Measure Theoretic Probability 1
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Lecture 24
Properties of Lebesgue Measure on R

Welcome to this lecture before we proceed with the discussion of this lecture let us first quickly recall what we have done in this week. So, we have finally finished identification between distribution functions on ℝ and the collection of probability measures on the measurable space real line together with the Borel σ-field. So, what we have seen is that given a probability measure we can construct the distribution function as per the usual definition.

But on the other hand, given a distribution function you can construct back the probability measure. So, this correspondence gives you the complete identification between these two classes of things alright. And we have also seen that this correspondence also extends to the class of Lebesgue Stieltjes measures which we discussed in the previous lecture. So, what you have seen is that this extension is connecting to the class of non-decreasing and right continuous functions on the real line.

So, here we are considering any function on the real line taking values in the real line. So, we consider such functions but with the properties that they should be non-decreasing and right continuous. And what we have seen is that for each Lebesgue Stieltjes measure we can construct corresponding such functions with appropriate parameterization but then on the other hand given such a function you can reconstruct the measure.

So, the main connection between these measures and the functions come in this following format; that the measure of $(a, b]$ is simply the increment of the function at the endpoints. So, if the interval is $(a, b]$ the increment is simply $F(b) - F(a)$ given the function F. And using that we followed the standard settings like we extended by finitely additivity and then we proved certain countable additivity by verifying continuity properties and extended it to the whole model σ-field by the Caratheodory's extension theorem.

So, in this way we have constructed a large class of examples so given any non-decreasing and right continuous functions you can now construct such measures. So, and what we mentioned was that this covers all probability measures, all finite measures and certain special class of σ finite measures which are known as this Lebesgue Stieltjes measures. And as a consequence of this construction we looked at the special example which we called as the Lebesgue Measure. So, in this lecture we are going to focus on the properties of the Lebesgue Measure and let us now move on to the slides.

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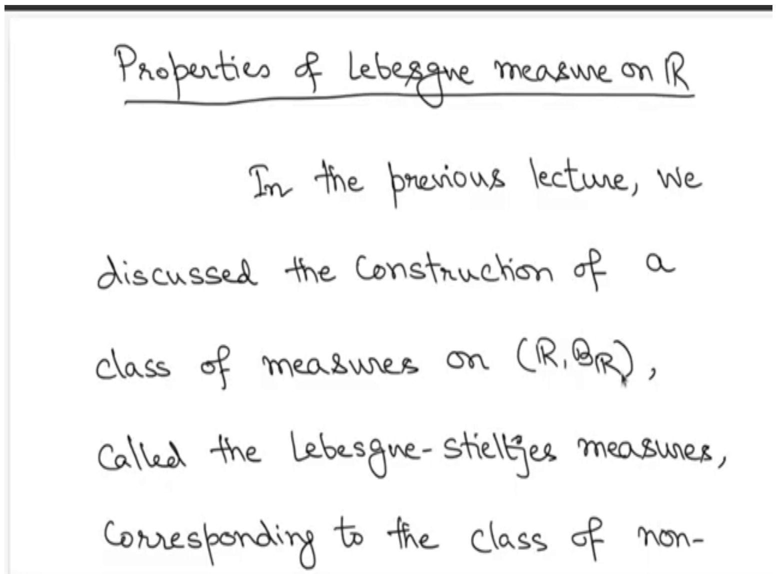

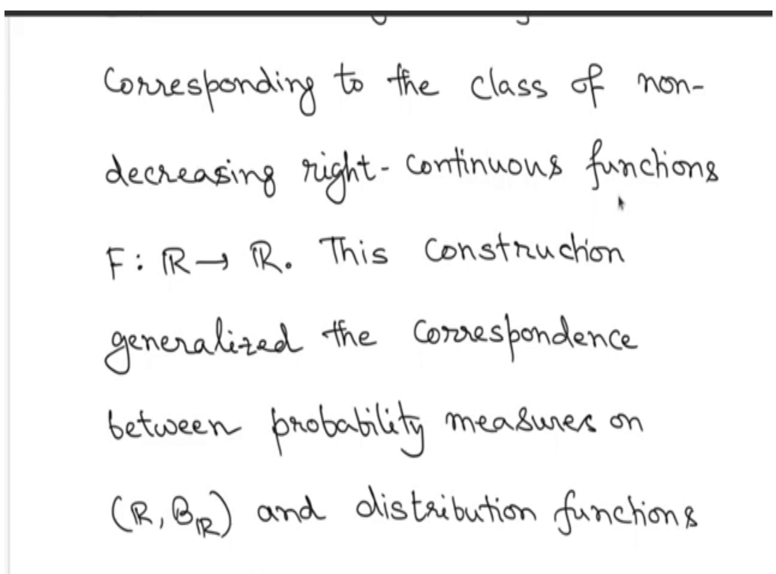

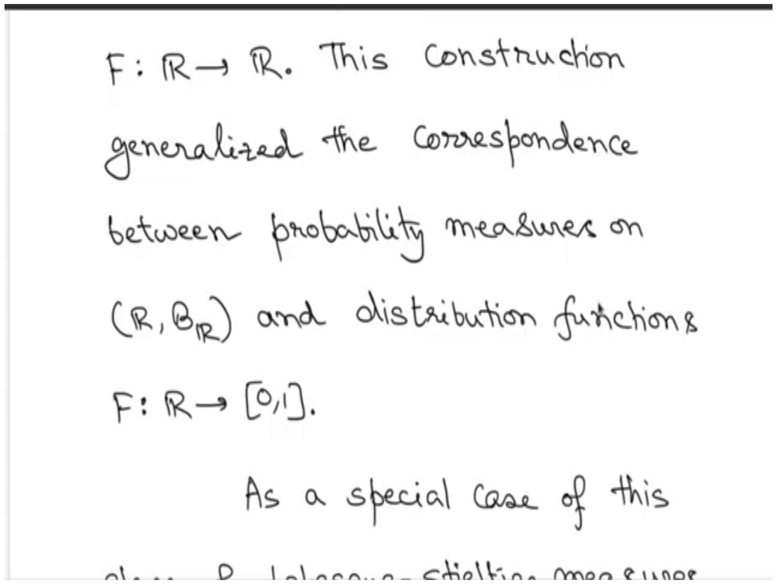

So, again a quick recall, so in the previous lecture we discussed this construction of this class of measures called the Lebesgue Stieltjes measures on the real line but this correspondence was through this class of non-decreasing and right continuous functions on the real line. This construction generalize this correspondence between probability measures and the corresponding distribution functions.

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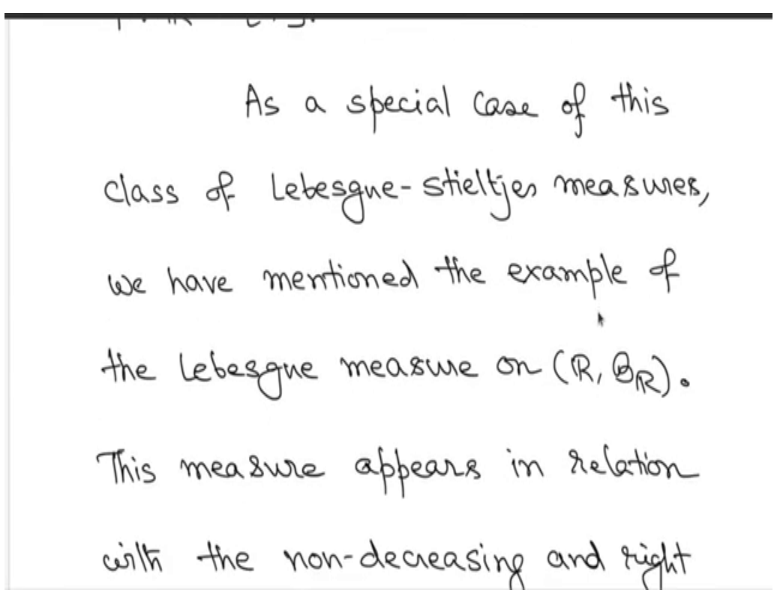

This measure appears in relation with the non-decreasing and right continuous function $F(x) = x, \forall x \in \mathbb{R}$.

In this lecture, we discuss various properties of this measure.

Note (19): we use λ to denote the

Now, let us move on to the special example that we just mentioned; so we look at this Lebesgue measure and this Lebesgue measure appears as a special example under the Lebesgue Stieltjes measures it is for the specific function; the identity function $F(x) = x$ itself. So, this is a nice non-decreasing and right continuous function and therefore as for the construction you will get a measure. So, the measure that you get is called the Lebesgue measure on the real line . And now we are going to focus on the various properties of this measure.

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properties of this measure.

Note (19): we use λ to denote the
Lebesgue measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. As mentioned in the previous lecture,

$$\lambda((a,b]) = b-a, \forall -\infty \leq a < b \leq \infty.$$

Therefore, λ associates the usual "length" as the size of the intervals.

Note (20): For any $x \in \mathbb{R}$, Consider

Note 20: For any $x \in \mathbb{R}$, consider the intervals $(x-\frac{1}{n}, x]$, $n=1,2,\dots$.

Since $\lambda((x-1,x]) = 1 < \infty$, we can apply the continuity from above (See Proposition 10 of week 2).

Now, $(x-\frac{1}{n}, x] \downarrow \{x\}$ and hence,

$$\lambda(\{x\}) = \lim \lambda((x-\tfrac{1}{n}, x])$$

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Now, $(x-\frac{1}{n}, x] \downarrow \{x\}$ and hence,

$$\lambda(\{x\}) = \lim_{n\to\infty} \lambda((x-\tfrac{1}{n}, x])$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0.$$

Note 21: Since any finite or

So, first of all the first thing that you should notice is that by construction the length of $(a,\ b]$ for any a b that is varying in this range you get the length as $b\ -\ a$ or the size that is attached to this is simply $b\ -\ a$. So, therefore you will immediately observe that the Lebesgue measure which we now denote by λ , so this λ associates the usual length on intervals. So, specifically for this left up and right closed intervals we have verified that by the construction itself we immediately get that the length or the size of this is the usual length. So, let us now move forward and see what are other interesting properties.

So, now speak any point in the real line call it x and consider this kind of a left open right closed interval, so what do we do? We look at $(x - \frac{1}{n}, x]$. Now, observe that if you choose the value of n as 1 so that is the largest set in this collection so as n varies this set string right so let us look at the case $n = 1$.

So, then you get the interval $(x - 1, x]$ but as per the definition of the Lebesgue measure what is the size of this; this is simply the length which is 1 and this is finite. So, now observe that this sequence of sets they are decreasing. So, therefore you could apply continuity from above provided this finiteness condition holds.

So, which you are now explicitly verified that for at least one of the sets in the sequence the size should be finite. So, here we apply this proposition 10 of week 2 which talks about continuity from above provided there is a finite size set, so here we have verified that there exists a set with finite size and therefore we now observe that the continuity from above should be true for this sequence of sets.

But now this sequence of sets decrease and decrease to the singleton set small x and therefore you will immediately claim by the continuity from above for this Lebesgue measure you get that the length of this singleton set is exactly the limit of the individual sizes or the lengths of these intervals; but then what are the sizes here? The sizes are simply $\frac{1}{n}$ and as $n \to \infty$ this goes to 0. So, by this argument we have verified that length or size of singletons under the Lebesgue measure is 0.

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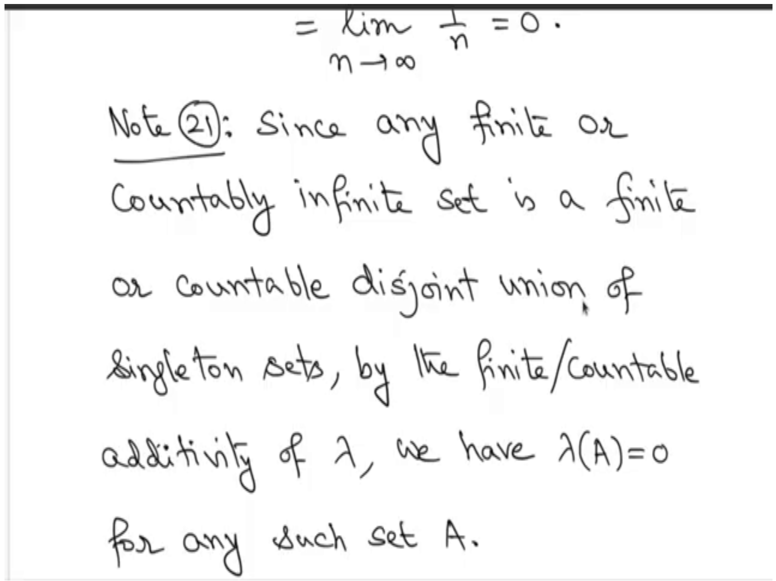

So, this is a important observation now but then you can use countable additivity or finite negativity to go from singleton sets to finite or countably infinite sets right because any finite or countable infinite set can be thought of as a finite or a countable infinite union, disjoint union of singleton sets.

So, just take each individual element in the set and think of it as the disjoint union of those things, so you will obtain any finite or countable infinite set on the real line. But then you use the finite or countable additivity of the Lebesgue measure λ and therefore you will immediately get that the length or the size of such sets if it is finite or a countable infinite set then you will immediately get that the length or the size is 0. So, this is simply using the fact that the length of singleton is 0 great.

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Note (22): For $a<b$,

$$\lambda([a,b]) = \lambda((a,b]) + \lambda(\{a\}) = \lambda((a,b]).$$

Similar arguments will show the following equality

$$\lambda([a,b]) = \lambda((a,b]) = \lambda((a,b))$$
$$= \lambda([a,b)) = b-a.$$

Note (23): There are uncountable sets

But then you will immediately use this fact that singleton sets has measure 0 to make other interesting observations. Like, you start with this closed interval now so we had already verified that the length or the size of left open right closed interval was exactly the usual length but then you look at this closed interval and write it as a finite disjoint union of this left and right closed interval and the singleton.

Singleton involving the left endpoint but use the finite entity of the Lebesgue measure then use the fact that the Lebesgue measure associates 0 mass to singleton sets and therefore you immediately get the length or the size of $[a,\ b]$ is the same as $(a,\ b]$, so that is great. And similar arguments will then show that all these intervals $(a,\ b]$, $[a,\ b)$, $[a,\ b]$, $(a,\ b)$

has the same size under the Lebesgue measure.

And that is nothing but the usual length, so this basically completes that identification of the Lebesgue measure that it associates the usual length for intervals. So, we have already identified the length of intervals and we have already identified the length or size of singleton sets or finite sets or countable infinite sets. But beyond that what are the other interesting examples?

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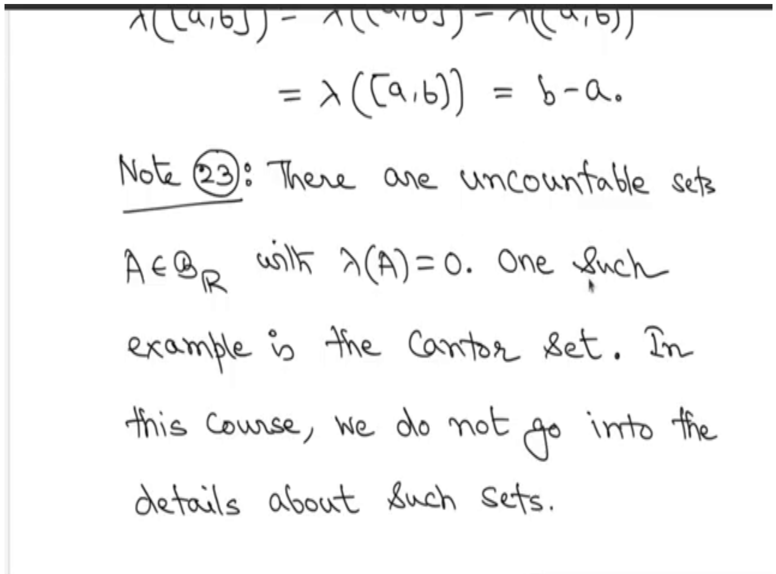

But then you should note that there are uncountable sets in the Borel σ-field such that the length is 0 so we do not go into details but just for an example this is there is this standard example called the cantor set and it can be shown that the cantor set is an uncountable set uncountable subset of the real line and its length is 0, the Lebesgue measure associates 0 mass to cantor set. Even though it is an uncountable set it associates 0 mass there.

So, we are not going into the details because we are focusing more on probability aspects of it so we will use this Lebesgue measure as mentioned in the previous lecture to discuss certain things about absolutely continuous random variables that we are going to see later on. So, we are not going to use much properties of such cantor sets or uncountable sets, so it should be good enough for us to know that there are such uncountable sets with mass 0. So, this is just an interesting fact, now let us move forward.

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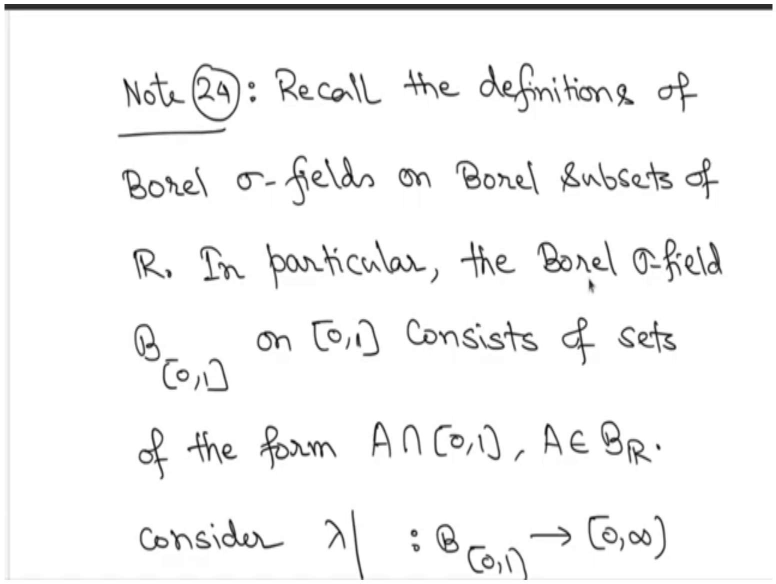

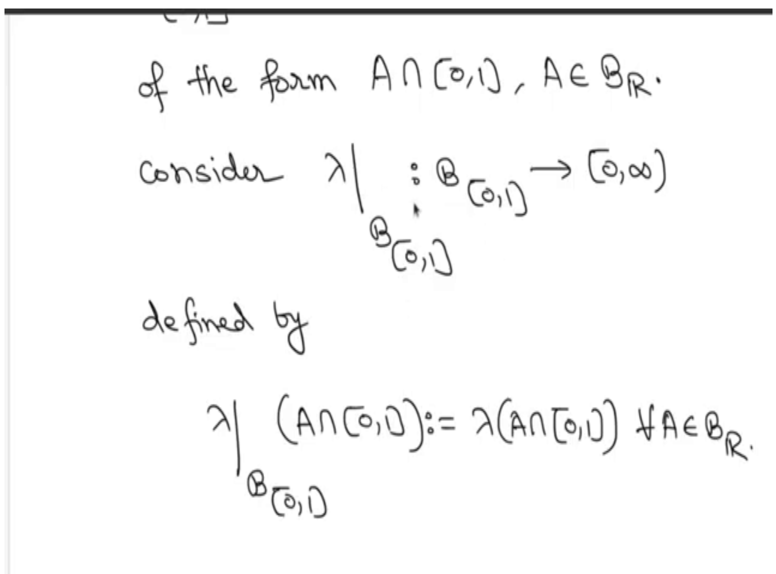

Now, recall the definitions of Borel σ-fields on Borel subsets of ℝ, so if you focus on let us say this kind of an interval $[0,\ 1]$ then you can define the Borel σ-field on $[0,\ 1]$ as follows so this is the collection of subsets of $[0,\ 1]$ which are of the form a intersected with $[0,\ 1]$ where a varies over the Borel subsets of the real line. So, choose all possible Borel subsets of the real line and intersect it with $[0,\ 1]$.

So, you will get all possible sets that are inside the Borel σ-field so that is how in fact the Borel σ -field on $[0,\ 1]$ is defined alright. But now what we are going to do is to construct a measure on top of this $\mathcal{B}_{[0,\,1]}$, so what do we do? So, we write it is this notation $\lambda|_{\mathcal{B}_{[0,\,1]}}$, so it will take any arbitrary subset of $\mathcal{B}_{[0,\,1]}$ and assign certain values.

So, let us see this; so how do you define this first of all? So, again any arbitrary set in this $\mathcal{B}_{[0,\,1]}$ looks like a intersected with $[0,\ 1]$ where a varies over all possible Borel subsets. So, what do you do? You simply look at the size of the set $A\ \cap\ [0,\ 1]$ under the Lebesgue measure, .

So, just look at the size of this set and associate that value for this set function so this set function that you define as λ restricted to this Borel σ-field. What is going to happen is that this set function that you have just defined turns out to be a measure on $\mathcal{B}_{[0,\,1]}$. So, this is left as an exercise, so this simply uses the non-negativity of the Lebesgue measure that is already given to you.

So, these values are non-negative and you then use countable additivity for a pairwise disjoint sequence of a ns you applied to the controllable negativity of the Lebesgue Measure λ and prove the countable additivity for the restricted function.

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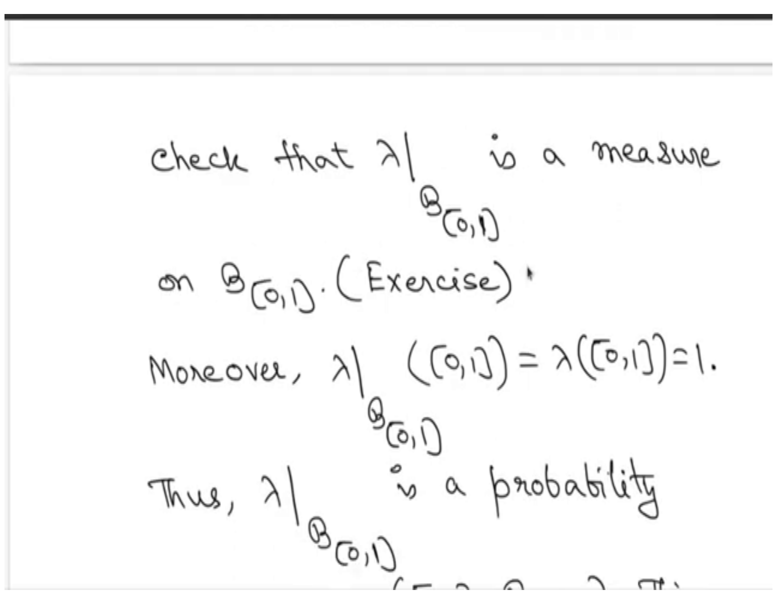

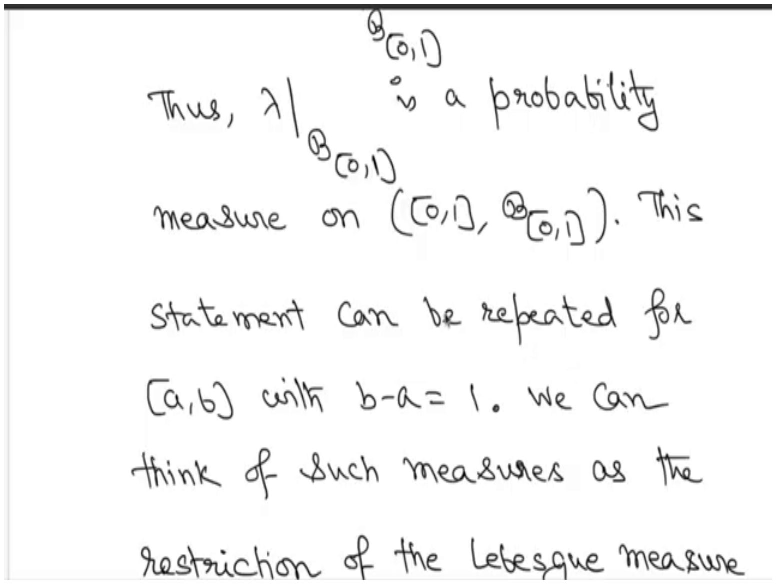

So, please check that this restricted function becomes a measure on $\mathcal{B}_{[0,\,1]}$. But then you also should note that what is the size of the whole set under this measure. So, it turns out that restricted function this λ restricted to 0 1 of the interval 0 1, $[0,\ 1]$ is nothing but 0 1 intersected with 0 1 itself this is as per the definition, right. So, you are just have to look at the what is the size associated to this interval by the Lebesgue Measure and that is nothing but the usual length which is 1. So, therefore what you ended up getting is that $\lambda|_{\mathcal{B}_{[0,\,1]}}$ becomes a probability measure on this measurable space .

So, you have verified this is a non-negative set function its countable additive by the properties of the Lebesgue measure and finally you have shown that the length or size of the whole set is equal to 1 and therefore this example gives you certain nice ways of constructing probability measures from other infinite class of measures, alpha, other infinite measures .

So, in particular for Lebesgue measures we have seen this but then you can repeat this construction or this statement for intervals of the form $[a,\ b]$ with length $b\ -\ a\ =\ 1$. So, in that case again you restrict it to the appropriate intervals and you can again show that such intervals if you restrict the Lebesgue measure there we will again get a probability measure . So, this will now be on $[a,\ b]$ with appropriate Borel σ-field on $[a,\ b]$ and there if you restrict the Lebesgue measure you are going to get probability measures . We are going to see usage of this later on.

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restriction of the Lebesgue measure to the Borel subsets.

Note (25): (A subset of $\mathbb{R}$ which is not in the collection $\mathcal{B}_{\mathbb{R}}$)

Using set theoretic arguments, specifically the axiom of choice and

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But now here is an interesting example , so now we are going to construct a subset of the real line which is not in the collection Borel σ-field. So, therefore it is going to show that the power set which is the collection of all subsets of the real line this is strictly bigger than the Borel σ -field, so there are subsets of the real line which are not in this Borel σ-field which are not Borel subsets of the real line. So, how do you show? So, this requires certain set theoretic arguments specifically using certain things called axiom of choice and equivalence relations.

If you use that you can actually construct a sequence of sets let us call them as a n which are pairwise disjoint and their union will cover $(0, 1]$. Moreover, we can also show that if you can choose the sets from the Borel σ-field, so applied it is not clear, so if the sets are in the Borel σ -field then you can show that the Lebesgue measure associated to the sets are equal . So, all these sets have the same length or same size under the Lebesgue measure. So, such a construction is possible provided the sets are inside the Borel σ-field . But then what is going to happen is this.

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such that if $A_n \in \mathcal{B}_{\mathbb{R}}\ \forall n$, then

$$\lambda(A_1) = \lambda(A_n)\ \forall n.$$

In this case,

$$1 = \lambda((0,1]) = \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \lambda(A_1)$$

which leads to a contradiction. Thus $A_n \notin \mathcal{B}_{\mathbb{R}}$. To avoid the complexity

$$1 = \lambda((0,1]) = \sum_{n=1} \lambda(A_n) = \sum_{n=1} \lambda(A_1)$$

which leads to a contradiction. Thus $A_n \notin \mathcal{B}_{\mathbb{R}}$. To avoid the complexity

of the construction of such sets, in this course we do not go into the details. The interested reader

So, observe this equality. So, you have the length of the interval $(0,\ 1]$ is 1 that we have already seen but if the sets A_n's are pairwise disjoint and their union is exactly this interval $(0,\ 1]$ then use countable additivity of the Lebesgue measure to write it as this summation $\sum_{n=1}^{\infty} \lambda(A_n)$. But as we said if the sets A_n are in the Borel σ-field then what you can do is that you can show that the Lebesgue measure associated to these sets remain the constant which we write as $\lambda(A_1)$ the first set. But now you are led to a contradiction, so contradiction is this you are having a constant term you are adding it infinitely many times.

So, it is a constant sequence you are adding it up and you are saying that the sum is 1, so now there are two possible choices which you can do. So, if the Lebesgue measure associated to this set is 0 then the sum must be 0 so the equality does not hold; if this Lebesgue measure associated to this set is positive then by argument principle this summation will diverge and diverge to infinity. So, it will not be equal to 1, so this leads to a contradiction.

And therefore, you will immediately claim that the sets a n are not in the Borel σ-field so this was the assumption that we made and which we implied that you can talk about the Lebesgue measure of this sets. So, that is the issue here , so if the sets are in the Bordel σ-field then you can construct such sets with same Lebesgue measure . So, that is leading to the contradiction. So, therefore there exists at least one such set which is not in the Borel σ-field . So, again to avoid this complexity of this construction of such sets we are not going into the details of this .

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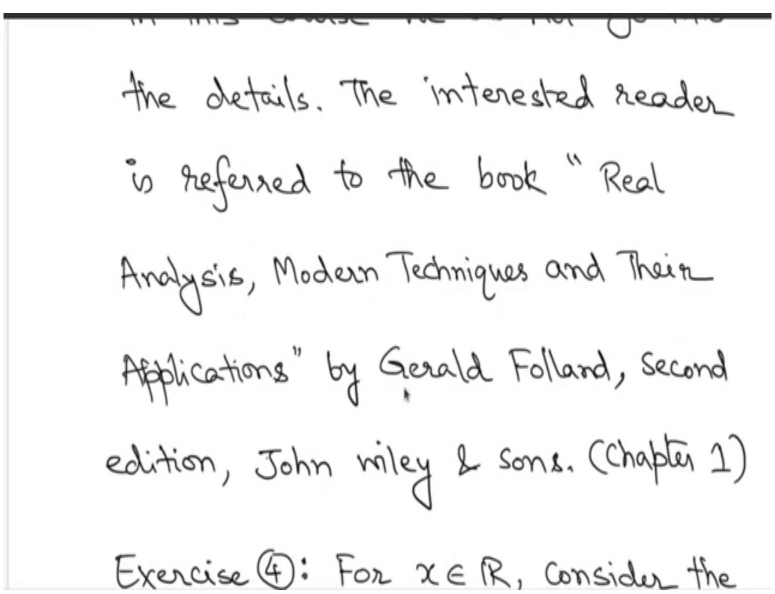

So, but I am mentioning our reference here so you can go to a book by Gerald Folland, the book title is Real Analysis, Modern Techniques and Their Applications it is a second edition of the book you can look at the chapter 1 of this. It is published by John Wiley and Sons; take a look at this if you are interested. We end this lecture by listing certain other interesting properties of the Lebesgue measure.

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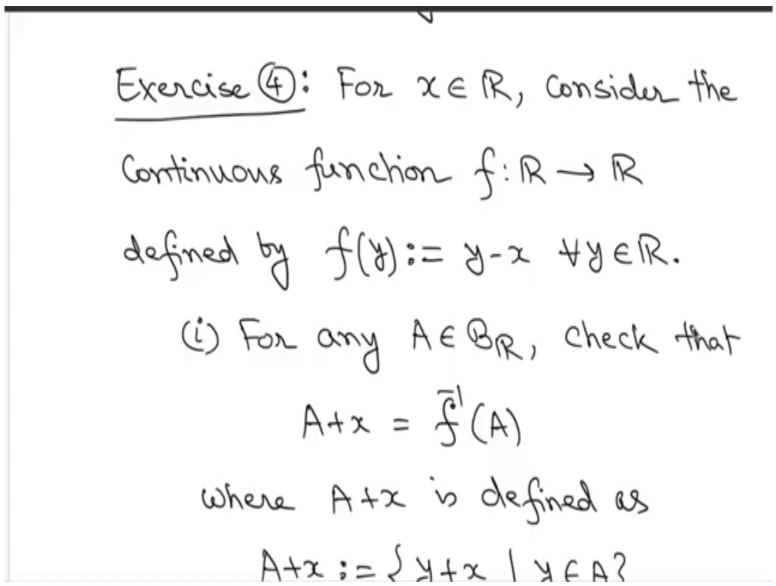

So, here you start with a point x in the real line and consider the continuous function that maps the point y to $y - x$. So, look at this function from $\mathbb{R}$ to $\mathbb{R}$, so this is a nice continuous function. So, now here are the steps in the exercise; so the first steps is that you verify that for any set A in the Borel σ-field the set $A + x$ can be written as the pre image of the set under the function f .

So, you are given this explicit function please compute the pre-image and show that this is $A + x$. So, now you will ask what is $A + x$? So, $A + x := \{y + x / y \in A\}$, so here you are just shifting the set A by the point x. So, this x you are fixing before so that is the first part of the exercise.

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(ii) Show that $A+x \in \mathcal{B}_{\mathbb{R}}$ for all $x \in \mathbb{R}$ and $A \in \mathcal{B}_{\mathbb{R}}$.

(iii) Show that the set function $\mu : \mathcal{B}_{\mathbb{R}} \to [0,\infty)$ defined by

$$\mu(A) := \lambda(A+x), \forall A \in \mathcal{B}_{\mathbb{R}}$$

is a σ-finite measure.

is a σ-finite measure.

(iv) Check that $\mu(A) = \lambda(A)$, for all A in the field of finite disjoint union of left-open right-closed intervals in $\mathbb{R}$.

(v) Show that $\mu(A) = \lambda(A) \; \forall A \in \mathcal{B}_{\mathbb{R}}$.

Note (26): Exercise (4) implies that

So, the second part says you should try to show that for all possible x in the real line and all possible Borel subsets the set $A + x$ is also in the Borel σ-field. So, that means that given any Borel subset of the real line if you translate it by x then you will get back another Borel subset so $A + x$ is again a Borel subset please check this. Now, using this observation you can now define a set function, let us call it as μ.

So, what you do; if that μ define the size of the set arbitrary Borel set as the size of $A + x$ meaning whatever the size associated to $A + x$ by the Lebesgue measure, so you fix that point x look at $A + x$ look at the Lebesgue measure of that that is some non-negative, number you associate that value to $\mu(A)$, so you get a new set function, μ. You can try to check that this μ becomes a σ- finite measure. So, this is on the Borel σ- field.

But then interesting step is this that you can now try to check that measure associated to sets of the form which are finite disjoint unions of left open right close intervals for such sets you can try to verify that the Lebesgue measure associates the same mass as $\mu(A)$, so for sets which are finite disjoint union of left open right closed intervals you will get the same length or the same size under both these measures .

So, first observe this verify this directly and then using this you can try to show that for all Borel sets these two measures agree. So, again we are saying that first verified for left open right close intervals then for finite disjoint unions and then to for all Borel sets please verify this. But then what does this exercise finally say? Look at the structure of $\mu(A)$ so that is defined as Lebesgue measure of $A + x$ but now you are saying that Lebesgue measure of $A + x$ is equal to Lebesgue measure of A for all Borel sets right, so that is what this last part of the exercise implies.

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Note 26: Exercise 4 implies that

$$\lambda(A+x) = \lambda(A) \ \forall x \in \mathbb{R}, A \in \mathcal{B}_{\mathbb{R}}.$$

This is stated as follows: The Lebesgue measure λ is translation invariant.

So, therefore what you end up getting is that for any point x in the real line and any set in the Borel subset of r you get that the Lebesgue measure associates the same mass as the Lebesgue measure of the shifted set or the translated set . So, now we state this property of the Lebesgue measure as follows; that the lebesgue measure λ is translation invariant .

So, if you translate the set a by some point you will not see any difference in the size , it associates the same size or the same length, alright. So, we have listed some of the very interesting properties of the Lebesgue measure in the next lecture we are going to see certain higher dimensional versions of the discussions that we did in this week. So, in particular we are going to discuss this connection between probability measures and distribution functions in higher dimensions. And we are also going to mention certain facts about the Lebesgue measure , so we stop here.