Measure Theoretic Probability 1
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Lecture 25
Distribution Functions and Probability Measures in higher dimensions

Welcome to this lecture, this is the final lecture of week 5. We start this lecture by moving on to the slides and recalling whatever we have done in this week.

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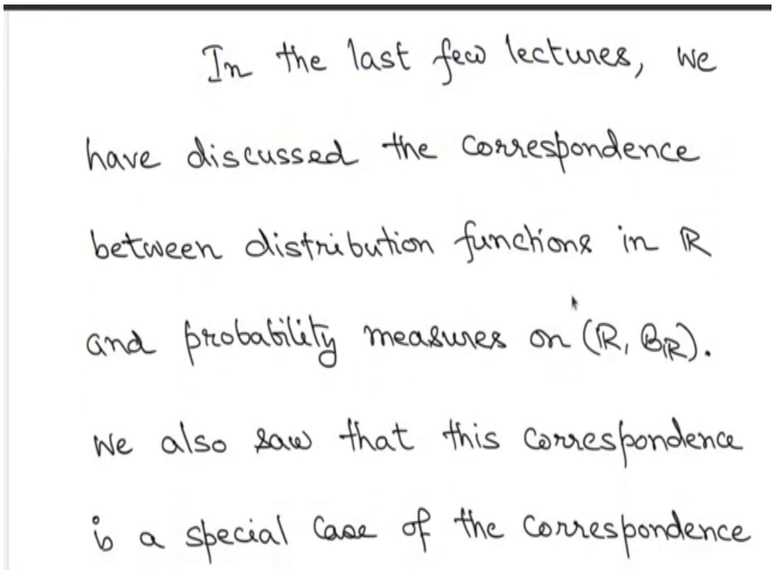

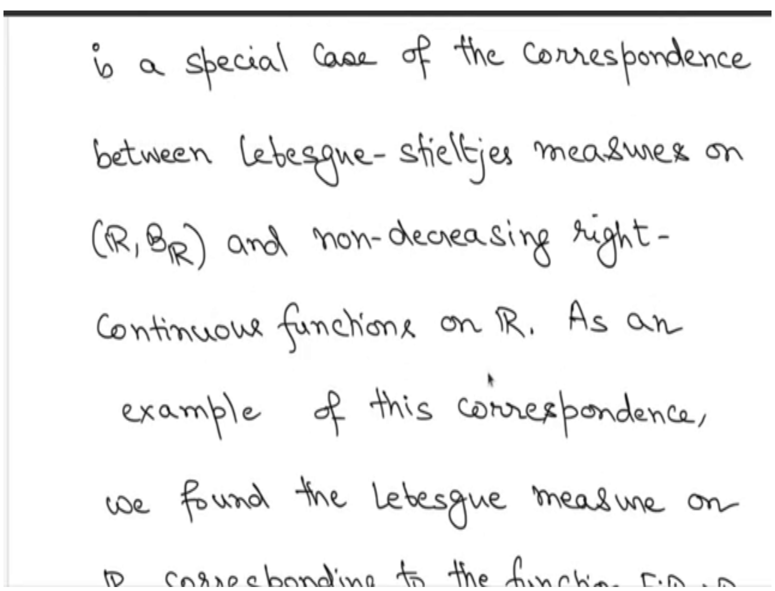

So, in the last few lectures specifically for the lectures in this week, we have discussed the correspondence between distribution functions defined on ℝ and corresponding probability measures on this measurable space real line together with the Borel sigma field .

We also saw that this correspondence between the distribution functions and probability measures is a special case of a more general correspondence between Lebesgue measure on this measurable space and non-decreasing right continuous functions defined on real line . So, this is appears as a special case of the correspondence between Lebesgue Stieltjes measures and the class of functions which are non-decreasing and right continuous. And in fact, as a consequence or as an example of this correspondence between Lebesgue Stieltjes measures and non-decreasing right continuous functions.

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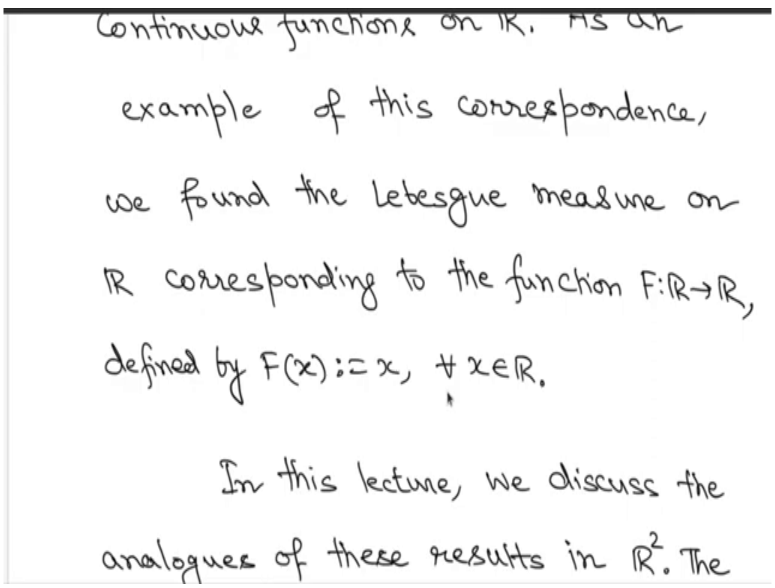

We have found this example of Lebesgue measure on the real line and this corresponds to the function if defined on the real line taking real numbers as their values defined by $F(x) = x$. So, this is the identity function. So, if you look at this right continuous and non-decreasing function by that construction given earlier, you would be able to construct a measure which you are now going to call as a Lebesgue Stieltjes measure. So, in the previous lecture, we have discussed extensively about properties of this Lebesgue measure and we have seen that Lebesgue measure assigns the usual lengths to the intervals. So, you can think of a Lebesgue measure as the extension of the usual length to all the other Borel subsets on it.

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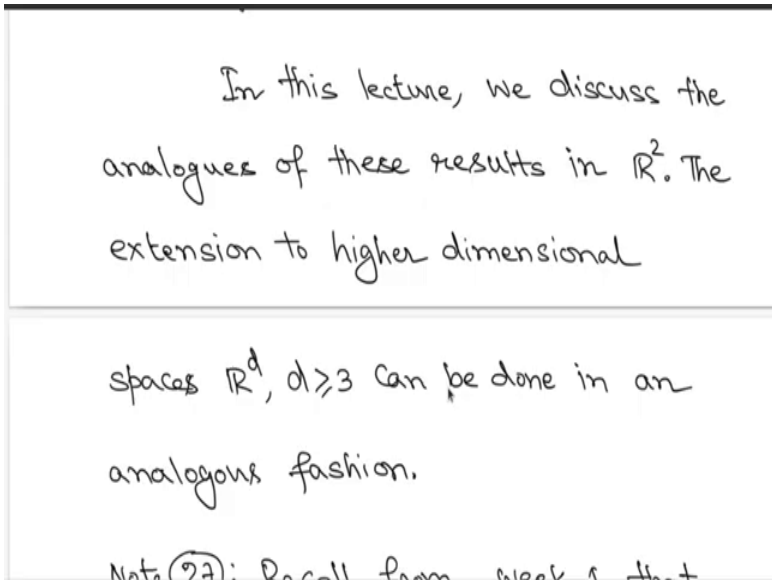

So, in this lecture, what we focus on is the extension or analogues of these results in higher dimensions. So, specifically for simplicity, most of the results that are stated in this lecture are on dimension two. So, you are setting the results on $\mathbb{R}^2$, but extension to higher dimensional spaces $\mathbb{R}^d$ for $d \geq 3$ can be done in an analogue fashion. So, the techniques remain exactly the same, you just have to introduce the appropriate notation to take care of the increase in dimensions.

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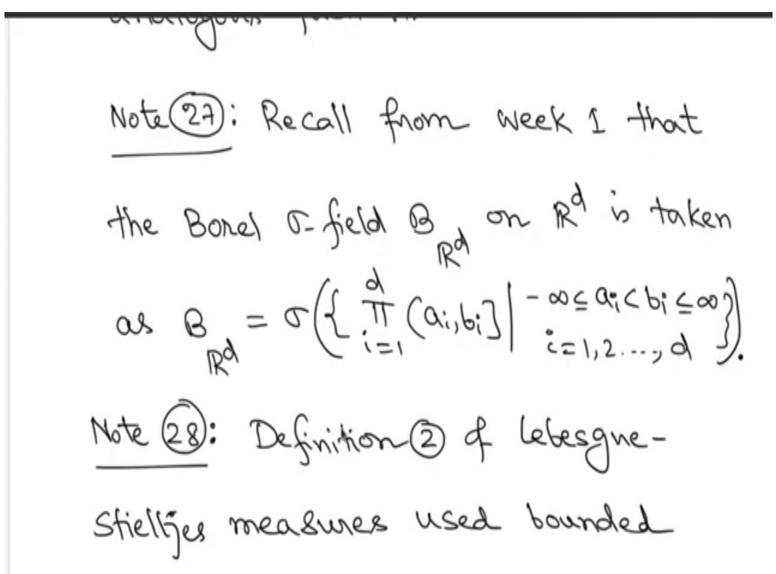

Now, recall first that the Borel sigma field on rd, which you would write it as $\mathcal{B}_{\mathbb{R}^d}$, so, this one is defined as $\sigma\left(\left\{\prod_{i=1}^{d}(a_i, b_i] \mid -\infty \le a_i < b_i \le \infty,\ i = 1, 2, ..., d\right\}\right)$. So, this is a typical set in $\mathbb{R}^d$ and for this collection of sets try to generate the minimum sigma field and that is what we call as the Borel sigma field. So, this we had defined in week one. But now, we would like to figure out the corresponding versions in higher dimensions for this Lebesgue Stieltjes measures and the corresponding non-decreasing right continuous functions.

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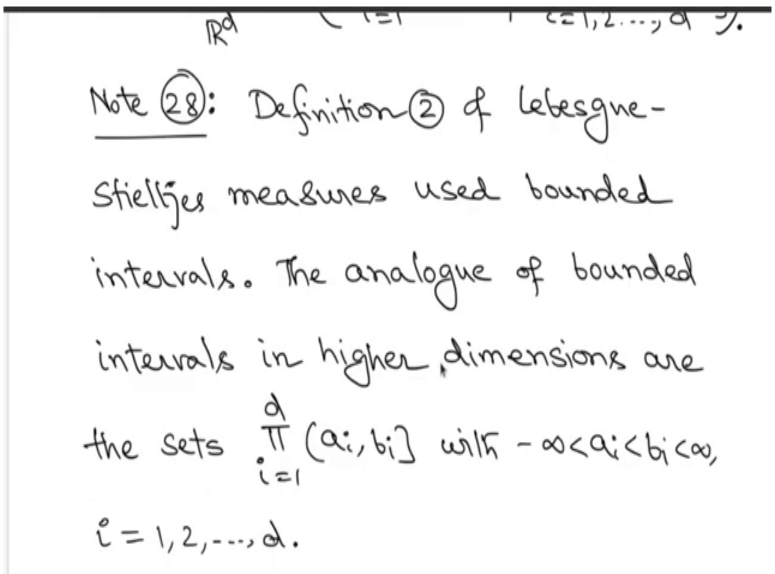

So, first of all, let us take up the case of Lebesgue Stieltjes measures. So, when we define the Lebesgue Stieltjes measures on the real line, we took help from this bounded interval. So, we said that these are the measures defined on the real line together with that Borel sigma field, such that these measures should associate finite mass to bounded intervals.

But the analog for bounded intervals in higher dimensions we have to identify before going into the definition of Lebesgue Stieltjes measures in higher dimensions. So, what are the corresponding analogs? So, what do we use? We use this default product, as just talked about in the Borel σ period is generators. So, you again take such left open right close intervals, D, many of them take the product of this.

So, that is a second $\mathbb{R}^d$. So, take such things as bounded intervals provided the limit points ai bi are now real numbers. So, in general for the case of generating sets, you allow these a_i or b_i is to be ∞ of course, these are subsets within $\mathbb{R}^d$. So, ∞ and $-\infty$ kind of points are not included, but still you have to incorporate sets of the form $[a, \infty)$. So, that is what we had understood as the notation.

So, now, what we are saying is that specifically when you are talking about bounded intervals in higher dimensions, you look at this type of set. So, these are what are the analogs of bounded intervals in higher dimensions. So, you use this bounded $(a_i, b_i]$. So, a_i, b_i here are real numbers. So, take the default product of such intervals great. So, now we are ready to define what are the Lebesgue Stieltjes measures on the d dimensions.

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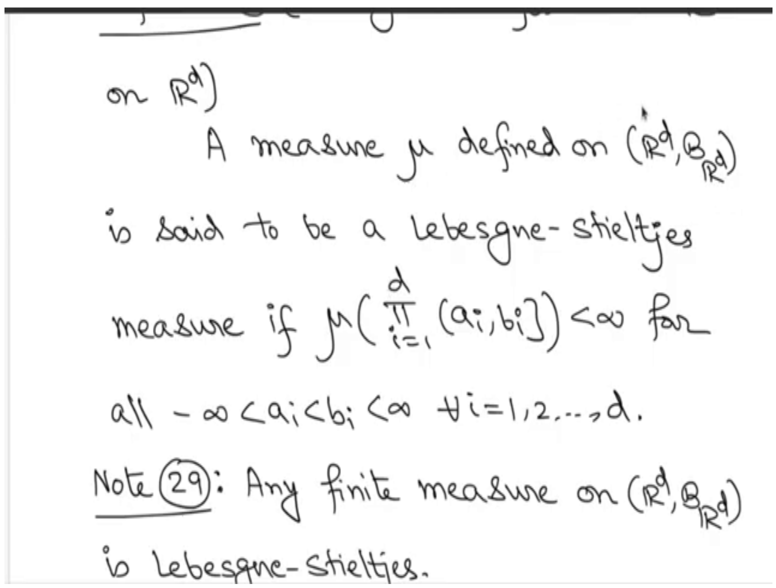

So, take a measure μ defined on this measurable space $\mathbb{R}^d$ together with Borel sigma field. So, you say that this is a Lebesgue Stieltjes measures if it associates finite mass to all such type of sets . So, you take this a_i, b_i to be real numbers take the default product whatever that set is. So, look at the size of that provided by μ if for all subsets μ assigns finite mass then you say that this measure μ is a Lebesgue Stieltjes measure .

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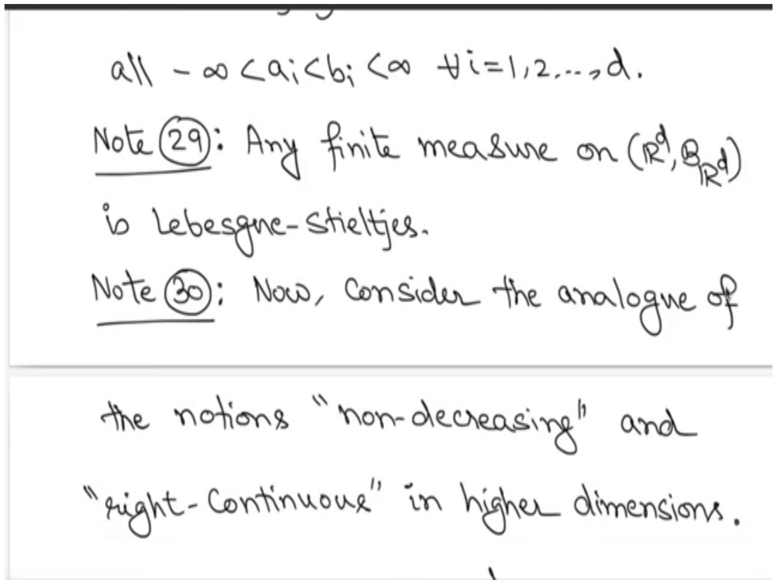

So, now again as done for the dimension one case you would ask for examples, and a quick observation will tell you that any finite measure on these measurable space is Lebesgue Stieltjes but as exactly done in one dimension, so, we are now going to look for infinite measures which are also Lebesgue Stieltjes. So, we will come to that in a minute, but then we also have to consider the corresponding version for the functions .

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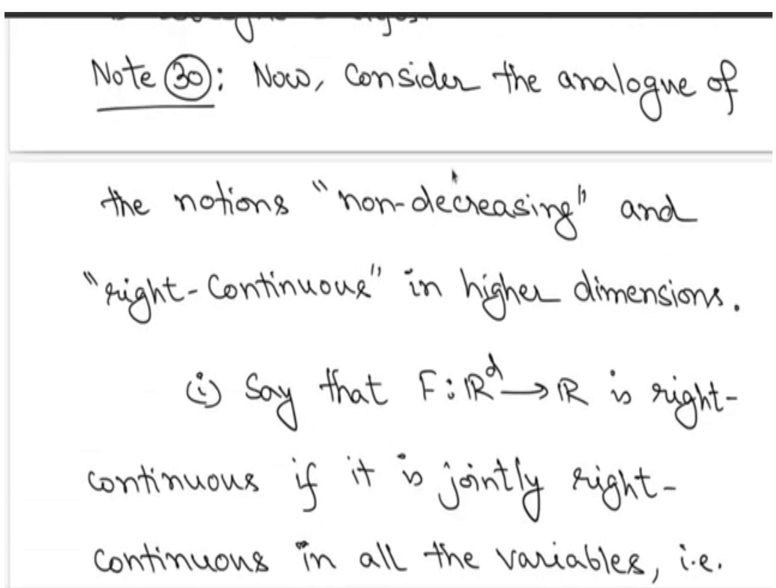

$$\lim F(x_1^{(n)},\dots,x_d^{(n)}) = F(x_1,\dots,x_d).$$

$x_i^{(n)} \downarrow x_i$

$i=1,2,\dots,d$

This is the same condition that we saw in Note 9 of week 4 for the distribution function of an $\mathbb{R}^2$-valued random vector.

So, you have to figure out the corresponding analogues for these terms non-decreasing and right continuous. So, what do they mean in higher dimensions? So, let us start with the case of right continuity. So, you say that a function defined on $\mathbb{R}^d$ taking real numbers as their values is right continuous if it is jointly right continuous in all the variables. So, what do I mean?

So, I am saying that look at a point $x_1, \dots, x_d$ in d dimensions. So, this is a vector So, corresponding to each of the coordinates approximate it from the right so, you have approximated it from the above each coordinate has a sequence which is approximating for above and then for each coordinate since you have the sequence you can construct this vector once more, which is basically approximating this vector from above in some sense.

So, coordinate wise it is approximating from above so, for all such things for all such sequences, you would like to have that this limit is exactly the function value at the point so, if it is happens you say that it jointly right continuous if all the variables. So, except in one of the coordinates, if you fix all the other coordinates, then you are basically getting back the right continuity in that variable.

So, all of that is covered under this joint like continuity, and this is of course, the same condition that we have seen in note 9, of week 4. So, where we are discussing the distribution function of $\mathbb{R}^2$ valued random vector or $\mathbb{R}^2$ valued probability measure . So, the idea remains the same there exactly was this condition in two dimensions that we had discussed. So, just go back and compare this condition. So, this was exactly the same condition that we had written down there .

So, this is the exact analog of right continuity in one dimensions as you extend to higher dimensions. So, you have to talk about jointly right continuous . So, therefore, for simplicity we are going to refer to this concept of joint right continuity as right continuity itself, but we understand that we mean joint right continuity.

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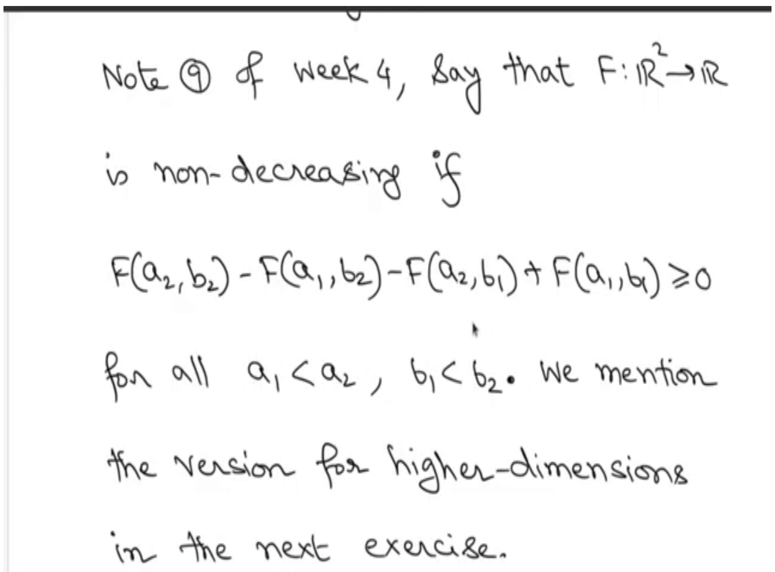

But now, let us come to the point about non-decreasing. Again, as understood from our first point about right continuity, these conditions should have some comparisons or similarities with whatever we have already seen for distribution functions corresponding to probability measures in dimension 2 or for random vectors in dimension 2. So, consider this case that you are looking at a function on $\mathbb{R}^2$ to $\mathbb{R}$. So, you say that this is non-decreasing, if this linear combination of values is non-negative.

So, for all $a_1 < a_2$, and $b_1 < b_2$. So, again you just go back and check the same condition holds for dimension two for distribution functions corresponding to probability measures. So, please check this. So, we had discussed this in week 4 already. So, taking motivation from that we now define the non-decreasing property to be exactly this. So, we are now going to mention what do we mean by non-decreasing in other higher dimensions. So, the here this is stated for two-dimensional case only. So, let us take up the case for d dimensions in the exercise.

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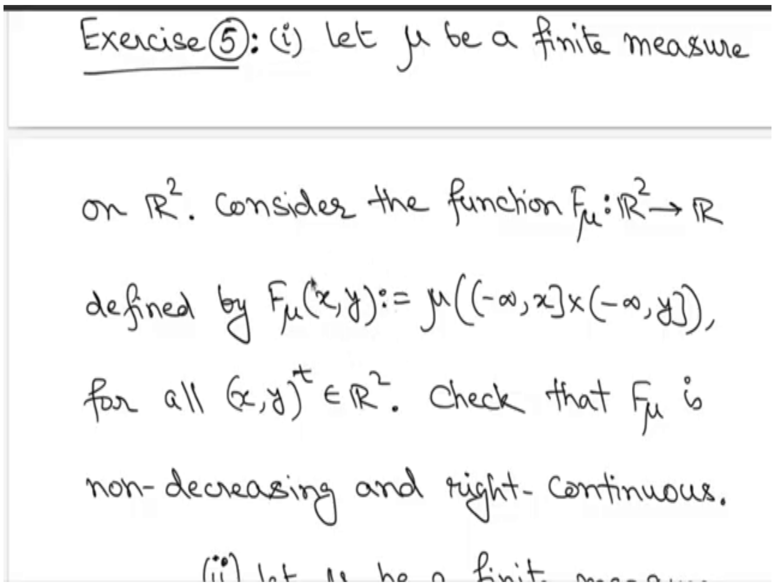

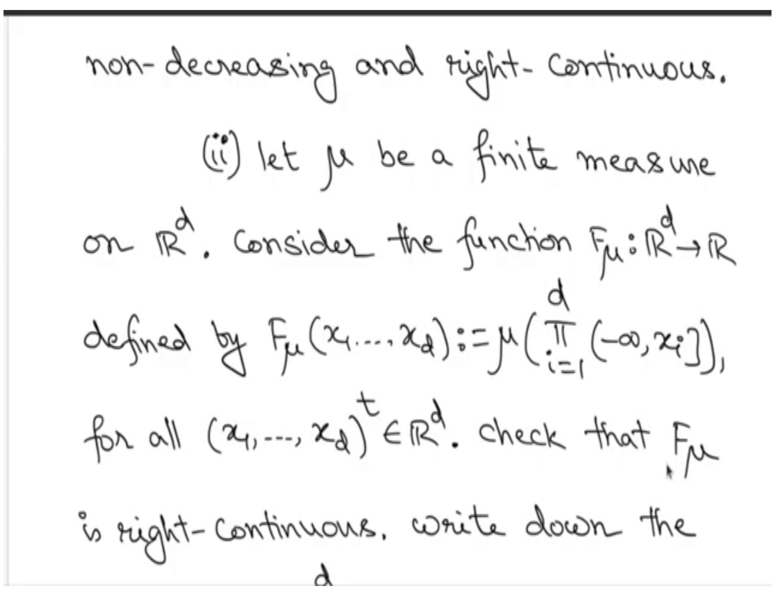

So, let us start with finite measures. So, let me we are finite measure on $\mathbb{R}^2$ and consider this function F_μ defined as; so, what do you do so? You again continue with the two dimensions, but here this is an important identification which is going to allow us to move to higher dimensions. So, here what we are doing we are choosing this point with coordinates (x, y) and assigning the value measure of this set. So, $(-\infty, x] \times (-\infty, y]$. So, look at that set look at the value associated to it or to the size associated to it by the measure μ. So, this is what we are considering here. So, since measure is finite this quantity is also finite, assign that value to the function at that point.

So, now, what you can check is that this function is non-decreasing and right continuous as for the definition given above. So, this is the exact connection between distribution functions and probability measures and we are extending it to finite measures and corresponding non-decreasing and right continuous function. So, we have got the exact analog, but you would ask what happens for general infinite measures which are Lebesgue Stieltjes. So, we are coming to that in a minute.

So, we are now coming to the D dimensional case, but now taking motivation from the two dimensional case that we just discussed in the first part of this exercise again define the function value at any point to be the size of such sets given by μ. So, size of subsets under the measure μ. So, again continue to assume that μ is a finite measure. So, these values are finite assign the value to this point. So, if you define the function if μ this way, then check that this becomes right continuous that is right continuous in all the variables together, jointly right continuous. Now, you would ask what is the corresponding version for non-decreasing?

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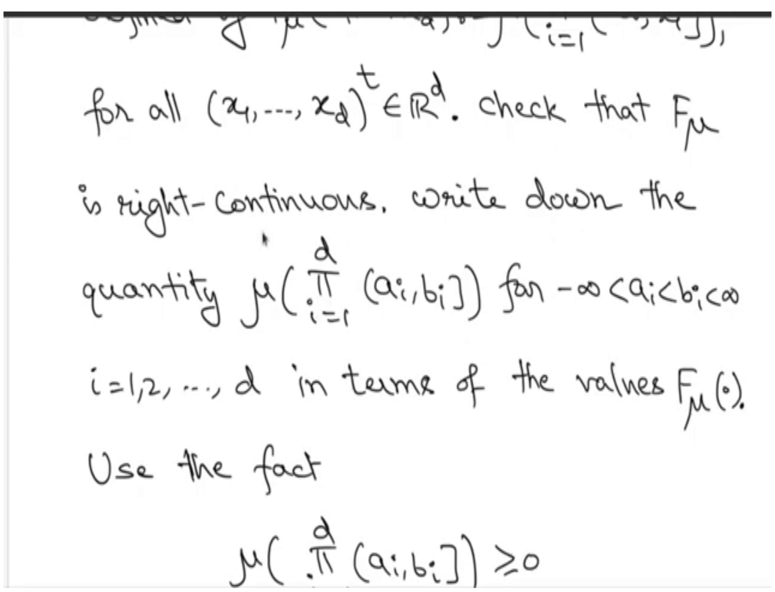

to get the non-decreasing property of F_μ.

(iii) Continue with F_μ as in (ii).

Fix $i \in \{1, 2, \dots, d\}$. Then, show that for every fixed $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_d$ in $\mathbb{R}$, the function

$$x \longmapsto F_\mu(x_1, x_2, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d)$$

To do this again we follow the same motivation that we did in the case of two dimensions. Again, we take the motivation from the case of probability measures; first write down this quantity which is the size of such product typesets. So, these are these left open right close intervals consider the default products here A_i's and B_i's are real numbers. So, these intervals give you certain finite bounded intervals in each coordinate and therefore, the default product of that is also a bounded set.

Now, what you should try to do is to write down this size under the measure μ in terms of the values of the function. So, basically what you did in two dimensions was that you look at the function values at the corners of these that set that you end up with right. So, in two dimensions you had this rectangle and you obtain the function values are on those points and wrote the size in terms of certain linear combinations of that.

So, here also in d dimensions, look at all these corners, try to evaluate the function values on there and try to write down this quantity the size of the set in terms of those values or those corners. And then once you have written down that formula in terms of the function value that these corners, so, you get some linear combination, use the fact that since μ assigns finite mass to all of these, but these finite mass must be non-negative by definition μ is given to be a measure.

So, therefore, this quantity has to be non-negative and hence, what you end up having is that, that linear combination for F_μ values of F_μ will be non-negative and that will give you the appropriate non decreasing property of the function . So, please try to work this out. But then you can continue with this function if μ and we are trying to make some connection with the usual non-decreasing property that we see in one dimension .

So, continue with this F_{μ} and fix a coordinate I. So, here the value of i is some values between 1 to d. Now, what do you do you forget about that coordinate look at all the other coordinates. So, here are the coordinate values are listed as x_1 up to x_d. So, except x_i. So, forget about that and fix all these other coordinate values . So, once you fix that, for this function F_{μ}, what you get is that once you keep varying the i-th coordinate which is now some x you get a function in one dimension. So, this is a function defined on the real line.

So, therefore, you can now consider this function and try to prove that as the given function is given to be non-decreasing what you have just formulated from the properties of the measure from the prior finite measure you can now try to show that this will also become non-decreasing in one dimension. So, therefore, non-decreasing in higher dimension implies non-decreasing in one dimension.

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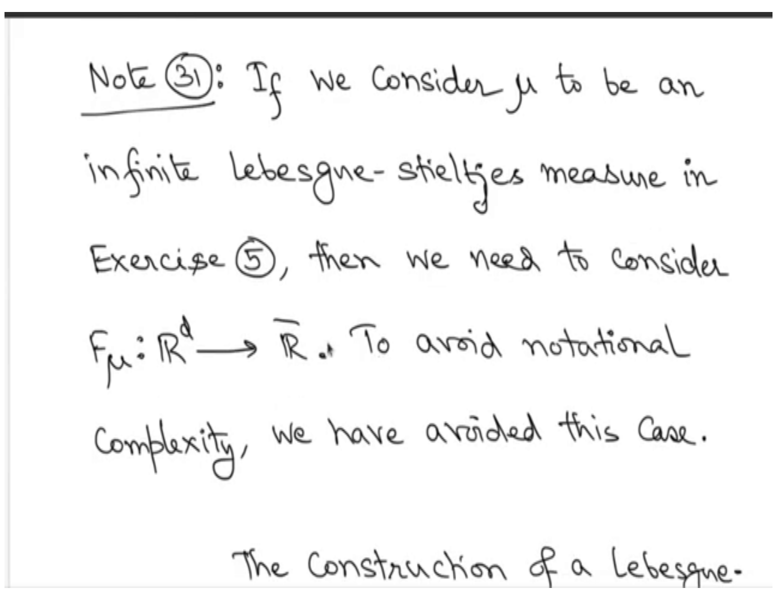

Now, if you consider μ to be an infinite Lebesgue Stieltjes measure then what will happen is that you have to consider the function to be taking values in the extended real numbers, because you are looking at size of sets which could be infinite. To avoid notational complexity, we have not stated the exercise for infinite Lebesgue Stieltjes measure. So, that was the reason for not stating this exercise for the infinite measure skills. But you can of course, do this thing, but you will have to consider the corresponding functions to be taking values in the set of extended real numbers. You can always spend time and try to formulate the appropriate version for these cases.

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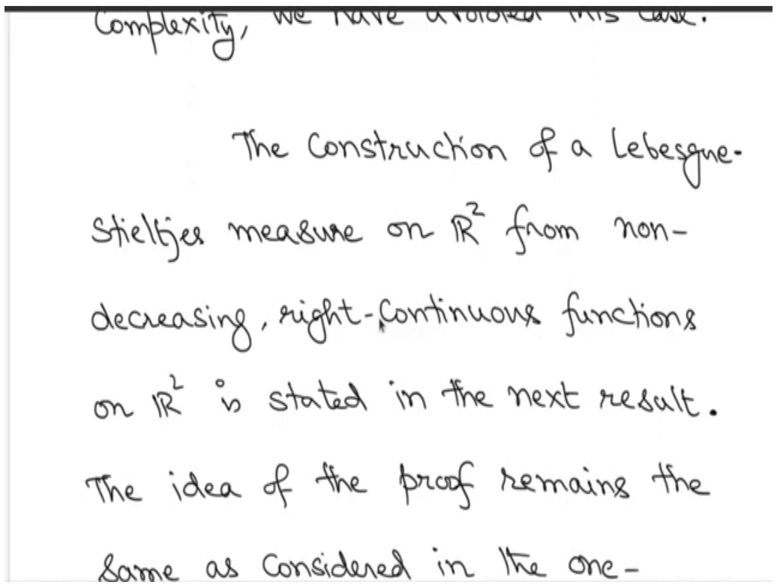

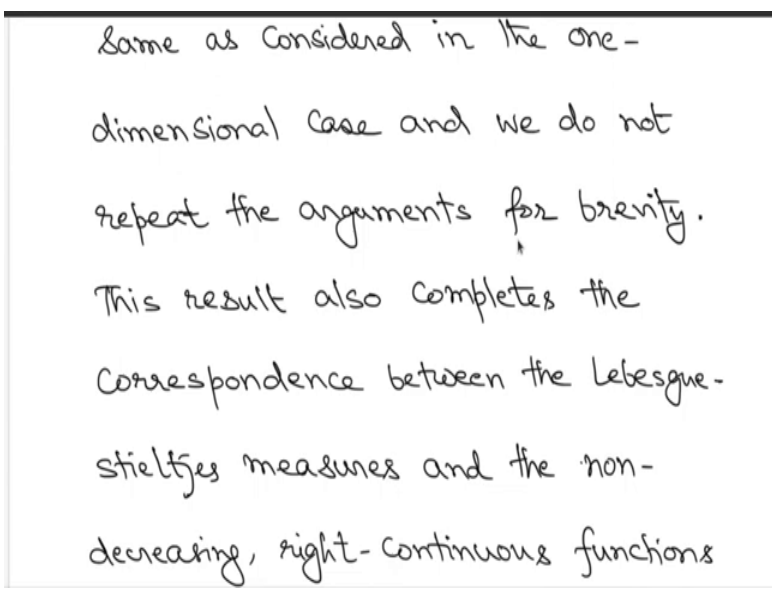

But now let us come to the main issue that we wanted to discuss. So, you would like to construct Lebesgue Stieltjes measures on $\mathbb{R}^2$ from non-decreasing right continuous functions on $\mathbb{R}^2$. So, in the exercise what we have already seen is that given the Lebesgue Stieltjes measures you can construct non-decreasing and right continuous functions on $\mathbb{R}^2$, we want to go the other way, we want to start with non-decreasing right continuous functions and construct the corresponding measures, this should be the exact same steps that we have followed in dimension one.

So, again the proof remains the same as considered in dimension one except certain appropriate notational complexity to take care of the dimensional matters. So, what do we do is that we do not repeat the same arguments and we focus mainly on the ideas behind this . So, this result what it does is that it completes the correspondence between Lebesgue Stieltjes measures in higher dimensions and the corresponding non-decreasing right continuous functions in this dimension . So, that is this idea.

So, we are considering the two-dimensional case and we are saying that once you have completed the construction of a Lebesgue Stieltjes measure corresponding to a non-decreasing right continuous function on $\mathbb{R}^2$ then you have finished the correspondence between the Lebesgue Stieltjes measures and non-decreasing right continuous functions in $\mathbb{R}^2$. So, let us move ahead and look at that statement.

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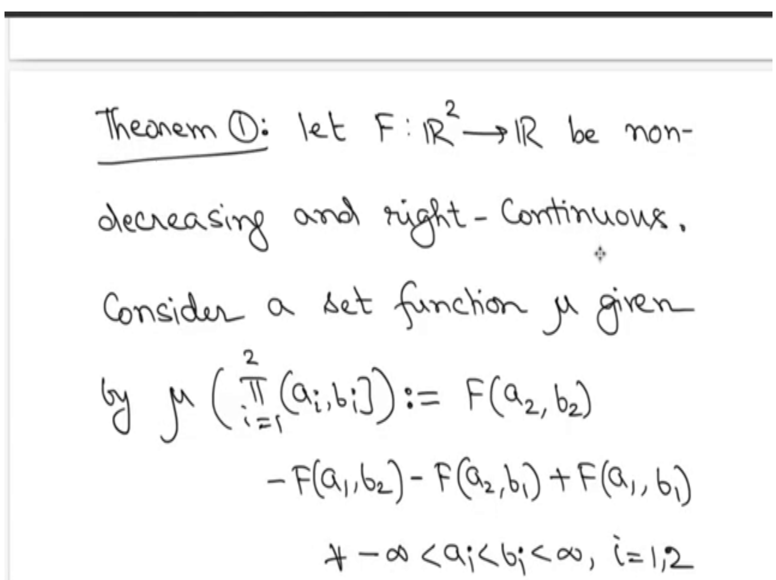

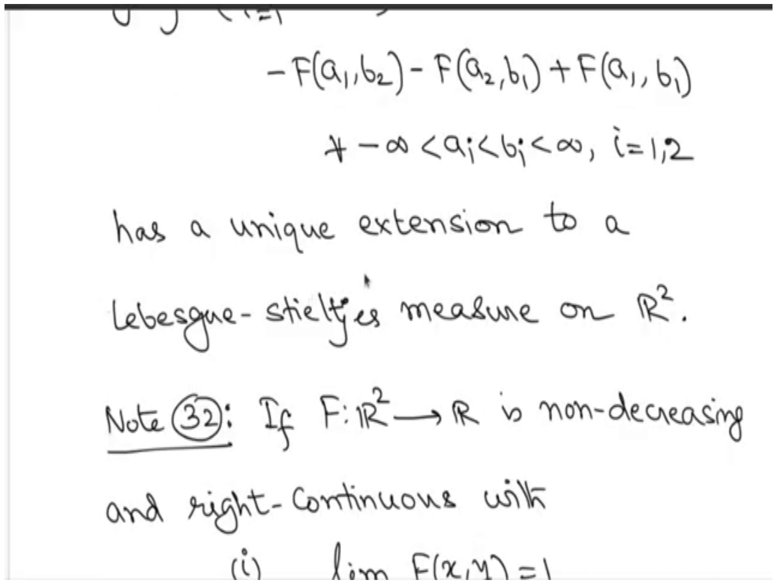

$$- F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)$$

$$\forall -\infty < a_i < b_i < \infty, \; i = 1, 2$$

has a unique extension to a Lebesgue-Stieltjes measure on $\mathbb{R}^2$.

Note (32): If $F: \mathbb{R}^2 \to \mathbb{R}$ is non-decreasing and right-continuous with

(i) $\lim F(x, y) = 1$

So, this is we were stating in this theorem, that let f be a non-decreasing right continuous function defined on $\mathbb{R}^2$, so, this is real-valued. And consider our set function μ, given us this. So, you look at such product type sets with ai and bi is finite. So, then for such things you consider the set function that associates these values. So, you are not saying anything about other types of sets, let us start with a set function that is just defined on such product type sets.

Now, what you can do, you can consider find a disjoint union of such things and extend the set function by finite additivity or so-called finite relativity that you want under the measure and so on, try to verify the appropriate continuity property is get countable additivities and so on. So, what will happen is that, starting from such a set function, you should be able to construct this unique extension to a Lebesgue Stieltjes measure on $\mathbb{R}^2$. So, that is what this statement says that given such a non-decreasing and right continuous function on $\mathbb{R}^2$, you can construct this set function. So, this is a very, very important observation, .

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Note (32): If $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is non-decreasing and right-continuous with

(i) $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = 1$

(ii) $\lim\limits_{x \to -\infty} F(x,y) = 0 \quad \forall y \in \mathbb{R}$

(iii) $\lim\limits_{y \to -\infty} F(x,y) = 0 \quad \forall x \in \mathbb{R}$

then the Lebesgue-Stieltjes measure obtained from Theorem (1) will be a probability measure. This also completes the identification between probability measures and appropriately defined "distribution functions" in $\mathbb{R}^2$.

And then let us assume certain more conditions on that function. So, start with this non-decreasing and right continuous function, but additionally, assume certain limiting values. So, limits at ∞ as x and y both simultaneously go to ∞, if that limit value is one, and if by fixing one of the coordinates and letting the other one go down to $-\infty$, if you get the limit value 0, for such functions, if you now continue this construction of Lebesgue Stieltjes measure, you will get back a proper dimension.

Okay. And that is exactly what you need, because for probability measures you exactly had that the corresponding distribution functions has these properties. So, please go back and check again the discussions in week 4, we have described exactly these properties. So, therefore, this also completes the identification between probability measures in $\mathbb{R}^2$ and appropriated defined distribution functions in $\mathbb{R}^2$.

So, by appropriately defined distribution functions, we basically mean that this function should be defined on $\mathbb{R}^2$, these functions should be non-decreasing right continuous with those specified limit values. Once you have such functions, you will be able to construct probability measures. And of course, starting with probability measures you exactly get back these type of distribution functions. This completes the complete identification between probability measures and appropriately defined distribution functions, great.

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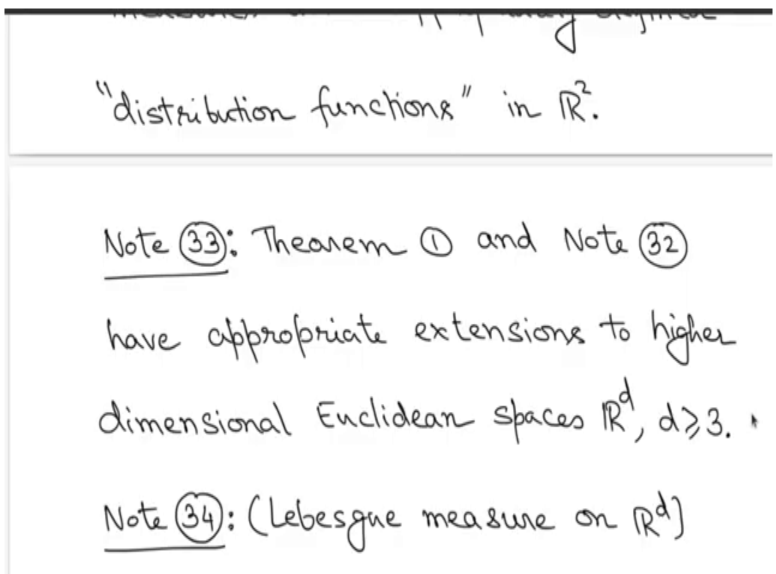

So, now we are going to spend some time on explaining certain more details in higher dimensional cases. So, now, in theorem 1, we have stated the result in dimension 2 and in Node 32, we have mentioned this special case that works for probability measures. So, again in dimension 2, what you should expect is that there are appropriate extensions to higher dimensional Euclidean spaces, when the dimension is greater or equal to 3.

So, what you will expect is that you will start with the appropriately defined right continuities, meaning they should be jointly right continuous, then for the functions, you have to define the corresponding non-decreasing property. So, how do you define that? So, again go back to those product type sets, lets me go back to the dimension to case once more.

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<u>Theorem ①:</u> let $F: \mathbb{R}^2 \to \mathbb{R}$ be non-decreasing and right-Continuous. Consider a set function μ given by $\mu\left(\prod_{i=1}^{2}(a_i, b_i]\right) := F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)$ $\forall -\infty < a_i < b_i < \infty,\ i = 1, 2$ has a unique extension to a

<u>Note ㉝:</u> Theorem ① and Note ㉜ have appropriate extensions to higher dimensional Euclidean spaces $\mathbb{R}^d$, $d \geq 3$.

<u>Note ㉞:</u> (Lebesgue measure on $\mathbb{R}^d$)

The Lebesgue-Stieltjes measure $\lambda^{(d)}$ on $\mathbb{R}^d$

So, what you defined was this product type sets right and looked at the size of this. So, if μ is a genuine measure, genuine Lebesgue Stieltjes measure then this corresponding linear combination would be non-negative. And that is what we had actually mentioned in exercise 5. So, let us go back once more. So, in exercise 5 we looked at such linear combinations of the function values at these corners right and we said that these appropriate linear combinations must be non-negative for the non-decreasing property of a F_{μ}. And this appeared because these kinds of product type sets for this left products of left open right close intervals which is now a box that should get assigned non-negative mass .

So, that is simply following from the properties of the measure and these appeared due to the motivation from the case of probability measures. So, for the case of probability measures, we looked at the corresponding distribution functions and he would exactly get back that same fact. So, the motivated by that we looked at such non-negative values for such sizes and the corresponding linear combinations of the corresponding functions, function values at those corners of the appropriate box .

And the idea here was that we are using the inclusion exclusion principle to get that appropriate linear combination. So, let us go back to that comment once more. So, the idea is this after you do things in dimension 2, then you expect that these things should go through in dimensions 3 onwards, but with appropriately defined notations. So, what do we expect is that with the appropriately defined right continuity and non-decreasing properties, you should be able to construct the corresponding Lebesgue Stieltjes measures so, that is the idea.

And again, you would ask what happens for property measures and distribution functions again, as a special case of this correspondence between leverages measures and the corresponding non-decreasing right continuous functions, you expect that the same results will go through for any d dimensional case when you are considering probability measures and their corresponding distribution functions.

So, again what you have to consider for the distribution functions that in addition to non-decreasing and right continuity the joint right continuity, you have to assume the appropriate limit values. So, limit values would be like this that the limit at infinity as the all the coordinates go to ∞ simultaneously, you should get the limit to be 1 and if you fix all coordinates except 1 the that specified coordinate goes down to $-\infty$ you will get the limit value 0.

So, for such functions you can construct probability measures these are the analogs of distribution functions in dimensions higher than 1. So, in dimension 2 we have stated exactly these conditions in the Note 32. So, again you can appropriately formulate the definition of distribution functions in higher dimensions without any reference to probability measures or random vectors, you can simply define it through this non-decreasing ness, joint right continuity and the appropriate limit follows.

And once you go through this construction, you will be able to construct the corresponding probability measure and that will give you the complete identification between the class of probability measures in dimension d and the corresponding distribution functions in dimension d. So, this is again extending all those ideas that we have already discussed multiple times in dimension 1.

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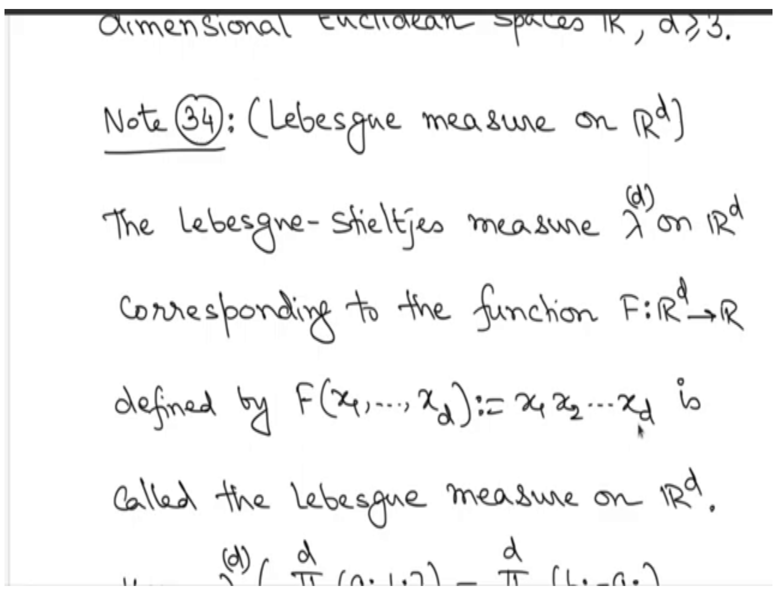

defined by $F(x_1, \ldots, x_d) := x_1 x_2 \cdots x_d$ is called the Lebesgue measure on $\mathbb{R}^d$.

Here, $\lambda^{(d)}\left(\prod_{i=1}^{d} (a_i, b_i]\right) = \prod_{i=1}^{d} (b_i - a_i)$,

which matches the usual "area / volume" in $\mathbb{R}^d$. $\lambda^{(d)}$ has properties similar to $\lambda^{(1)}$ – the Lebesgue measure

which matches the usual "area / volume" in $\mathbb{R}^d$. $\lambda^{(d)}$ has properties similar to $\lambda^{(1)}$ – the Lebesgue measure on $\mathbb{R}$, such as singletons have zero size/mass.

Now, again, what we have done in dimension 1 is that we have considered the special case of Lebesgue measures that comes from Lebesgue Stieltjes measures. So, you would expect that there is a corresponding version of Lebesgue measures in dimension d. So, recall that in dimension 1, the Lebesgue measure came out to be the appropriate length function. So, it associated length usual length two intervals. And this is basically extending that length function to all the other possible Borel sets in the real length.

So, that is what Lebesgue measure did in dimension 1. And since there are the standard areas and volumes in higher dimensions, what you should expect that there should be a appropriate analogue of Lebesgue measure in higher dimensions. And that is exactly what we are going to discuss that yes, there is this thing that there is a specified Lebesgue Stieltjes measure which we now denote by $\lambda^{(d)}$, to take care of the dimensional matter, .

So, we consider this Lebesgue Stieltjes measure $\lambda^{(d)}$, defined on $\mathbb{R}^d$. So, basically, it is defined on $\mathcal{B}_{\mathbb{R}^d}$, but just for simplicity, you say that it is defined on $\mathbb{R}^d$. So, what will happen is that you have to choose the appropriate function, same dimension one, you considered the function $f(x) = x$, but in dimension d, you have to consider this function now.

So, take any point $x_1, \ldots, x_d \in \mathbb{R}^d$ and specify the value of the function at that point to be the product of the coordinates, , so if $x_1, x_2, \ldots, x_d$ is the point then you look at the product $x_1 x_2 \ldots x_d$, so that is a real number, assign that value to the function at that point. This is it you get the function. So, now, what will happen is that this function is of course, non-decreasing and jointly right continuous in all the variables.

So, this is a very nice function, and therefore, corresponding to the specified function, you can now construct Lebesgue Stieltjes measure. But then these Lebesgue Stieltjes measure, we now call as the Lebesgue measure on $\mathbb{R}^d$ for the specified choice of the function, whatever Lebesgue Stieltjes measure that you get, you call it as the Lebesgue measure on $\mathbb{R}^d$, so, what happens to the product typesets?

So, the size associated to such product type sets is exactly this. So, you can easily go back and check that now these is exactly the function value increments and products of that. So, that is the values that will appear for the size of these type of product sets, products have left open right close intervals. And you again identify that this exactly matches with the usual area or volume in Rd.

So, in dimension 2, what you get is this product of two intervals, this is a rectangle in $\mathbb{R}^d$. And that is exactly the area of that rectangle in $\mathbb{R}^2$. So, if you are considering dimension 2, then a rectangle has exactly this area. So, you are looking at the length of each side and multiplying them . So, this is exactly the same analog that is given by the Lebesgue measure on dimension 2, in dimension 3 onwards you get the volume. So, that is again the usual volume that we expect.

So, for such sets, you already get the usual volume or area, but what Lebesgue measure is saying is that it can associate similar sizes to all possible Borel sets in these higher dimensions. And then what you should expect is that this Lebesgue measure that you just defined on dimension d this should have properties similar to the dimension one case. Lebesgue measures on dimension one has some nice properties and similar properties you can show for the Lebesgue measure on dimension d, such as signal turn as size or mass.

You can show all these nice properties that you have discussed for Lebesgue measures and similar things extend to dimension d. So, whatever results you have seen in dimension 1 analogs of this will go through for dimension d. So, please try to check what are these usual properties now for dimension d case. So, we stop here and we will continue the discussion in the next lecture.