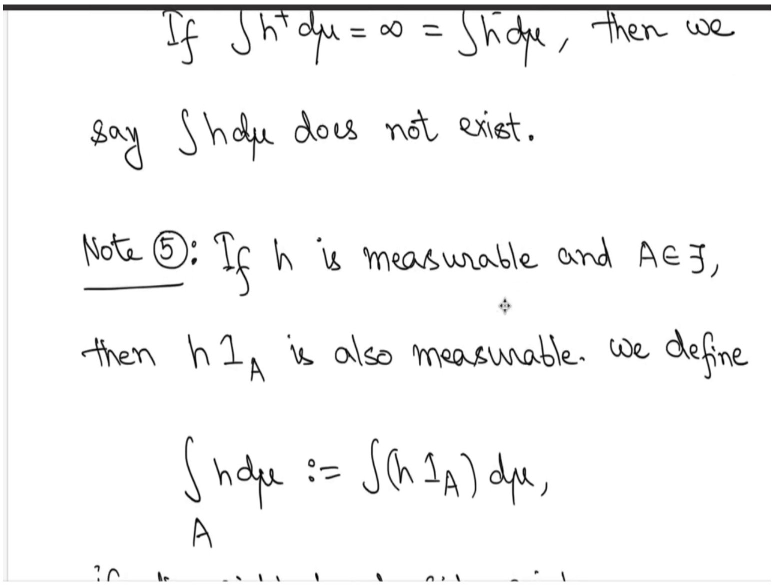

Now, with this in hand, we have now completed our description of n integration procedure, but now we are going to talk about properties of this, but here there are certain interesting points that you should be aware of.

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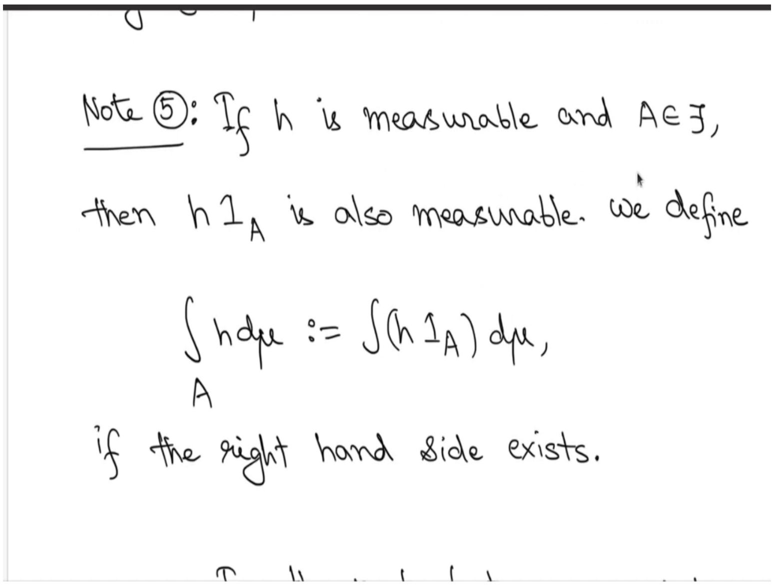

So, if h is measurable, and you take a set coming from the domain side, remember 1_A will be measurable then. So, you can consider the product $h\,1_A$. So, this is a nice measurable function now, so you can now talk about integration of $h\,1_A$. So, look at this integration. So, if the right hand side exists, you are now going to say that whatever that value is, that is the value of $\int_A h\,d\mu$.

So, you are now not integrating over the whole set, you are just saying you are going to concentrate your attention on the values of h on the set A. So, here what is happening is that outside the set A 1_A is 0, so on points which are coming from A^c 1_A is 0, so the product of function value of h and 1_A for those points will be 0.

So therefore, that will not contribute to your integrations. So, that is basically the idea that we are following. So, with this in hand what we are saying is that whatever is this value of this integration, if you can define this right hand side, if you can compute this value, whatever this is you assign it to $\int_A h\,d\mu$. So, this is the important notation that we are going to use later on.

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In the next lecture, we shall look at the properties of the integral above. We use this integration method to define expectation or mean of an RV.

Definition ① (Expectation or Mean of an RV)

So now, in the next lecture, we are going to focus on the properties of this integration procedure. So, it will take some time we will have to develop several properties of this integration procedure over different lectures. One of the important things that we need to verify, which has not been proved as of yet is that the linearity structure. So, we have built in the linearity structure when we moved from indicators to simple functions and then from non-negative measurable functions to general measurable functions.

So, this is being built in, but for non-negative measurable functions, we use certain supremum structures, and it is not clear if the linearity holds there. So, you have to be careful with this point and we are going to discuss such properties in the following lectures.

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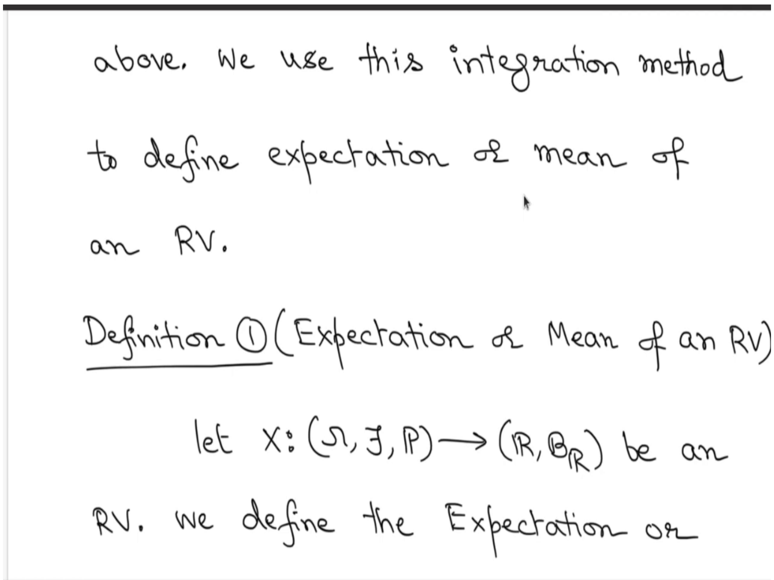

But we are going to use this integration procedure to define expectation or mean of random variables. So, what is this.

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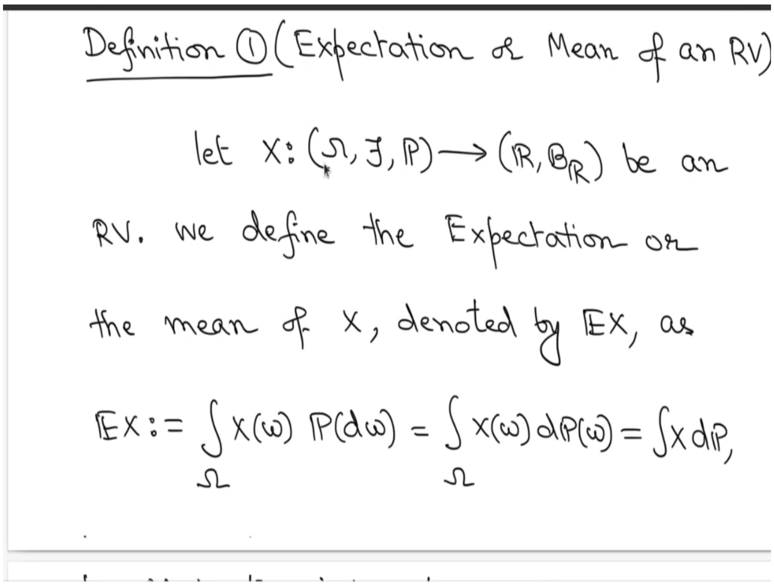

So now, consider this definition. So, take a random variable, so therefore, it is defined on some probability space taking values in the real line together with the Borel $\sigma -$ points, so that is corresponds to the measurable structure of the function X. Define the expectation or the mean of the random variable X, which you are going to denote by expected value of X or $\mathbf{E}\, X$.

$$\mathbf{E}\, X := \int_{\Omega} X(\omega)\, \mathbb{P}(d\omega) \; = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega) \; = \int X\, d\mathbb{P}$$

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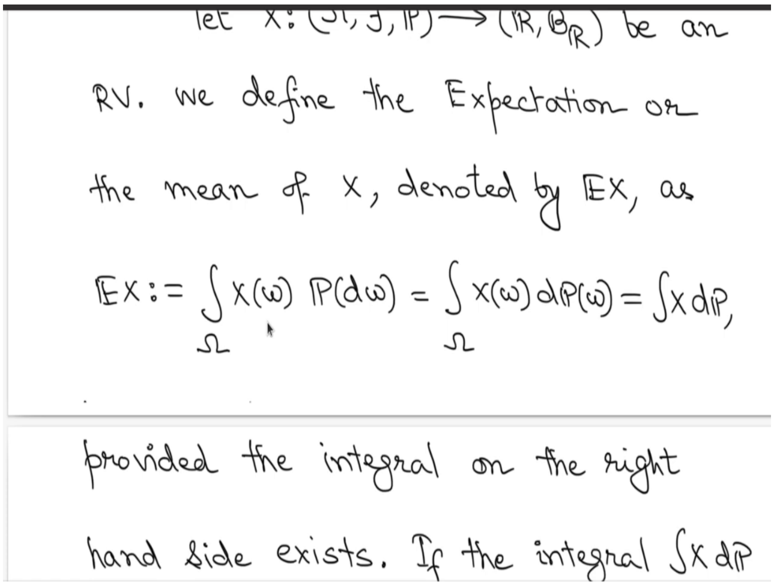

So, this is a shorthand notation, and if you can define this integration as part of the integration that we discussed in the previous part, if you can define this assign that value to this quantity, so that is the notation that you are going to use, if you can define this integration, then you are going to say that that is the expectation or mean of this random variable X.

So, be careful there are two situations when we can define these expected values of X. First situation when X is integrable in that case integration of X^{+} and X^{-} should be finite. So, integration of X will therefore be finite, but there is another case when X is quasi integrable in that case, we are now saying that we will still consider the expectation or the mean which are now taking values either $+\ \infty$ or $-\ \infty$.

So, be careful this is an extension from the ideas that you have already seen in your basic probability theory, but here we are allowing the expectation value to take $+\ \infty$ or $-\ \infty$. So, this is the case that will happen when X is quasi integrable. Of course, if the integrations of X^{+} and X^{-} with respect to the probability measure becomes infinite, if both the integrations become infinite for X^{+} and X^{-}, then you of course cannot define the integration of X.

So, that is the case that we are going to throw out, but from usual steps that you have discussed in your basic probability, we are now making an addition that when X is quasi integrable, you can still consider expected value of X as per the definition given here.

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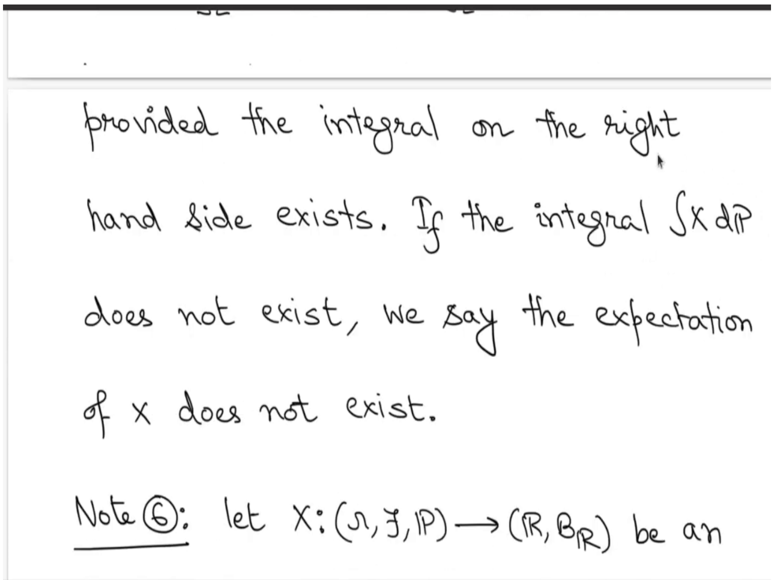

So again, if the integration does not exist, we say the expectation of X does not exist. So, just as simple as that, otherwise expectation will exist if X is integrable then $\mathbf{E}\,X$ will be finite. Otherwise, if X is quasi integrable then $\mathbf{E}\,X$ will be taking the values $+\infty$ or $-\infty$.

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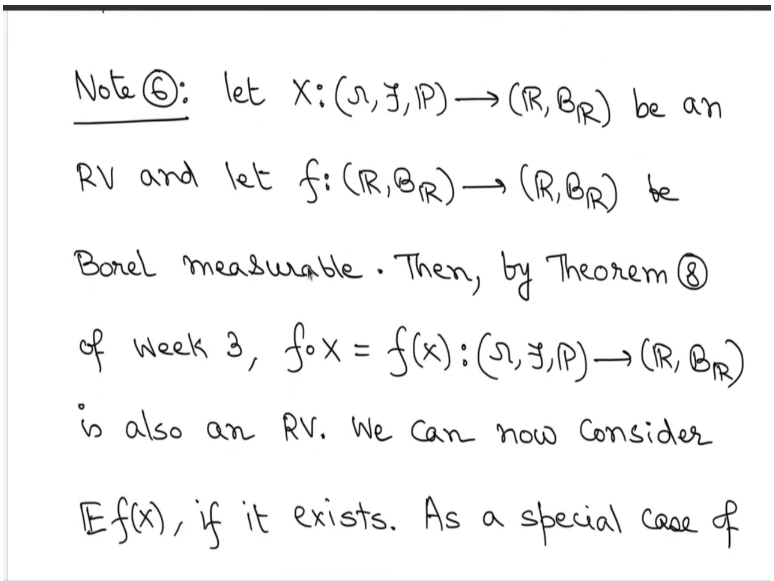

Important fact is this, that if you take a random variable and apply a measurable function on top of it. So, compose with given measurable function f. So, if the composition is well defined, it will give you a random variable, again defined on the same probability space.

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RV and let $f : (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be Borel measurable. Then, by Theorem 8 of Week 3, $f \circ X = f(X) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is also an RV. We can now consider $\mathbb{E} f(X)$, if it exists. As a special case of this observation, we consider the n-th

You can now consider expected value of that random variable now. So, these kind of structures will follow from the discussions that we did for measurable functions in week 3. So, you can now try to consider these $\mathbf{E}\, f(X)$.

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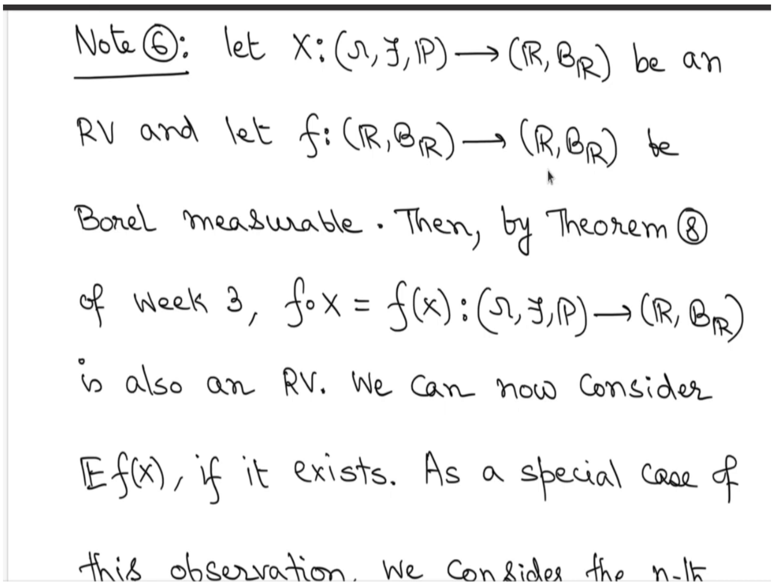
is also an RV. We can now consider $\mathbb{E} f(X)$, if it exists. As a special case of this observation, we consider the n-th moments of X about any point $c \in \mathbb{R}$ as $\mathbb{E}(X-c)^n$, provided they exist.



What we are saying is that as a special case of this observation, you can now choose your functions, which are nice continuous functions defined on the real line. f is taken from the real line to real line itself, so it is a nice function.

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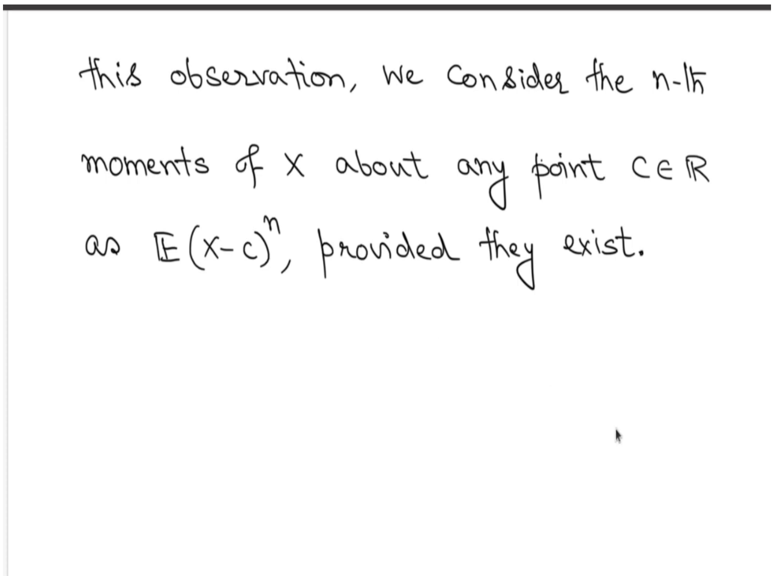

So, you could take functions of the form $(x - c)^n$. And once you compose with X, so you will get expressions of this form. So here what we are saying is that we are going to consider integrations of these kind of measurable functions, these kind of random variables and these we are able to consider as the n-th moment of the random variable about any point c taken in the real line.

As long as these exist you can talk about these moments. So, in the later lectures, we first discuss properties of the integration procedure, and then prove the usual properties of the moments that we know about.

And you will see that these properties will now be proved for general random variables, we are not going to restrict our attention to especially the discrete cases or other continuous random variables, we are going to prove the inequalities or bounds that you know about for moments of random variables for general random variables, we do not have to stick to discrete cases or other special cases. This is a very nice advantage provided by the integration theory that we have started discussing. We will continue this discussion in the next lecture.