Measure Theoretic Probability 1
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Lecture – 22
From Distribution Functions to Probability Measures (Part 2)

Welcome to this lecture. So, in this week, we have been discussing the construction of probability measures from distribution functions. This will allow us to complete the correspondence between probability measures on the real line or in some higher dimensional Euclidean space, with the collection of or the class of distribution functions on our or a appropriate higher dimensional Euclidean space. So, we had left out a part of the proof, in the previous lecture. So, let us go to the slides and see what status we left the proof. So, we will have to look at the proof and look at the status of the proof.

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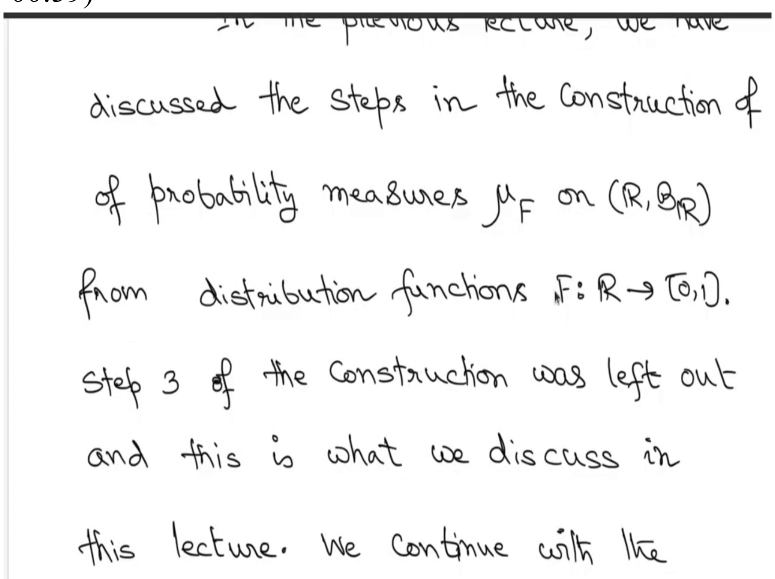

So, we have started this construction of probability measures, which we denoted by μ_F for a given distribution function F. So, here we are working in dimension one. So, here distribution function is a function taking values between 0 and 1, it is a non-decreasing right continuous function with limits at $+\infty$ and $-\infty$, being 1 and 0, respectively. So, for such functions, we want to construct this set function. And we want to show, that this is non-negative countably additive and assigns the total mass 1 to the set real line. So, so, let us recall the setup that we have done so far.

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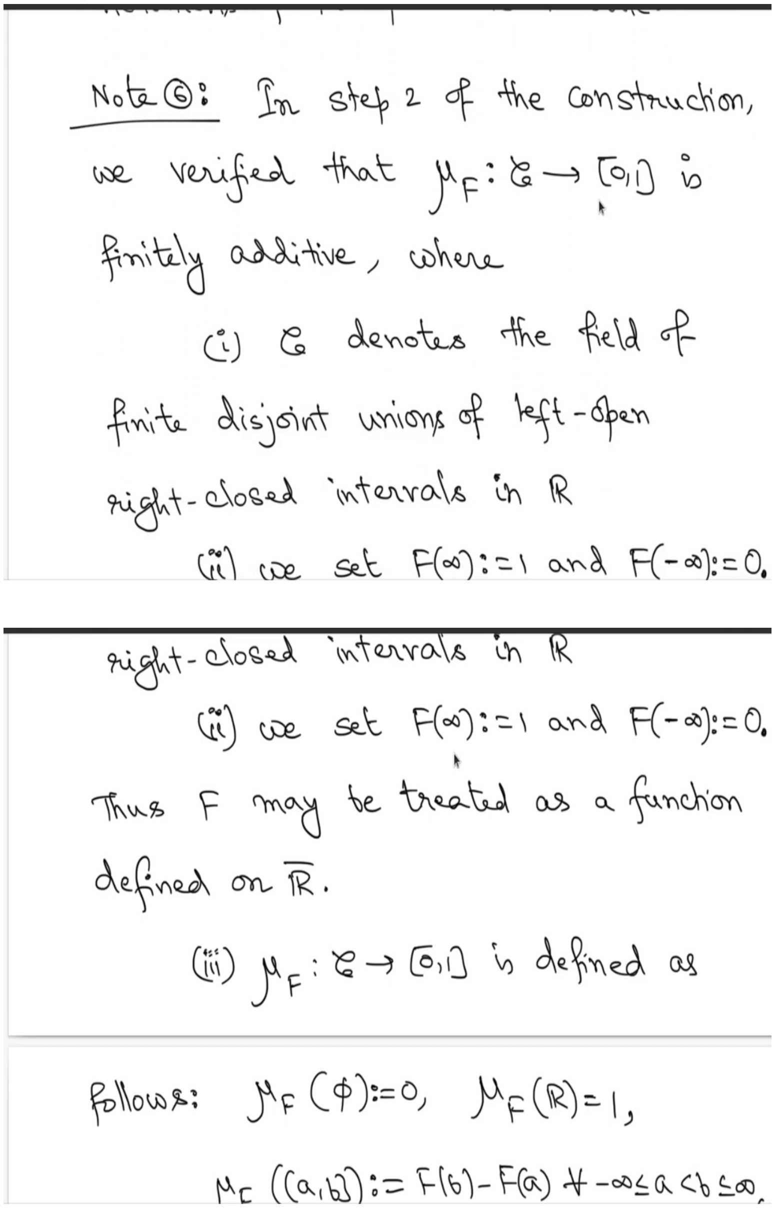

So, what we do is that, we had described this set function by specifying its values on the field of finite disjoint unions of left open right closed intervals in the real line. So, remember this fields $\mathcal{C}$ we had discussed in week one. And that generates the poodle similar field on ℝ. So, this fact we are going to use and apply Caratheodory's extension theorem to get the relevant extension from the field to the generated σ-field which is the borel σ-field.

But then what did we do, we first observed that since the function is non-decreasing, and has limits at $+\infty$ and $-\infty$ being 1 and 0, we can define or extend the function at the points $+\infty$

and $-\infty$, by specifying their values as 1 and 0. So, thereby we are getting this function F as a function on $\bar{\mathbb{R}}$. It is still a non decreasing right continuous function. And it still has the appropriate limits. So, that is the thing and then we also have the fact that F still takes values between 0 and 1. So, you are treating it as a function on the extended real line.

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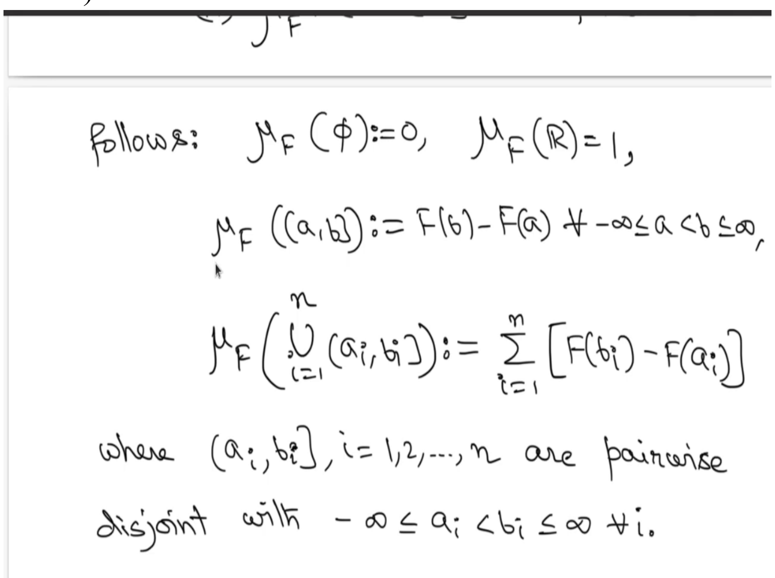

And for such a function, what we did, we looked at this set function. So, and we specified the values for the sets coming in this field of finance disjoint union of left open right closed intervals. So, what did we do? So, in preparation for this set function to be a probability measure, we assigned the value 0 to the empty set, value 1 to the real line. And for left open right closed intervals, we specified the structure of finite additivity by specifying these values.

So, for an interval of the type $(a,\ b]$, we looked at the increment of the function. So, by that I mean $F(b)\ -\ F(a)$. So, this value, so these values we assigned to the this set. So, this left open right closed interval. So, this is the value that we specified. But then in preparation for it to be finitely additive, we said that finance visual union of such intervals should be specified the value as the sum of the individual values, which is nothing but the sum of all these increments. So, this will allow us to finish the definition on the field of finite disjoint union of left open right closed intervals.

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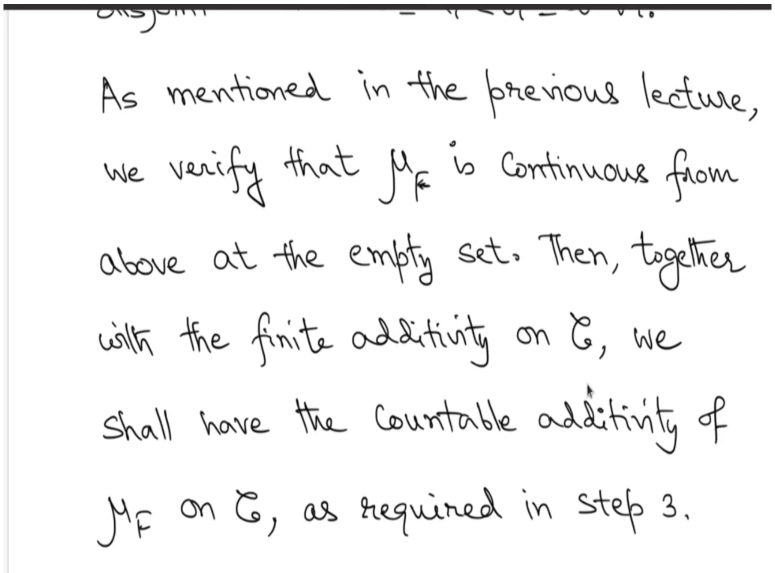

But then we verified that, this μ_F turned out to be a non-negative finitely additive set function on this field. And we wanted to verify a continuity property of this set function. In order for us to claim that μ_F is countably additive on the field's (())(04:59). So, that is the trick that we are using. So, instead of directly trying to prove countable additivity we first show finite additivity.

And then verify appropriate continuity properties of this set function on the field, to allow us to claim countable additivity. But, then we will see that this is also not an easy task, we have to go through certain technical arguments and prove the relevant continuative properties. A direct proof of countable additivity may be much more difficult. So, let us go and see how we can show the relevant continuity property, which is continuity from above at the empty set.

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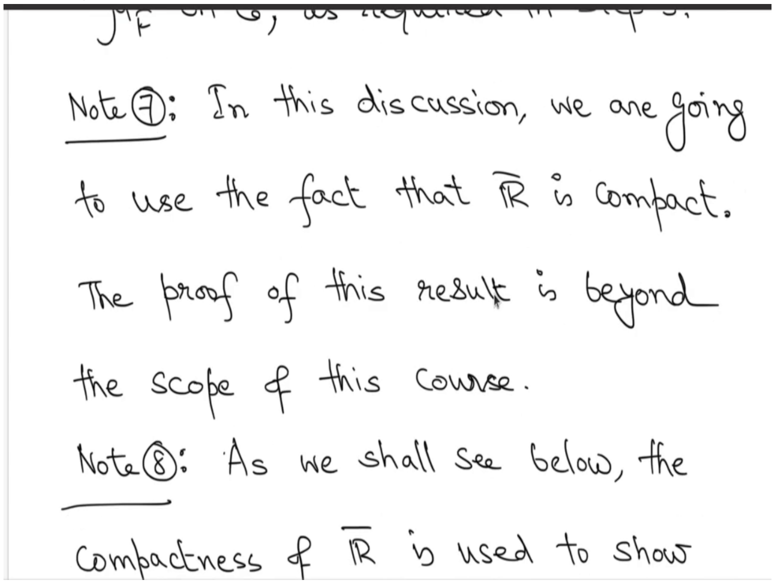

So, before we go forward, let us make a few comments. So, the first thing is that we are going to use the fact, that the extended real line is compact. So, what did we do on the real line? We attached extra points, which are $+\infty$ and $-\infty$. You can think of these as some kind of limit points. And once you add these limit points, this set, the set of extended real numbers which is denoted by $\bar{\mathbb{R}}$, becomes compact. So, the proof of this result is beyond the scope of this course and we just assume it for our discussion. So, we are not going to prove this fact.

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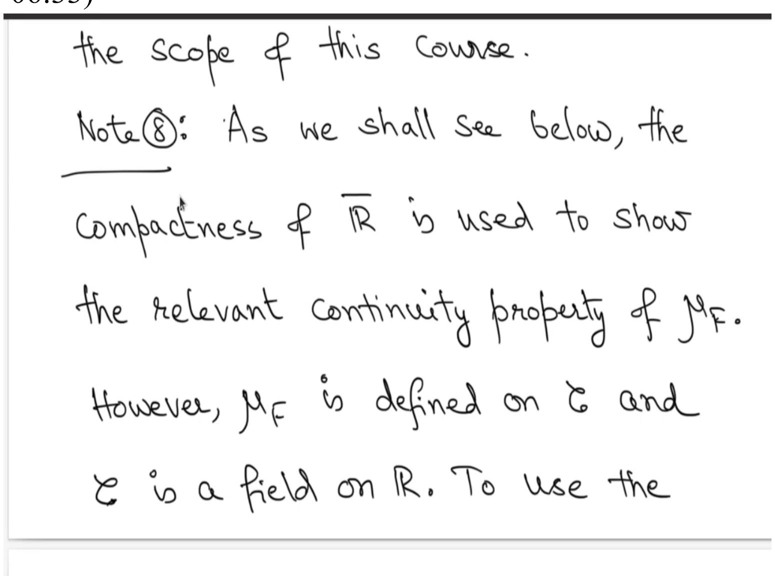

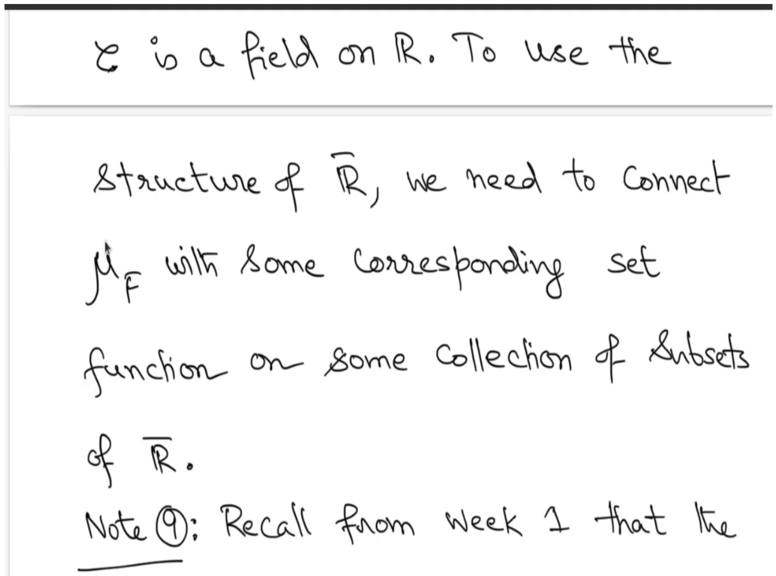
$\mathcal{C}$ is a field on $\mathbb{R}$. To use the structure of $\bar{\mathbb{R}}$, we need to connect μ_F with some corresponding set function on some collection of subsets of $\bar{\mathbb{R}}$.

Note ⑨: Recall from week 1 that the

Now, as we are going to see below, this compactness is going to help us prove the relevant continuity probability. But then remember, the original set function that we are dealing with that is so far defined on the field of finite disjoint union of left open right closed intervals in the real line. So, you are still within the realm of the real line, and you have not gone to the extended real line. So, therefore, to use this structure of the extended real line, what you need to do is to make some connection with a appropriate set function on $\bar{\mathbb{R}}$, on the extended real line.

So, what we do? We are going to connect this μ_F which is defined on the field of finite disjoint unions of left open right closed intervals in $\mathbb{R}$ with an appropriate set function defined on some collection of subsets of the extended real line. So, this is what we do, and once you make the connection, only then you are able to use the compactness properties of the extended real line. So, this is the procedure that we are going to use.

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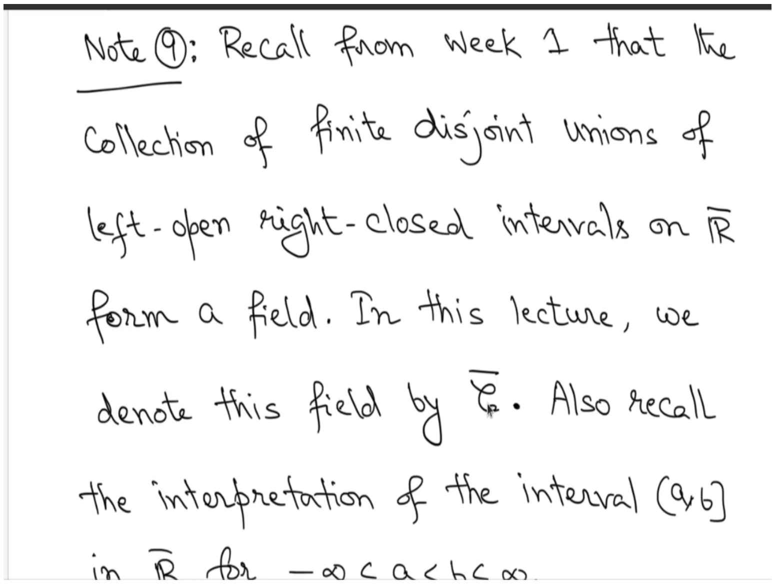

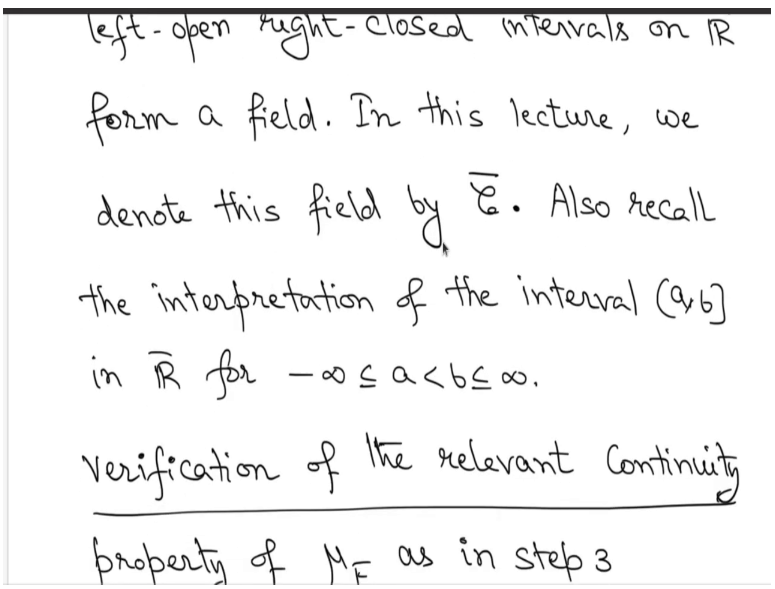

Now recall, that we already have a good candidate of this nice collection of subsets on the real line. So, the candidate is, that the finite disjoint union of left open right closed intervals on the extended real line. And remember, this again forms a field. So, if you look at this collection of left open right closed intervals, take finite disjoint unions of those, then it will form a field on the extended real line.

Of course, you have to remember the appropriate interpretations for $(a,\ b]$, because here we are allowing the points $+\ \infty$ and $-\ \infty$ to be in these intervals, the extended real line contains the points $+\ \infty$ and $-\ \infty$. So, just to distinguish this field, that we could talk about on the external real line, we are going to denote it by $\bar{C}$. So, the field on extended real line, the $\bar{\mathbb{R}}$ is going to be denoted by $\bar{C}$. So, as you can expect, there is some kind of a connection between the fields C, which is on the real line and the field $\bar{C}$ on the extended real line $\bar{\mathbb{R}}$.

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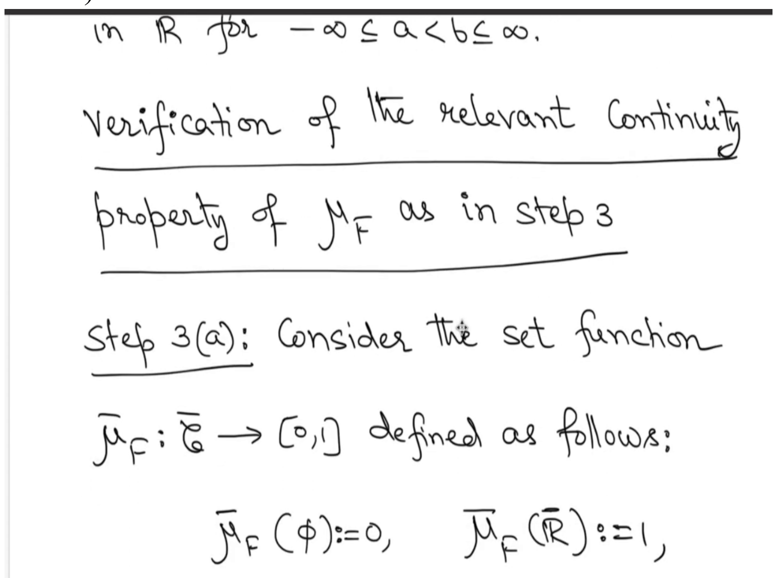

Now, remember, we are going to use these intervals $(a,\ b]$, but this $(a,\ b]$ can appear as a subset of the real line, in which case you have to use the appropriate interpretation for b being $+\ \infty$ or a being $-\ \infty$. Similarly, when you are talking about this interval on the extended real line, be careful about this notation. Then these points $+\ \infty$ or $-\ \infty$ may be inside your set. So, now, we are going to complete the proof that we left out in the previous lecture.

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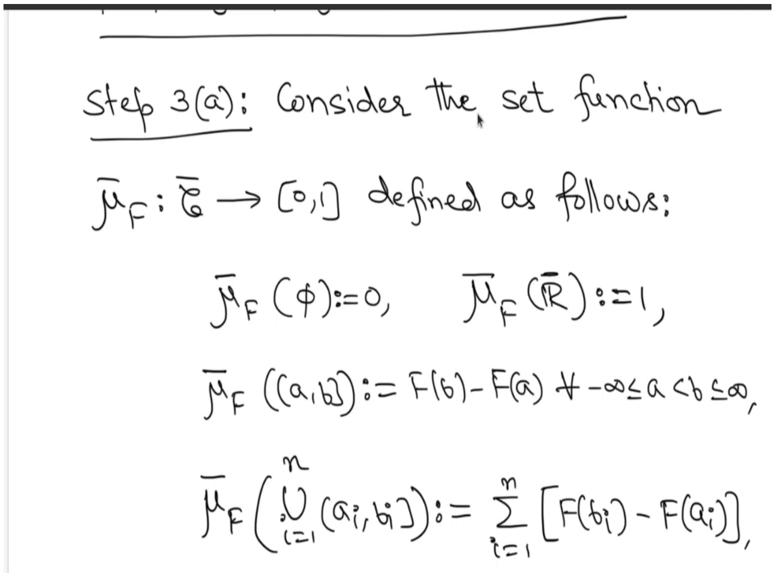

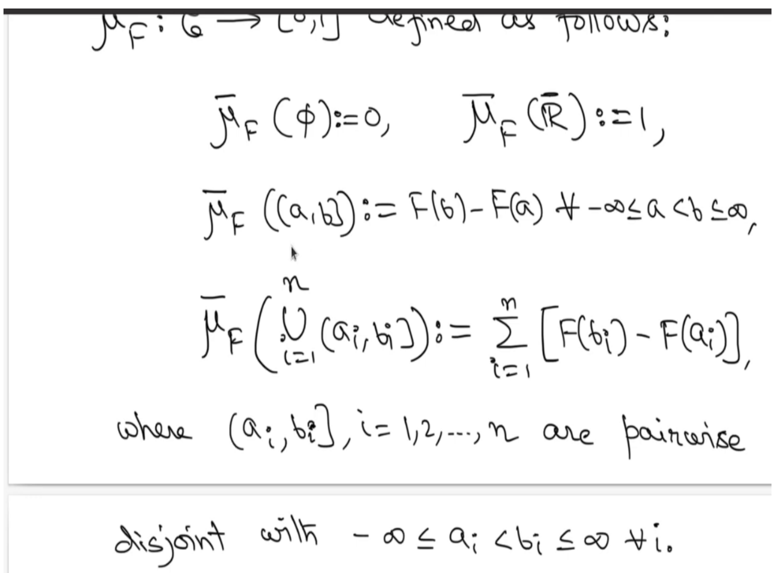

So, we are going to verify the relevant continuity property for the μ_F as left out in step three. So, we split this part into several further sub steps. So, we start with this observation, that we can consider a set function on the field of finite disjoint union of left open right closed intervals on the extended real line. So, on $\bar{\mathcal{C}}$, so, we are defining this set function and we are denoting it by $\bar{\mu}_F$.

As you can expect, there should be some connection with μ_F that we have defined. But let us see how we define this. So, again we are going to set it to be 0 for the empty set, we are going to set the value 1 for the extended real line. And for intervals of this type, we are going to use this increment structure that we have already set for the μ_F. So, similar structure we are just going to retain for this $\bar{\mu}_F$.

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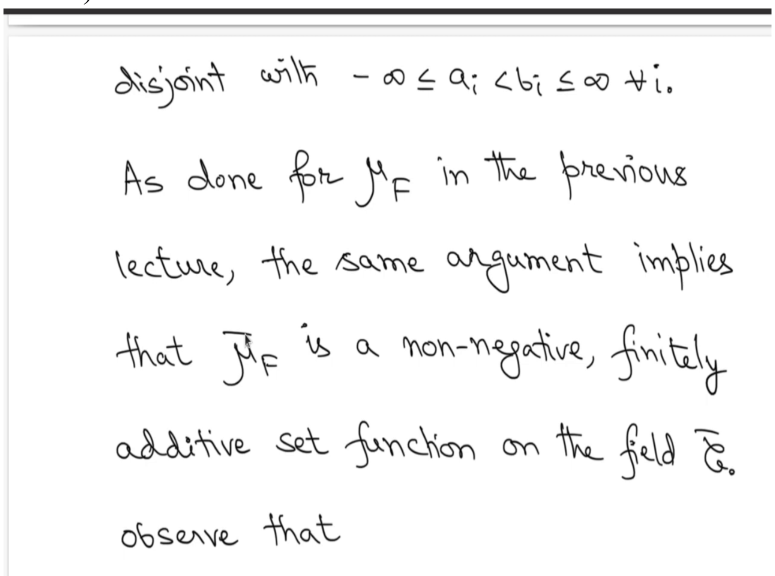

So, then what you can immediately observe that, as argued for μ_F you can use the same argument and show, that $\bar{\mu}_F$ that you have just mentioned is a non-negative set function. This will also become finitely additive. But then very important, this is still defined on the field $\bar{\mathcal{C}}$, which is a field on $\bar{\mathbb{R}}$, the extended real line.

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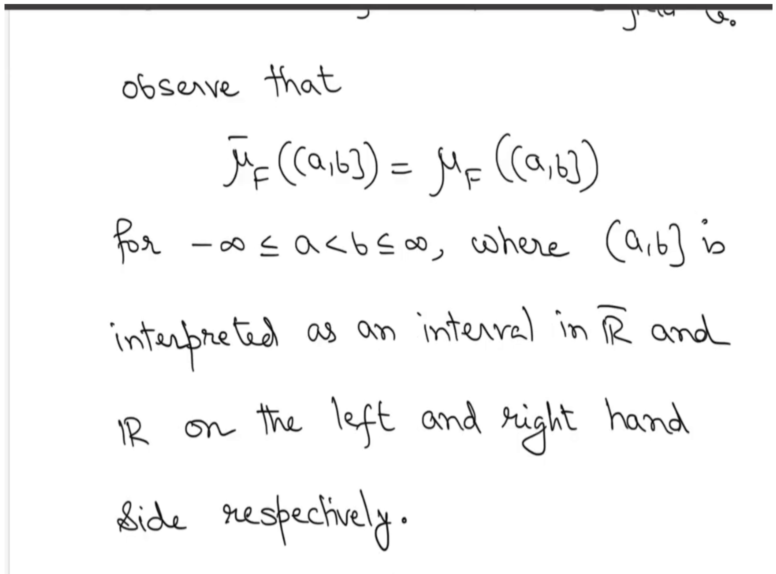

observe that

$$\bar{\mu}_F((a,b]) = \mu_F((a,b])$$

for $-\infty \leq a < b \leq \infty$, where $(a,b]$ is interpreted as an interval in $\bar{\mathbb{R}}$ and $\mathbb{R}$ on the left and right hand side respectively.

But now, observe that these two set functions, $\bar{\mu}_F$ and the μ_F. So, $\bar{\mu}_F$ and μ_F, agree on left open right closed intervals. So, before we state this proof of this result, let us first clarify what exactly we mean, here by left open right closed intervals. So, on the left-hand side, you choose a and b within this range. So, a and b could be $+\infty$ or $-\infty$.

But on the left-hand side, you have to be careful that these sets should be interpreted in the sense of sets inside the extended real line and therefore, may contain the ∞ points. On the right-hand side, again you look at the same notation left open right close intervals for such a b, but here we are going to treat these intervals as intervals within the real line. And therefore, they will not contain the points $+\infty$ or $-\infty$.

Even though the left end point a and the right end point b may be taken to be $+\infty$ or $-\infty$. As per our notational justification, that we have been following so far, as long as these sets are within the real line, we are not including the points $+\infty$ or $-\infty$ inside. So, therefore, on your left-hand side, when you are applying a $\bar{\mu}_F$, you will treat the set as a set inside the extended real line.

Whereas, on the right-hand side, when you are looking at this interval, you should treat it as a set inside the real line. So, that is the proper notational justification. But now, what is the proof for this fact? Look at the left-hand side, that is defined as per our definition given above as the increment of the function values. That is simply $F(b) - F(a)$. But that is the exact same value when you look at the right-hand side.

So, again the size of this set under μ_F is exactly $F(b) - F(a)$. Since both are defined through the increment of the function values, we are easily able to compare and see that these two functions attach the same size to such left open right closed intervals, but with the appropriate notational justification that the $(a, b]$ has to be treated as a set inside the extended real line, on the left hand side and on the right hand side it has to be treated as a set inside the real line.

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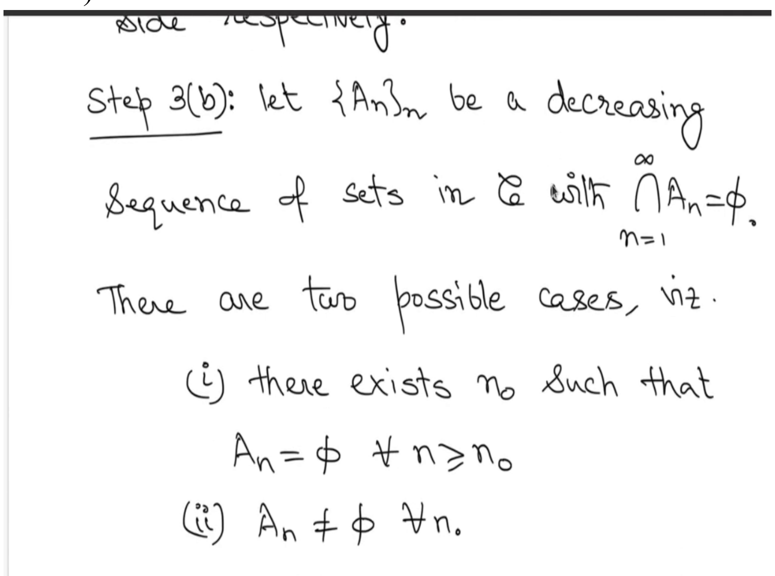

But now let us start off with the argument. So, we want to show what? We want to show that given any decreasing sequence of sets inside the field, that is on the real line. So, this is the finite disjoint union of left open right closed intervals inside the real line. So, for that field, we want to show, that the set function that we have defined on this field, this we want to show to be countably additive, and we want to show it is continuous from above. To do that, we start off with a general sequence, $\{A_n\}$ inside this field, which decrease and decrease to the empty set.

So, this whole intersection, this countable intersection is the empty set. So, here we split the case into two possible sub-cases. So, the in the first sub case what we are doing is that since this sequence of sets that we are considering are decreasing, then there might exist a suitable integer, large enough integer such that after a certain stage all the sets are just empty sets. So, this can happen and, in this case, of course, the intersection of all these sets is empty.

But, there may be an opposite case that in the case of all such natural numbers n, your sets A_n which is a finite disjoint union of left open right closed intervals is not empty. So, these are certain non-trivial sets, but they decrease and decrease to the empty sets. So, there are these two cases.

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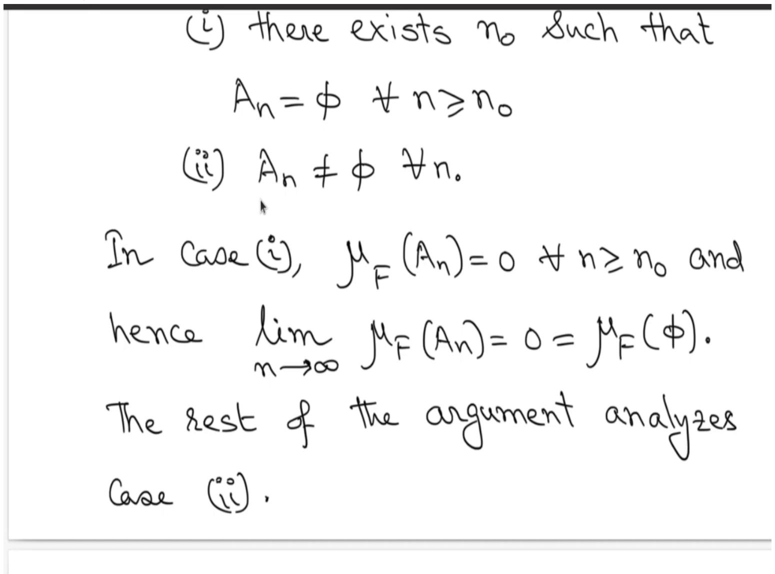

So, the first case Roman one, is easy to handle. So, observe that if you look at the size of these sets according to say, set function that we have defined. So, since A_n's are empty sets after this stage in n_0, then it will get them value 0, as per the definition that we have already considered. So, therefore, if you consider the limit of these sizes, it will also be 0 and agree with the value associated to the empty set. But then, the rest of the argument, we are going to focus on the case Roman two, where the sets A_n, this sequence of sets A_n is non-trivial. So, they are non-trivial meaning, they are not empty sets, but they decrease and decrease to the empty set, very good.

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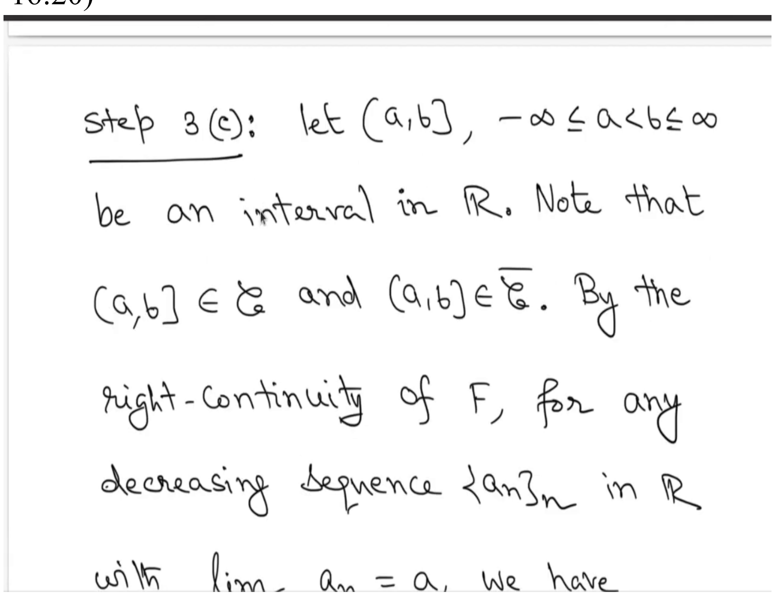

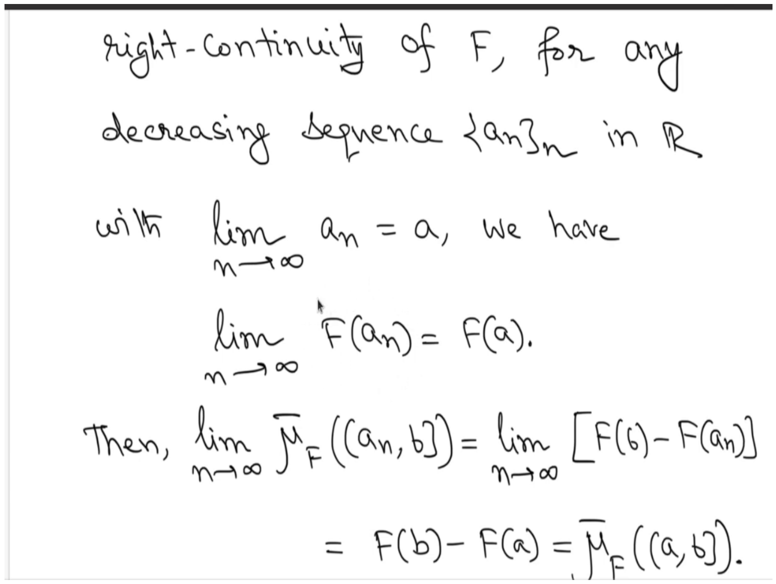

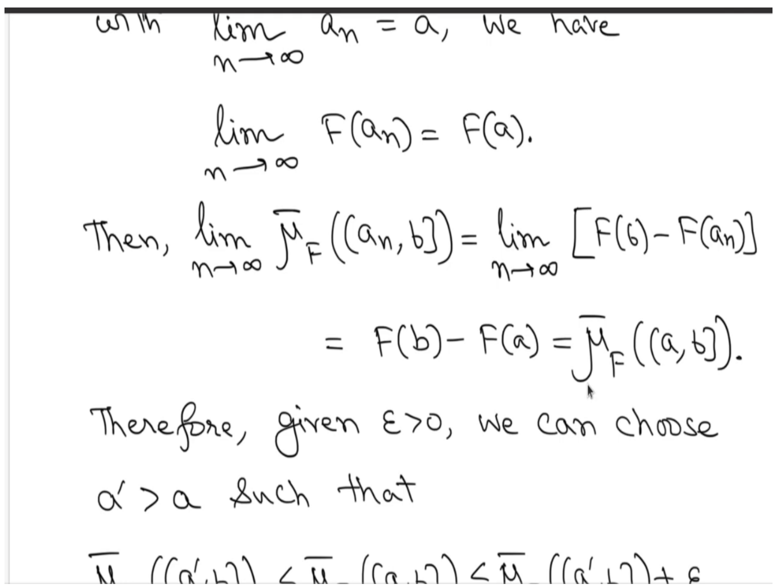

So, let us now make another observation. So, here we are starting off with some interval left open right closed interval inside the real line. What we observe first is that this set is inside the field, that we have considered. But then this set is also inside the extended real line. So, just to be clear, when you fix this a and b, if a and b are real numbers, then there is no problem. So, that is the type of sets we are considering here.

But if $b = \infty$, then you have to make the appropriate justification. But then what we are going to consider is this left endpoint a. And we use right continuity of the given distribution function F.

What do we do? We choose a decreasing sequence of real numbers. So, here we are not using extended real numbers, we are going to use a decreasing sequence of real numbers, call it $\{a_n\}$.

We can construct such a sequence, such that the limit is that number a, the left endpoint of the interval, we can choose such a sequence of real numbers. And by the right continuity of the distribution function, we can also claim that $F(a_n)$, the function values at a_n, we will go down to the function value at a. So, this is simply using the right continuity of the distribution function.

So, now observe that the value associated to these sets $(a_n, b]$. So, this is also a left open right closed interval. And you can think of it as inside the real line as well as the extended real line. So, if you think of it as a set inside the extended real line, then what is the value associated to it by the $\bar{\mu}_F$. So, that is nothing but this increment of the functions. Now, take limits as n goes to ∞.

So, you have to consider this limit now. but then here, only limit that you need to be concerned about is $F(a_n)$. But then by the right continuity $F(a_n)$ converges to $F(a)$. So, therefore, this is nothing but the size of the set under the set function $\bar{\mu}_F$. So, we are focusing on $\bar{\mu}_F$ here. So, we will see the reason why.

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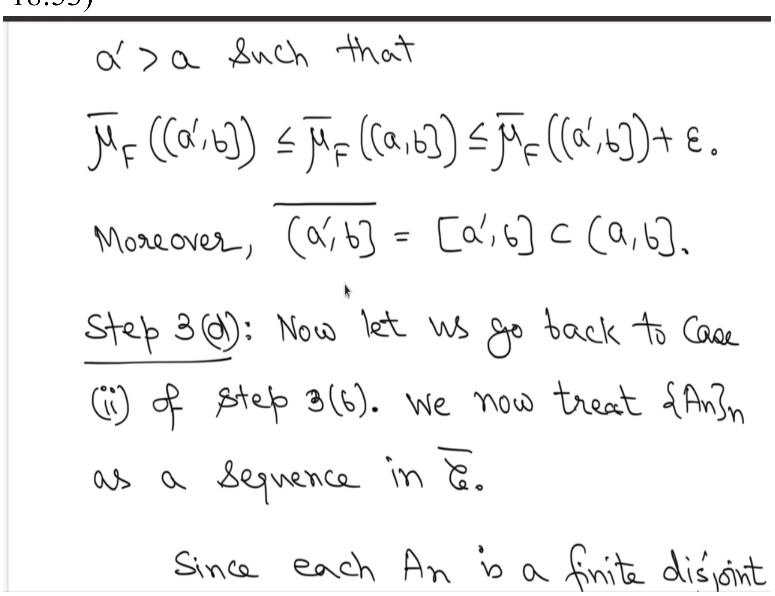

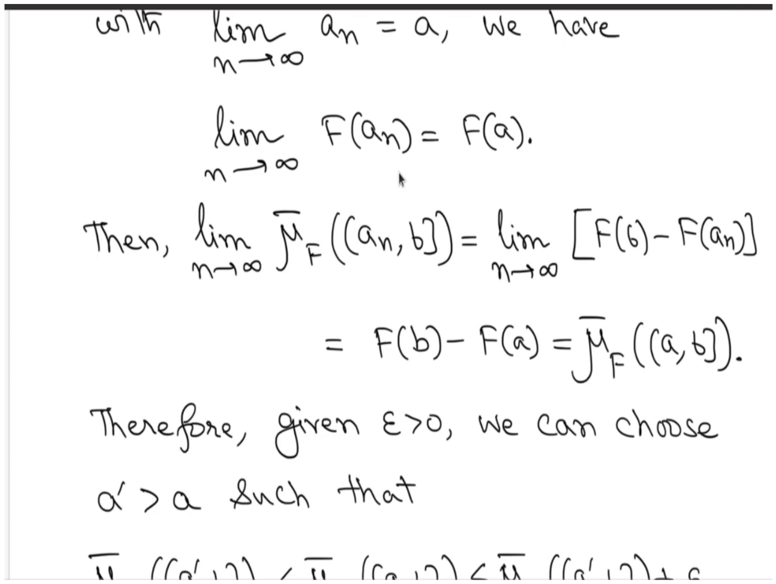

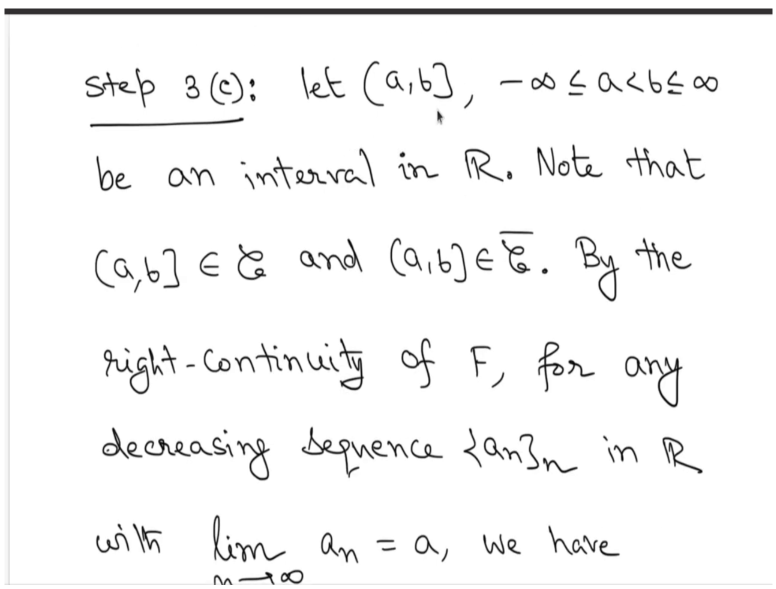

So, what we are saying is this, that given this $\epsilon > 0$, we can choose some a' which is slightly larger than a. So, this could be chosen as one of the a_n's that approximate the values a from above. So, we can choose such a value a', by approximating from above. Such that, this set gets associated smaller mass, so that is very clear. So, $a' > a$. So, the set $(a', b]$ is contained in $(a, b]$. So, even for finitely additive set functions like this, you can easily check that this inequality will hold.

And what we are going to observe is that by this right continuity argument, we can actually approximate it from above side, in this sense that you can choose this value a', so that you get a slightly smaller set inside the interval (a, b). So, you are now choosing this interval, $(a', b]$, which is a slightly smaller interval contained inside. But its size is very close, in the sense that given this $\epsilon > 0$, the size of this plus ϵ will be an upper bound for the size of $(a, b]$.

So, why is this again? So, this is purely for the reason that this limit a is this limit value. So, this limit is exactly this size. For exactly this reason, this value plus ϵ for an appropriate choice of a', you can get this inequality. So, moreover absorb another fact that the closure of this left open right closed interval with this approximating interval that I have just constructed is closure of $(a', b]$.

But that is contained inside $(a, b]$. That is by construction, a' is always a real number, that is why, construction. So, these observations that we did says that, given any interval $(a, b]$ to being considered any such interval, then you can look at the size of this interval according to $\bar{\mu}_F$. And you can obtain a slightly smaller interval, such that it is still a left open right closed interval, but it approximates the size of the given interval with a pre-assigned error bound. So, given any $\epsilon > 0$, you can figure out such a interval, which is contained inside $(a, b]$.

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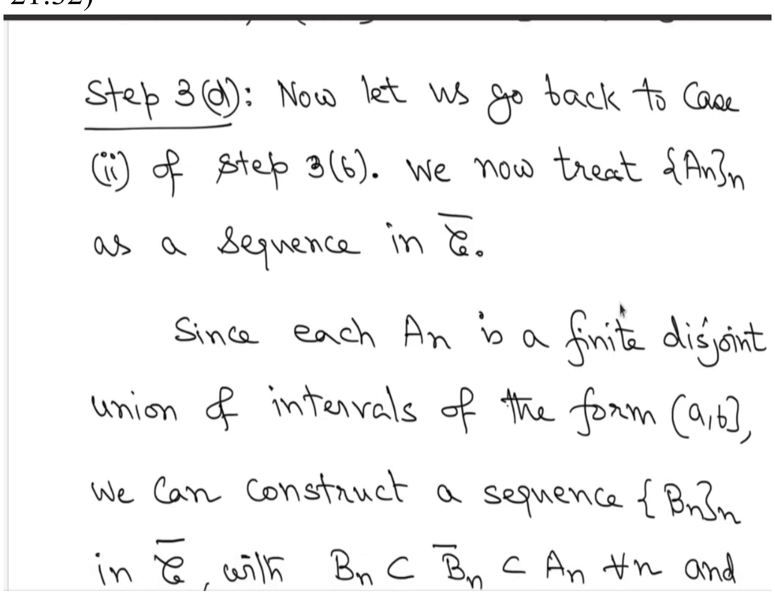

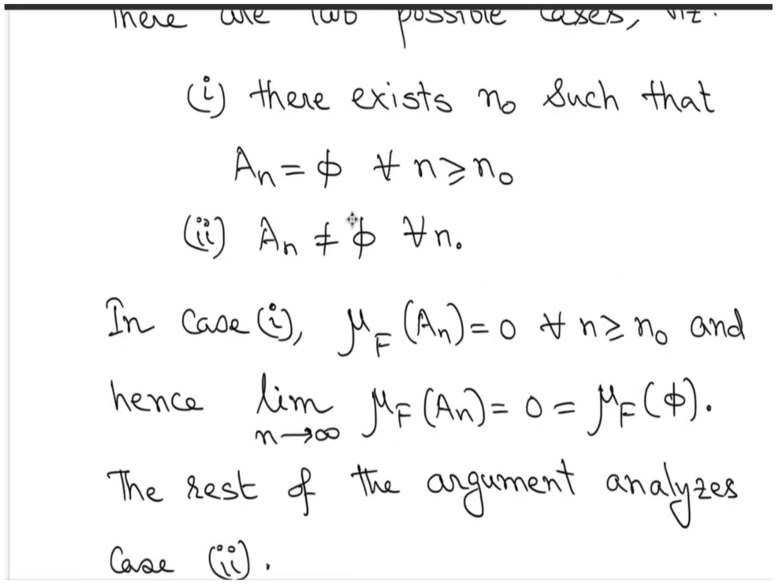

So, now we go back to the step two. So, what was this, so, we wanted to show the continuity from above for this sequence A_n. So, that was the step. And what we are assuming now, that all these A_n's are not the empty sets, but they decreased to the empty set. So, the complete intersection, the countable intersection is the empty set. So, let us start that argument once more.

So, here are these left open right closed intervals or their finite disjoint unions can be considered as a sequence in $\bar{C}$, which is a field on the extended real line. But since each A_n is either left open right closed interval or a finite disjoint union of such intervals, we can now construct a sequence B_n, again in inside the field. Such that, $B_n \subset \bar{B}_n$, so, that is fine. But $\bar{B}_n$ should be contained in A_n.

Such that, the sets B_n that sizes approximates this in this format. So, remember here A_n's and B_ns are sets inside the real line. So, you can treat them as sets inside $\bar{C}$ or otherwise. So, here $\bar{\mu}_F$ or μ_F, both will assign the same values. But then, interestingly enough, by the construction that we mentioned in the previous step, even if A_n's are one single left open right closed interval or a finite disjoint union of them, for each such interval, you can approximate it from inside by a pre assigned error bound.

And by that, you can construct this sets B_n. So, if A_n's are finite disjoint unions of left open right closed intervals, for each interval, you take that smaller interval content inside, take their finite disjoint unions, then they will give you this sets B_n for you with this as pre assigned error bound. So, you can choose this ϵ accordingly beforehand. And you can set this error bound by yourself.

So, that is not a problem. So, for each n, B_n, which is again, a final disjoint union of left open right closed intervals, but purely contained inside the real line. So, that is why there is no distinction here between μ_F and $\bar{\mu}_F$.

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for any $\varepsilon > 0$ chosen beforehand.

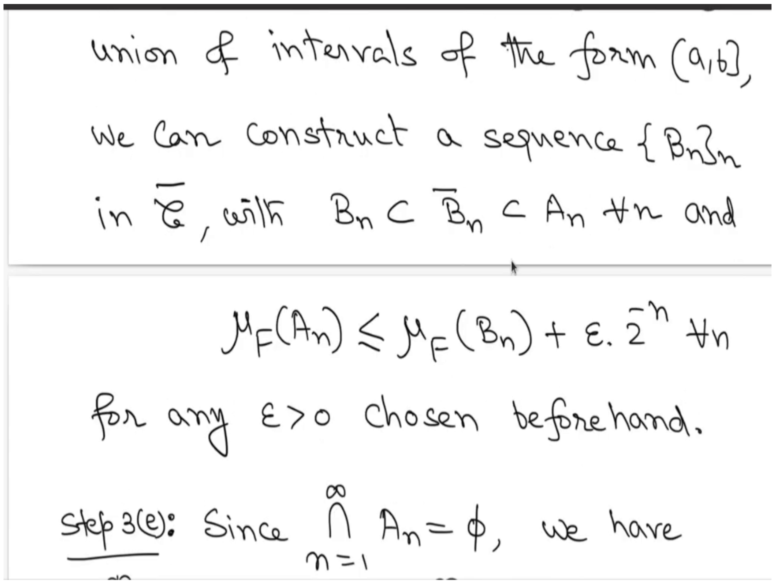
Step 3(e): Since $\bigcap_{n=1}^{\infty} A_n = \phi$, we have $\bigcap_{n=1}^{\infty} \bar{B}_n = \phi$, i.e. $\bigcup_{n=1}^{\infty} (\bar{\mathbb{R}} \setminus \bar{B}_n) = \bar{\mathbb{R}}$.

Therefore, the open sets $\bar{\mathbb{R}} \setminus \bar{B}_n$, $n = 1, 2, \ldots$ covers the compact set $\bar{\mathbb{R}}$.

Then there exists n_0 such that

$\bigcup_{n=1}^{n_0} (\bar{\mathbb{R}} \setminus \bar{B}_n) = \bar{\mathbb{R}}$ and hence $\bigcap_{n=1}^{n_0} \bar{B}_n = \phi$.

Then, using the inequality in step 3(d) we have $\bar{\mu}_F(A_{n_0}) < \varepsilon$ and hence $\bar{\mu}_F(A_n) < \varepsilon \ \forall n \geq n_0$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n \to \infty} \bar{\mu}_F(A_n) = 0$.



But then, observe that the complete intersection, the countable intersections of A_n's are given to be the empty sets. And $\bar{B}_n \subset A_n$. So, their countable intersection is also the empty sets. Now, think of these $\bar{B}_n$ as sets inside the extended real line. You can now say that the union of the complements, which we write is in this notation, $\bar{\mathbb{R}} - \bar{\mathcal{B}}_n$. So, this is the complement of $\bar{B}_n$ inside the extended real line.

So, here what is happening is that these complements if you look at the union of that, that will be the extended real line. So, this is simply just flipping this relation by doing complementation. So, this intersection becomes a union of the complements. But then, $\bar{B}_n$ are closed sets. So, $\bar{\mathbb{R}} - \bar{\mathcal{B}}_n$, the complements will be open sets. And what we are saying is that the union, the countable union of these open sets is covering the whole of the extended real line.

But since, extended real line is compact, so, here we appeal to the compactness of the extended real line, you must find finitely many such open sets whose union will be exactly the extended real line. So, simply stated there exists a natural number n_0 such that the union of first n_0 of these sets will cover the extended real line. So, you do not have to look at n from 1 to ∞, you do not have to look at all the sets, you can stop at a finite stage.

You can figure out such n_0, such that this finite union will already cover the whole of extended real line. But then, now, again go back to the complemented version of this statement. So, this union becomes a intersection. So, then you immediately are able to claim that intersection of these finitely (())(26:01) $\bar{B}_n$ is the empty set. So, this is simply complementing this relation on both sides, very good.

So, now, we have this fact, that the finite intersection of these first few $\bar{B}_n$ is already giving you an empty set. Now, if you use the upper bound, that is mentioned in step III(d). So, let us go back, says in step III(d), remember that $\mu_F(A_n)$ will be less or equal to this quantity. So, if you use a appropriate set theoretic argument, what we will be able to show is that $\mu_F(A_n)$ will be small.

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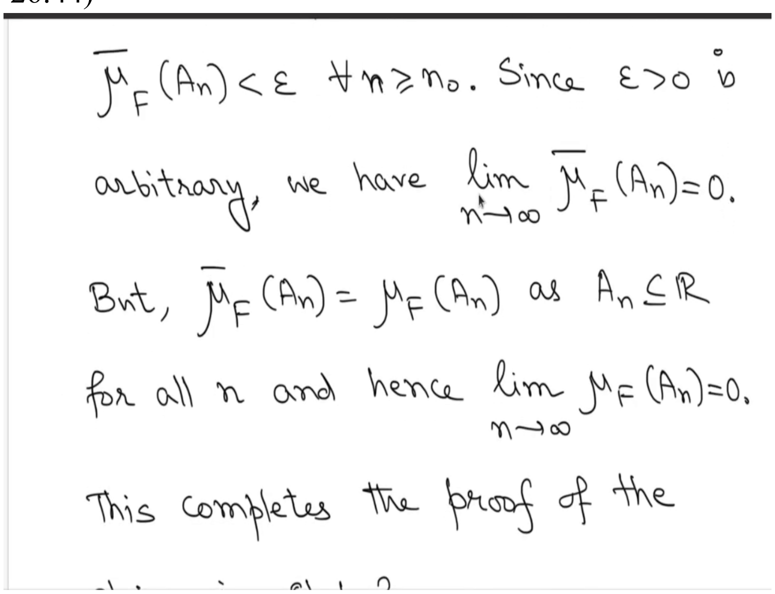
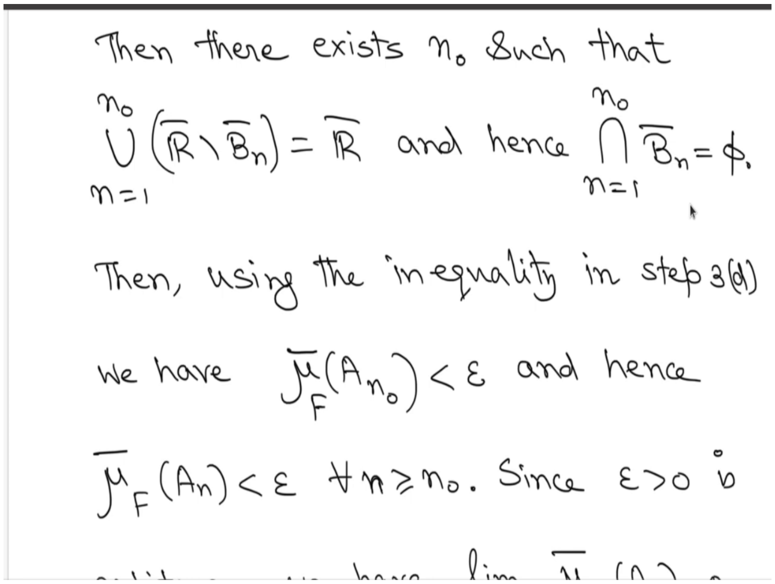
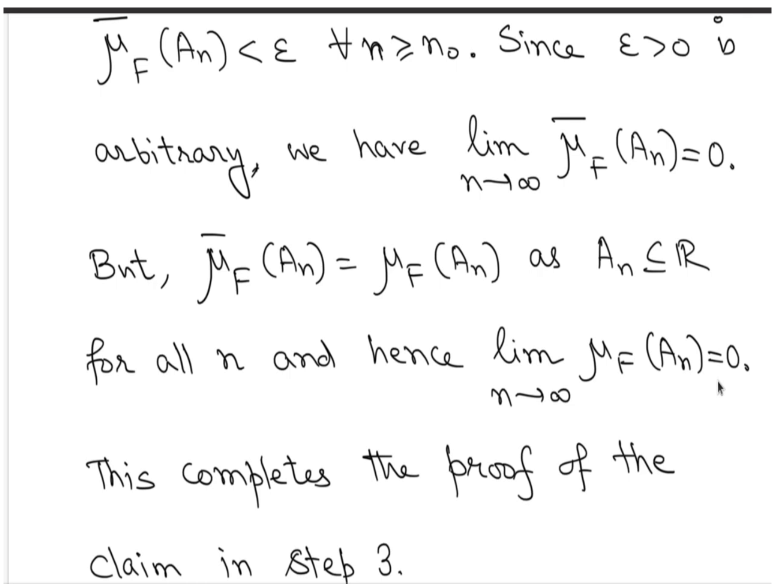




Then there exists n_0 such that $\bigcup_{n=1}^{n_0} (\overline{\mathbb{R}} \setminus \overline{B}_n) = \overline{\mathbb{R}}$ and hence $\bigcap_{n=1}^{n_0} \overline{B}_n = \phi$.

Then, using the inequality in step 3(d) we have $\overline{\mu}_F(A_{n_0}) < \varepsilon$ and hence $\overline{\mu}_F(A_n) < \varepsilon \ \forall n \geq n_0$. Since $\varepsilon > 0$ is



$\overline{\mu}_F(A_n) < \varepsilon \ \forall n \geq n_0$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n \to \infty} \overline{\mu}_F(A_n) = 0$.

But, $\overline{\mu}_F(A_n) = \mu_F(A_n)$ as $A_n \subseteq \mathbb{R}$ for all n and hence $\lim_{n \to \infty} \mu_F(A_n) = 0$.

This completes the proof of the claim in Step 3.

So, that is all we are basically doing. So, if you go about doing that argument, you will be able to show, that for the set A_{n_0} you will get this upper bound. So, you have to do it appropriate set theoretic way by looking at all these inclusions between the sets. You have to do a bit of argument, try to work this out, but from the statement that the final intersection of all these $\bar{B}_n$ is the empty set, you can then try to show that $\bar{\mu}_F$ associates the value smaller than ϵ to the set A_{n_0}.

But then for the later sets, the sets come afterwards. This sets A_n are contained inside A_{n_0}, because this is a decreasing sequence and therefore their sizes will be smaller. So, therefore, for this A_{n_0} onwards, all the sizes are smaller than epsilon. But what do we do? We had fixed that ϵ beforehand, and then we have somehow figured out this value of n_0. So, therefore, what we are saying is that given any $\epsilon > 0$, we can figure out this n_0.

Such that the size of the sets after the stage onwards is smaller than epsilon, and that is exactly the definition of the limit saying that the limit of $\bar{\mu}_F(A_n)$ will be 0. So, the value assigned to the sets A_n by the set function μ were that limit will be 0. But then, as long as you consider these sets within the real line, $\bar{\mu}_F$ and μ agree. So, this is something we had already observed. And therefore, you can immediately claim that $\mu_F(A_n)$ is exactly equal to 0.

So, what did we do? We started with a decreasing sequence of sets An, inside the field of finite disjoint union of left open right close intervals in the real line. So, for such sequence of sets, we have shown that the, the limit value for this sizes is 0. So, therefore, this will show you the continuity from above at the empty set. And hence, it will now complete the proof of that claim in step three.

And the later steps we have already discussed in the previous lecture, that afterwards you claimed countable additivity of the set function μ_F. And then appeal to the Caratheodory's extension theorem to get the appropriate extension to the whole of real line with borel σ-field, that measurable space. So, this completes the proof of that claim.

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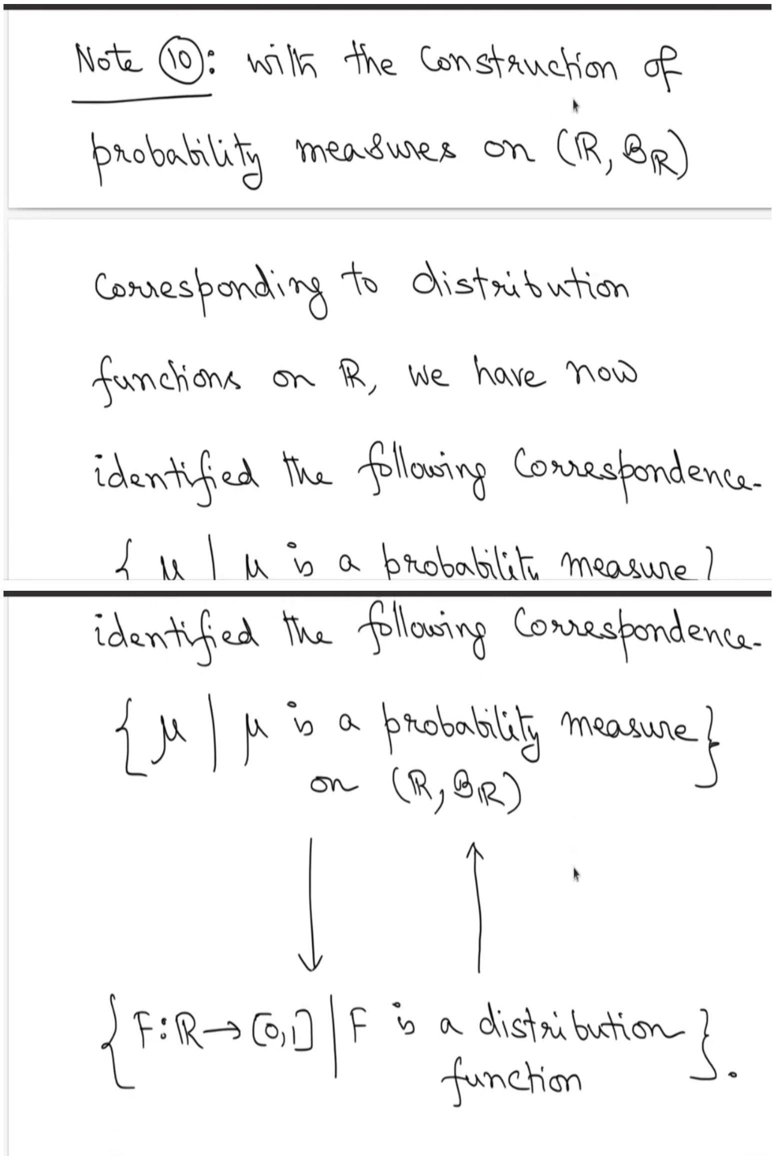

But then, note that with this construction, we have now constructed these probability measures corresponding to distribution functions. And just to formally write down the correspondence, that we have mentioned multiple times, now, everything is proved. So, what we have now proved is the following correspondence, that on one hand, you if you consider the probability measures on the real line with the borel σ-field.

And on another hand, you consider this class of distribution functions. So, by that I mean it is a function defined on the real line taking values between 0 and 1. It is a non-decreasing right continuous function limits at $+\infty$ and $-\infty$ being 1 and 0 respectively. So, there is this nice correspondence between these two collections of sets or these two collections. And this is completing the connection, between random variables, their corresponding laws and their corresponding distribution functions. So, we have completed that identification now. We are going to discuss the higher-dimensional analogs in a later lecture in this week.

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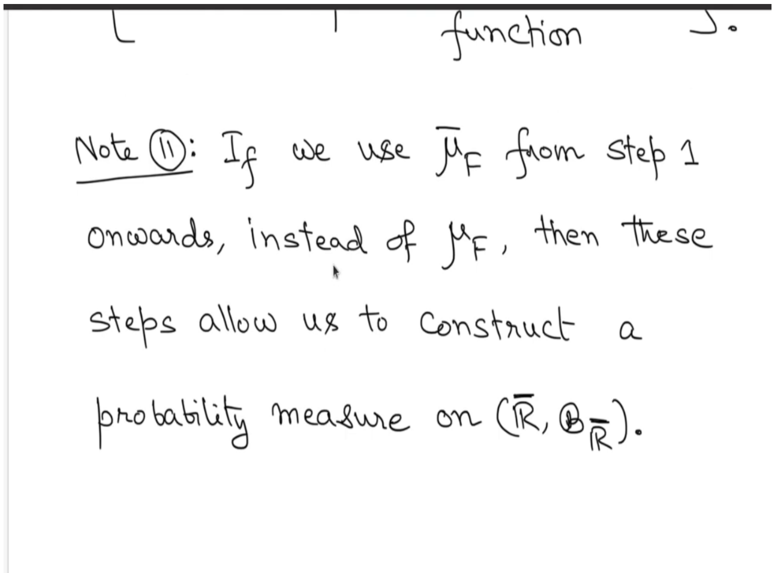

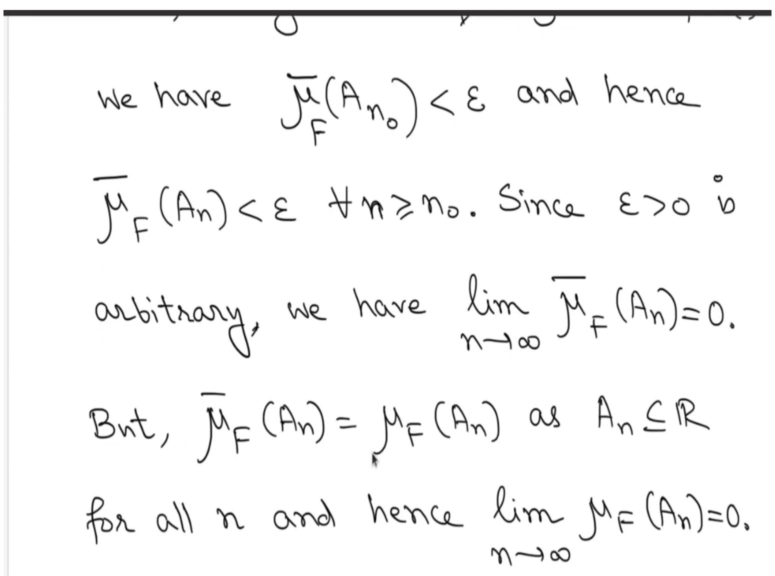

Another comment, before we stop is that if we use $\bar{\mu}_F$ right from the beginning, so, here what did we do? In the previous lecture, we started with μ_F, which is defined on this field of finite disjoint union of left open right closed intervals inside the real line. But if you start with the extended real line, consider the finite disjoint union of left open right closed intervals inside the extended real line.

There you can start talking about μ word itself right from the beginning, right from step one. If you use that, then follow the argument, what you will be able to show is that this $\bar{\mu}_F$ as already mentioned is non negative, is finitely additive. Then you verify the relevant continuity properties, claim the countable additivity of $\bar{\mu}_F$ on the field of finite disjoint union of left open right closed intervals inside the extended real lines.

Then, we can appeal to the Caratheodory's extension theorem and get an extension to these measurable space, $\bar{\mathbb{R}}$ together with the borel σ-field of $\bar{\mathbb{R}}$. So, the same construction that we have discussed, but we have avoided most of the construction or most of the terms involving the extended real line, we had bypassed that. But if you had started doing that right from the step one, you could have constructed a probability measure on the extended real line together with the borel σ-field.

So, this is an important observation, that says that given a distribution function, first of all, you can consider it as a function defined on the extended real line by specifying the value 1 at ∞ and 0 at $-\infty$. So, you get that function, corresponding to that function, you can construct this probability measure, let us call it $\bar{\mu}_F$ on the extended real line. And that is also in correspondence with that function, that we have just talked about on the extended real line.

And what you can then observe is that, something that we have used in this proof that for certain types of sets, this $\bar{\mu}_F$ and μ_F agree. So, this you could extend using the principle of good sets, there are certain restrictions that we have mentioned earlier, that if you restrict, borel set inside the extended rear line. So, by that I mean take a borel set in the extended real line take it intersection with the real line.

So, that will now become a borel set inside the (())(32:59). So, that is the identification that we are, while writing these sets on both sides. So, $\bar{\mu}_F$ and μ_F will agree under that identification. So, what will happen is that if you had actually done this construction of $\bar{\mu}_F$ starting from step one, then you would have obtained a probability measure on the extended real line.

But, the thing is that this borel σ-field on the real line. If you take sets from that, and do the appropriate identification, you could restrict the probability measure from $\bar{\mathbb{R}}$ to the real line. So, that is another way of proving this. But we want to avoid talking about too much on the extended real line, that is why we stayed with the real line and we (())(33:42) almost throughout the proof, only to prove the relevant continuity properties we had brought in this $\bar{\mathbb{R}}$ and prove the relevant properties.

So, here we have appealed to the extended real line compactness. And therefore, we have managed to prove the relevant continuity properties. So, in later lectures, we are going to see extended versions of these result for higher dimensional Euclidean spaces. So, we will continue this discussion in the next lecture.