Measure Theoretic Probability 1

Professor Suprio Bhar
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 21

From Distribution Functions to Probability Measures (Part 1)

Welcome to this lecture, this is the first lecture of week 5. So, before we proceed, let us first recall what we have already covered. So, we have crossed the halfway mark of this course. In week one, we have discussed about measurable spaces. In week two, using that information from week one we consider measure spaces and in particular probability spaces. Then in week three, we considered measurable functions.

Then, using this information from measure spaces, in particular probability spaces, and measurable functions on top of this probability spaces, which we called as random variables. We defined the law or distribution of random variables in week four. So, then, in week four, we considered this law. And conversely, we have also shown that, given this law, which is a probability measure on the real line, for the case of random variables, or $\mathbb{R}^d$ for the case of a $\mathbb{R}^d$ valued random vector.

So, given such a probability measure, you can always construct a probability space and random variable or a random factor with that specified law. So, we had obtained this correspondence between random variables or vectors and probability measures. So, we had seen that correspondence. Then, from probability measures we went to a class of functions, which we called distribution functions corresponding to these probability measures.

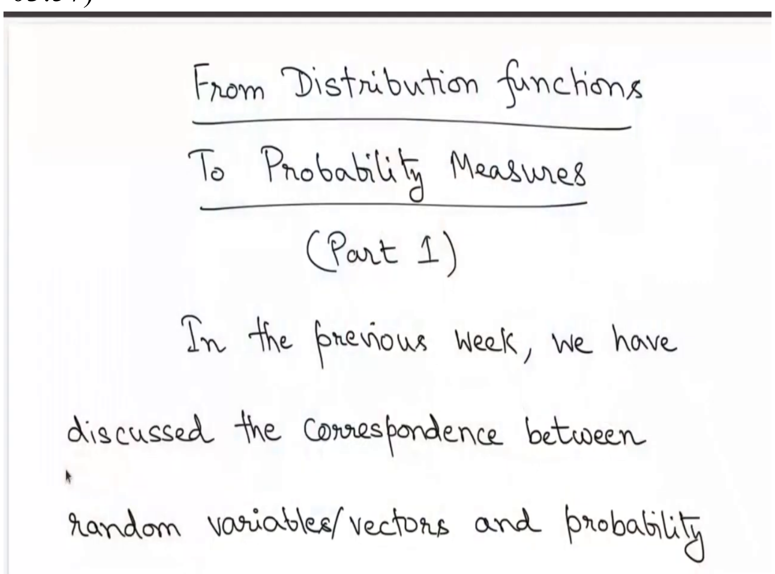
And using that connection, we had said that, we will define the distribution function of a random variable or random vector to be the distribution function corresponding to its law. We had stated the connection between random variables and vectors and probability measures. So, that was both-way correspondence, both-way connection, both-way correspondences, but, for the case of distribution functions, we had only obtained the one-way connection, one-way association from the probability measures to the distribution functions.

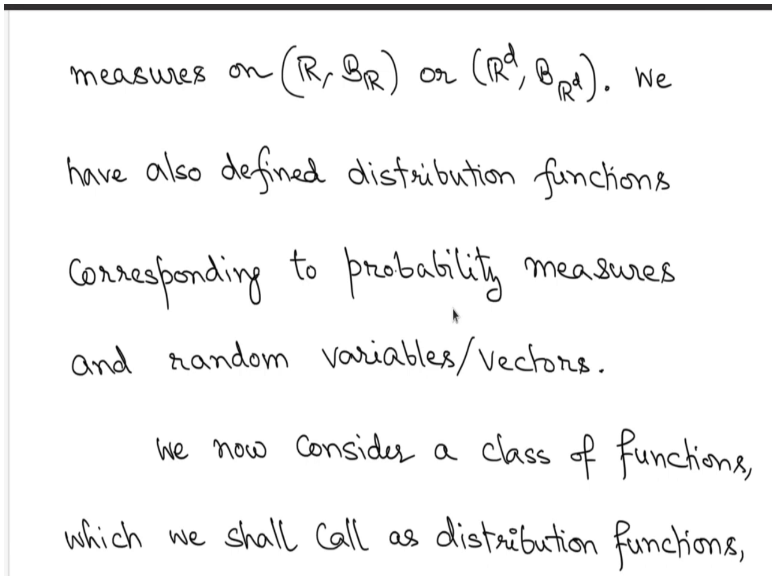
And we had remarked that, we are going to discuss the other way connection soon. So, this is what we discuss in week five. So, we are going to see that the standard properties of distribution functions, by that, I mean that it is a function defined on the real line, takes values between 0 and1, is non decreasing, right continuous, limit at ∞ is 1, limit at $-\infty$ is 0. So, these are the standard properties of the distribution function corresponding to a probability measure.

So, given such a function, which apparently, needs not be connected with such a probability measure, we are going to construct a probability measure. So, and that will finish that correspondence between the probability measures and distribution functions. Of course, whatever I have just said, is applicable for the dimension one case, meaning for real valued case. However, you can extend all of these ideas in higher dimensions, in $\mathbb{R}^d$.

In first two lectures, we are going to discuss the one-dimensional case. And then we shall appropriately extend these ideas and these results to higher dimensions. And that we shall discuss in the later part of week five. So, in the first two lectures, we are going to focus on the one-dimensional case. So, as before let us move on to the slides.

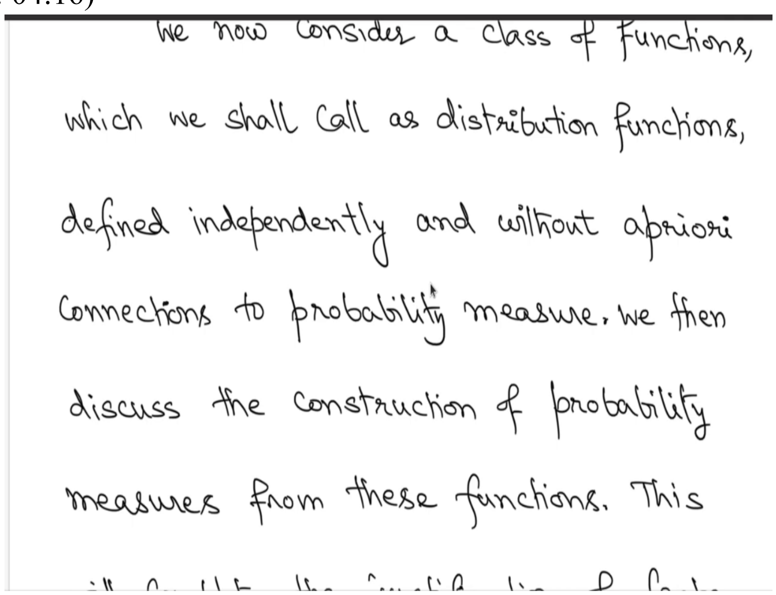
(Refer Slide Time: 03:57)





So, in the previous week, we had discussed this correspondence between random variables and vectors, and probability measures. And we had also defined as distribution functions corresponding to probability measures. And that connection we had used to define distribution functions of random variables or vectors.

(Refer Slide Time: 04:16)



Connections to probability measure. We then discuss the construction of probability measures from these functions. This will complete the justification of facts mentioned earlier in Week 4 (See Note 8 and 11). In this lecture, we

of this result.

Definition 1 (Distribution function)

Say that a function $F : \mathbb{R} \to [0,1]$ is a distribution function if F is non-decreasing and right-continuous and

$$\lim_{x \to \infty} F(x) = 1 \ , \quad \lim_{x \to -\infty} F(x) = 0.$$

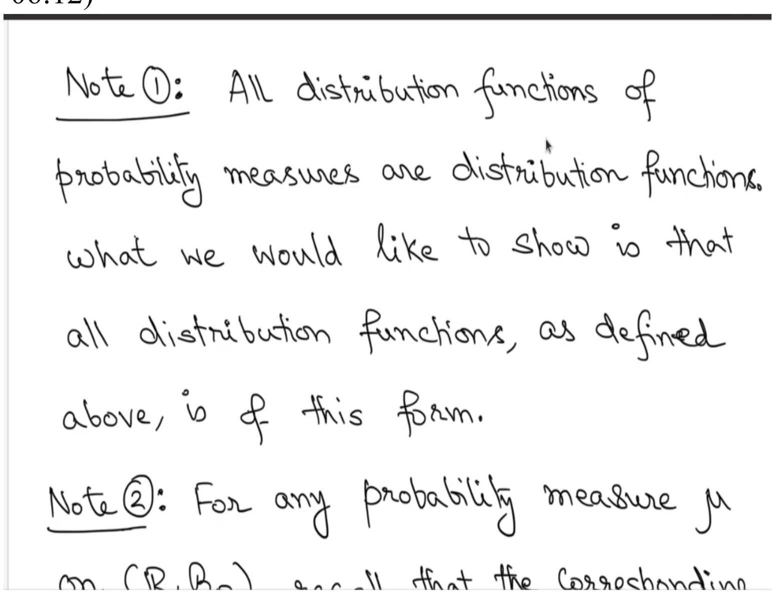
We are now going to consider a class of functions, which we are going to refer to as distribution functions. Now, this class of functions, these functions are defined independent and without any apriori connection with any probability measure. So, this is the idea. So, we will first start with certain properties of class of functions, with which there is no apriori connection to probability measures. But we are going to show that we can construct a probability measure and this function will turn out to be the corresponding distribution function with respect to this probability measure.

So, we are now going to discuss this construction of probability measures from this class of functions. So, this is going to complete that appropriate justifications for the results and facts that were mentioned earlier in week four. So, you can refer to notes eight and eleven, in week four notes. So, as mentioned in the introduction, we are going to focus on the1 dimensional version of this connection.

Meaning, given this class of functions, we are going to construct appropriate probability dimensions. And this is going to give us all possible probability measures on the real line, defined on the borel σ-field. So, let us first start with this basic definition. So, you consider a function defined on the real lie, taking values between 0 and 1. And you are going to refer to such a function as a distribution function.

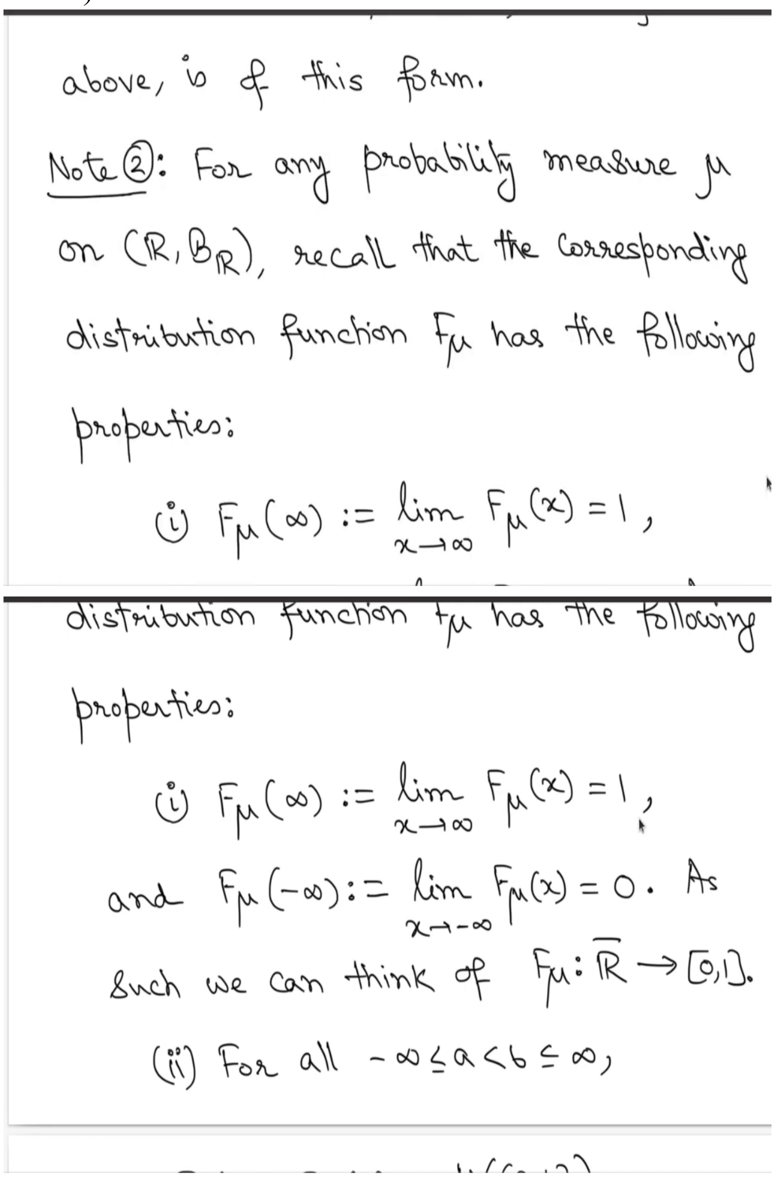
If the function is non-decreasing, right continuous and has this appropriate limits. Meaning, limit at ∞ is 1 and limit at $-\infty$ is 0. So, you call such functions as distribution functions. So, as you see here, there is no apriori connection with any probability measure. So, in this definition, no probability measure has been mentioned, no random variables has been mentioned.

(Refer Slide Time: 06:12)



But then, note that all distribution functions that you can find corresponding to probability measures, will satisfy these properties. So, therefore, all distribution functions of probability measures are distribution functions in the sense of the above definition. And what we are going to discuss, what I would like to show is that all distribution functions, actually give you all probability measures in this connection. That means, that all distribution functions as considered above again, is of this form.

(Refer Slide Time: 06:48)



As mentioned earlier, in this lecture, we are going to discuss the construction of probability measures from distribution functions. And in fact, we are going to see, that given this distribution function, once you construct the probability measure, this will also allow you to make this identification that all distribution functions appear as the distribution function of a probability measure of this type.

To do this, we first take some motivation from distribution functions corresponding to probability measures. So, let μ be a probability measure defined on this measurable space real line together with the borel σ-field. And recall, that the corresponding distribution function has some nice properties. So, the first property that we are going to focus on is the value at ∞ and $-\infty$.

So, let me rephrase, we are going to focus on the limits at ∞ and $-\infty$. So, here, we already know that the limit at ∞ exists. And it is one, limit at $-\infty$ also exists and its value is 0. Now, what we do is that we think of the distribution function as a function defined on the real line. So, that is as per the definition. But now, what you can try to do is to extend the domain by adding the two points $+\infty$ and $-\infty$.

So, what is the domain again? The domain is the real line, we are adding these two points $+\infty$ and $-\infty$ and therefore, we are going to get the bigger domain, which is the set of extended real numbers. And now, let us say we want to define or extend of a given distribution function on the extended real line. So, you have to associate the values for the points $+\infty$ and $-\infty$.

What you can choose to do is to associate the value one, the limit value at ∞ to be the value of the function at ∞. And similarly, look at the limit at $-\infty$, which is 0, you associate this value to the function value at $-\infty$. So, you are going to associate the values $+1$ to ∞, then associate the value 0 to the point $-\infty$. Once you do that, you can get the function extended to the extended real line.

So, you have just had to define the corresponding values to these extra points $+\infty$ and $-\infty$. And thereby, we obtained this distribution function on the extended real line. We are going to see the usage of this later on, in the construction yourself. But this is a quite a useful observation, that by using the limit values, you can extend that function defined here.

(Refer Slide Time: 09:37)

(ii) For all $-\infty \leq a < b \leq \infty$,

$$F_{\mu}(b) - F_{\mu}(a) = \mu((a,b]).$$

(iii) More generally, if $-\infty \leq a_i < b_i \leq \infty$ for $i = 1, 2, \ldots, n$ with $(a_1, b_1], (a_2, b_2], \ldots, (a_n, b_n]$ being pairwise disjoint, then

$$F_{\mu}(b) - F_{\mu}(a) = \mu((a,b]).$$

(iii) More generally, if $-\infty \leq a_i < b_i \leq \infty$ for $i = 1, 2, \ldots, n$ with $(a_1, b_1], (a_2, b_2], \ldots, (a_n, b_n]$ being pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) = \sum_{i=1}^{n} \left[F_{\mu}(b_i) - F_{\mu}(a_i)\right].$$

Furthermore,

But then, you have this interesting connection, that the increment of the function values for the points $a < b$, increments of the function values is exactly the, the size or weight or the measure of the interval, $(a,\ b]$ under the measure μ. So, this we have seen earlier. So, again, just to note that here, we are following the notation that a can be the point $-\ \infty$ and b can be the point $+\ \infty$.

However, μ is defined on the real line. So, you have to interpret this interval accordingly. So, for example, if b is ∞ here, so, closed ∞, you have to reinterpret it as $(a,\ \infty)$. So, we are following those same notations that were considered earlier. So, the probability measure μ is purely defined on the real line. So, that is the given probability measure. So, we cannot talk about points ∞ to be included in the intervals $(a, b]$.

And more generally, for such type of intervals, look at this $(a_i, b_i]$, for $i = 1$ to n. If these intervals are pairwise disjoint, you can now make this observation, that the size of these disjoint union of intervals is exactly given by the summation, which can be expressed exactly as the increments of the function values, the distribution of functions values. So, you look at these increments of the distribution function values and add them up. So, that is what we are doing.

(Refer Slide Time: 11:11)

Furthermore,

$$\mu(\mathbb{R}) = \lim_{n\to\infty} \mu((-n,n]) = \lim_{n\to\infty} [F_\mu(n) - F_\mu(-n)]$$

$$= \lim_{x\to\infty} F_\mu(x) - \lim_{x\to-\infty} F_\mu(x)$$

$$= 1 - 0 = 1.$$

(ii) For all $-\infty \leq a < b \leq \infty$,

$$F_\mu(b) - F_\mu(a) = \mu((a,b]).$$

(iii) More generally, if $-\infty \leq a_i < b_i \leq \infty$ for $i = 1, 2, \ldots, n$ with $(a_1, b_1], (a_2, b_2], \ldots, (a_n, b_n]$ being pairwise disjoint, then

(iii) More generally, if $-\infty \le a_i < b_i \le \infty$ for $i = 1, 2, \dots, n$ with $(a_1, b_1], (a_2, b_2], \dots, (a_n, b_n]$ being pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) = \sum_{i=1}^{n} \left[F_\mu(b_i) - F_\mu(a_i)\right].$$

Furthermore,

And furthermore, you also observe that the measure of the whole real line using the continuity from below, we can show it to be 1. Of course, the probability measure is defined on the real line, but then note that using the properties of the functions, we can show that the measure of the whole real line is 1. So, how do you show this? So, measure of the real line, you use the continuity from below and apply it to these kind of sets $(-\ n,\ n]$.

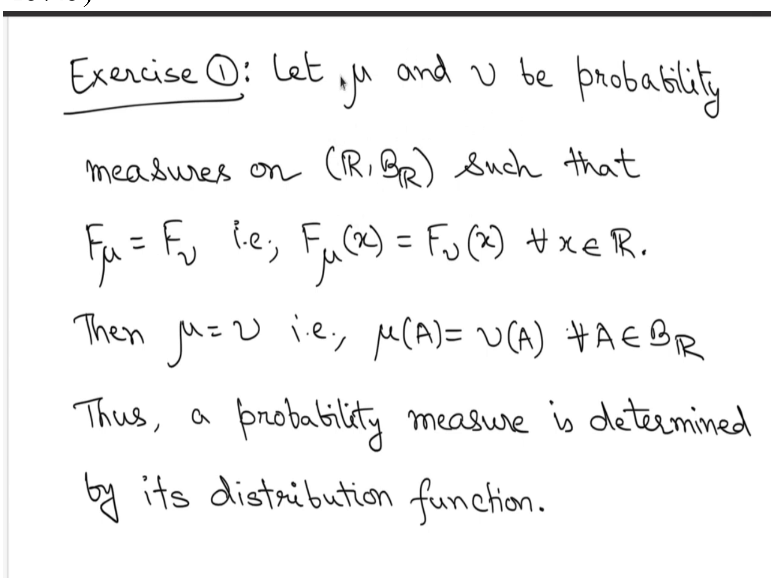
Then these sets increase to the whole real line. So, the size of these sets can be represented as increment of the function values like this. So, look at this function values now, and take the limit. So, these limits now exist as per the assumption, as per the given properties of distribution functions. Then therefore, you end up with this value to be 1 and this other limit to be 0. So, with that at hand, you have that the measure of the real line is 1.

Of course, as we said that the measure of the real line under the probability measure μ is exactly equal to 1. However, note that this 1 we are obtaining through properties of the function. Now, what is the need for this observation? So, let us go back once more. So, we said that measure of this type of intervals can be obtained as increment of the function values.

More generally, if you have finite disjoint union of left open right closed intervals, then the measure of that can again be written in terms of the function values. And then finally, we are saying that the measure of the whole real line can be written and computed by the properties of the function values. So, why you are stressing on the function values and the properties of the function, it will be clear in a moment.

So, what we are basically trying to say is that, given the function which appears on the right-hand side of all of your expressions, we can recover properties of the measure. And using this we hope to construct probability measures from the class of functions that we have started off with. So, here what we know is that if μ is a probability measure, then we are obtaining all these properties of the function from the probability measure. But using those properties, you can recover the measure of these sets. So, let us use these ideas. These identifications, we are going to use and construct the appropriate probability measures.

(Refer Slide Time: 13:43)



So, in this regard, it is an important observation, that if μ and ν are probability measures on the real line. Such that, their distribution functions are the same. That means, that for each point on the real line, the distribution function of μ and distribution function of nu, both agree, they match. If that is the case, then you can show that the measures must also match. That means, for all borel sets A the measure of A with respect to the measure μ and measure of A with respect to measure ν will match, $\mu(A) = \nu(A)$. So, you can restate this statement in words like this, that a probability measure is determined by its distribution function.

(Refer Slide Time: 14:26)

by its distribution function.

<u>Note ③</u>: Another way to state the result in Exercise ① is the following: Let μ and ν be probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu = \nu$ on $\mathcal{E}$, where

$$\mathcal{E} = \{(-\infty, x] \mid x \in \mathbb{R}\}$$

in Exercise ① is the following: Let μ and ν be probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu = \nu$ on $\mathcal{E}$, where

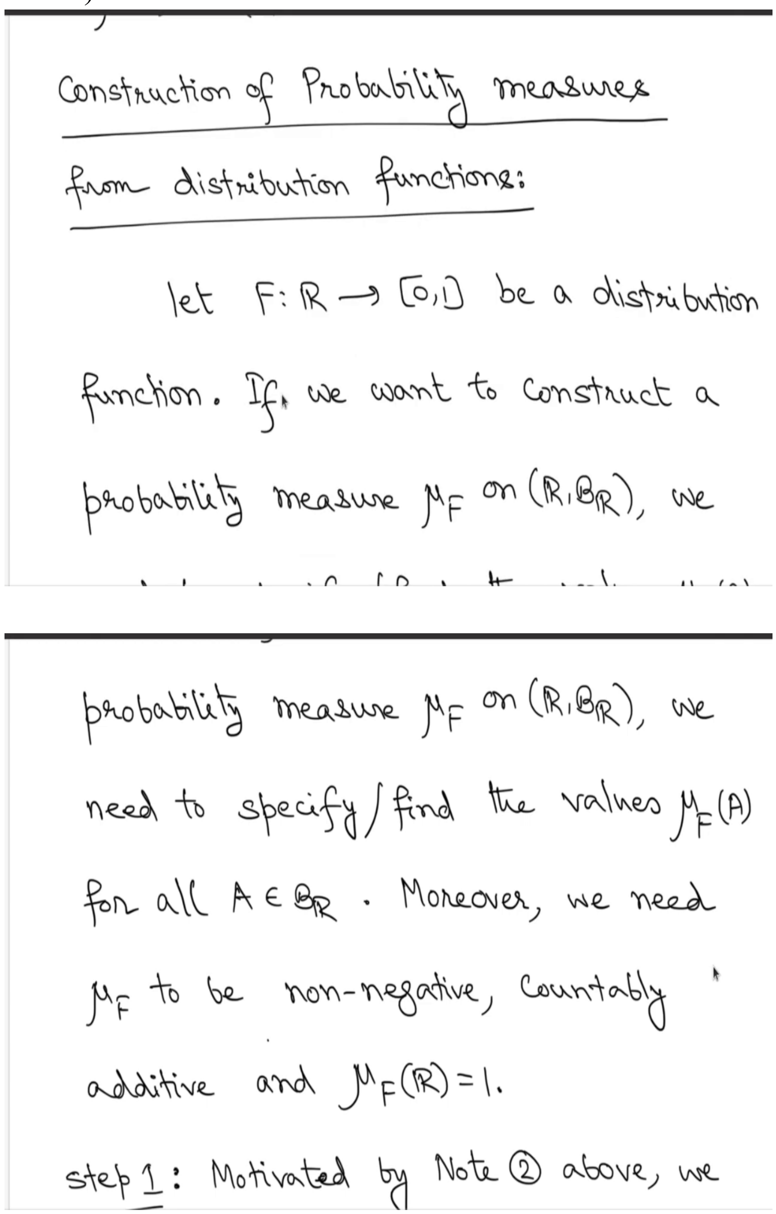
$$\mathcal{E} = \{(-\infty, x] \mid x \in \mathbb{R}\}$$

i.e, $\mu(A) = \nu(A)$ $\forall A \in \mathcal{E}$. Then $\mu = \nu$, i.e. $\mu(A) = \nu(A)$ $\forall A \in \mathcal{B}_{\mathbb{R}}$.

But then there is another way to state the same result, in this exercise. So, it is the following. So, consider these two probability measures once more. Let μ and ν be probability measures on the real line. Then you say that these two probability measures agree on a class of sets, which class of sets it is exactly? $(-\infty, x]$, when you say that these agree that means that for all subsets in your class, this specified $\mathcal{E}$, measure of a under μ and ν will agree.

So, in particular note that your sets A are exactly of this type of intervals. And therefore, they exactly give you distribution functions. But here we are not using the term distribution functions, we are referring directly to the measures. And we are saying if the measures agree on this collection of sets, meaning if the distribution functions agree under the hood, then the two probability measures will match. So, that is just a restatement of the previous exercise. Once you work that out, you will immediately get this statement.

(Refer Slide Time: 15:26)



So, with that at hand, we are now ready to start the construction of probability measures from distribution functions. So, let us start with a distribution function. So, we are still working in the dimension one case. Now, we want to construct a probability measure corresponding to the given distribution function. So, for the duration of the construction, we are going to refer to the constructed set function, which will be a probability measure to be μ_F.

Just to denote the dependence on the function F. And we want to construct it on the, this specific measurable space real line together with the borel σ-field. But then, before defining this probability measure, what we need first is that μ_F should be a well-defined set function on this measurable space. So, and to get a set function, what you need to specify are the values or the sizes for all these types of borel sets A coming from the borel σ-field.

So, you need to figure out the values $\mu_F(A)$. Moreover, once you have specified the set function, then you need to check the properties of a measure, which is that that set function is non-negative and countably additive. Finally, if you want to say that this set function thus constructed is a probability measure, then you also need to check that the size associated to the whole set is 1. So, once you have verified all these things, you can complete the construction. So, these constructions rely heavily on the motivation from the note to above.

(Refer Slide Time: 16:54)

additive and $\mu_F(\mathbb{R}) = 1$.

step 1: Motivated by Note ② above, we define, $F(\infty) := 1$, $F(-\infty) := 0$ and

$\mu_F(\phi) := 0$,

$\mu_F((a,b]) := F(b) - F(a), \ \forall \ -\infty \le a < b \le \infty$,

$\mu_F\left(\bigcup_{i=1}^{n} (a_i,b_i]\right) := \sum_{i=1}^{n} \left[F(b_i) - F(a_i)\right]$, if define, $F(\infty) := 1$, $F(-\infty) := 0$ and

$$\mu_F(\phi) := 0,$$

$$\mu_F\big((a,b]\big) := F(b) - F(a), \ \forall\, -\infty \le a < b \le \infty,$$

$$\mu_F\Big(\bigcup_{i=1}^{n} (a_i, b_i]\Big) := \sum_{i=1}^{n} \big[F(b_i) - F(a_i)\big], \text{ if}$$

$(a_i, b_i]$'s are pairwise disjoint,

and $-\infty \le a_i < b_i \le \infty, \ \forall i = 1, 2, \ldots, n$

So, in note two we had seen that for certain nice type of sets like left and right closed intervals, the size of the set according to the probability measure can be figured out from the properties of the values of the distribution function. So, therefore, motivated by that, we first do this. We first think of the function F as a function defined on the extended real line. So, apriori you are given all the values on the real line.

But then you want to figure out the values on the extended real line. So, that means you have to associate values for the points $+\infty$ and $-\infty$ these two extra points in the domain. And as motivated earlier, you associate the value 1 at the point ∞ and associate the values 0 at the point $-\infty$. So, therefore, you get the function F extended to the extended real line. We will see the usage of this.

But then our main target is to define a set function μ_F, which in preparation for it to be a probability measure, we should define that the size of the empty set is 0. So, we do that right at the beginning. But then we want to specify the size of certain nice type of sets. For example, take this left open right closed interval $(a, b]$. So, here a and b vary between these ranges. So, these are the standard ranges that we have used, and we are following the same notations.

For example, if $b = \infty$, we are going to take the set (a, ∞), because we are taking the sets $(a, b]$ to be within the real line. But then we want to specify the size of this left open right closed interval under the set function. And what we do? As motivated earlier, we will simply define it to be the increment of the function values. But then, in preparation for μ_F to be a probability measure, we first of all require it to be finitely additive.

And therefore, for finite disjoint union of left open right closed intervals, what you should expect is that the size associated to such finite disjoint unions will be simply the addition of the individual sizes. But the individual sizes you have defined it as the increment of the function values. So, therefore, the size associated to finite disjoint unions of left open right closed interval should be this summation. So, this is as expected.

(Refer Slide Time: 19:24)

$(a_i, b_i]$'s are pairwise disjoint,

and $-\infty \leq a_i < b_i \leq \infty, \forall i = 1, 2, \ldots, n$

$$\mu_F(\mathbb{R}) := \lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x) = 1 - 0 = 1.$$

This defines μ_F on the field $\mathcal{C}$ of finite disjoint union of left-open right-closed

$\mu_F(\phi) := 0,$

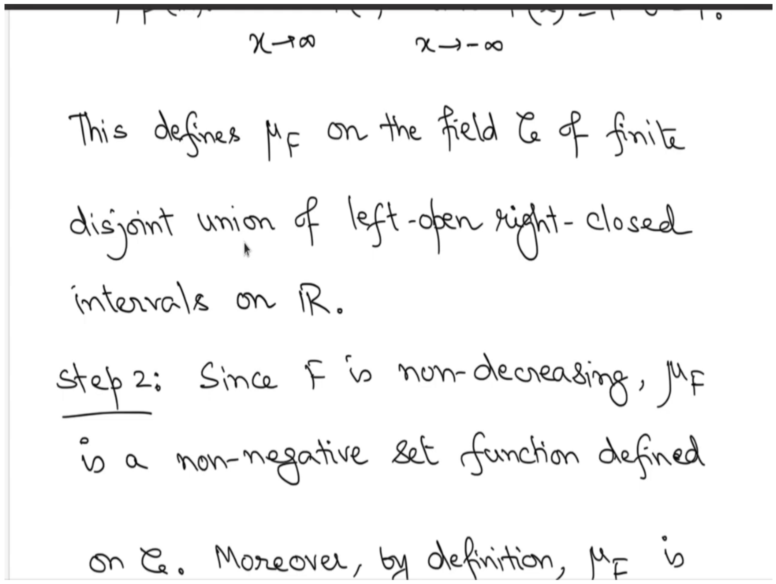
$\mu_F((a, b]) := F(b) - F(a), \ \forall -\infty \leq a < b \leq \infty,$

$$\mu_F\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) := \sum_{i=1}^{n} \left[F(b_i) - F(a_i)\right], \text{ if}$$

$(a_i, b_i]$'s are pairwise disjoint,

and $-\infty \leq a_i < b_i \leq \infty, \forall i = 1, 2, \ldots, n$

$\mu_F(\mathbb{R}) := \lim F(x) - \lim F(x) = 1 - 0 = 1$



And finally, in preparation for it to be a probability measure once more, you require that the size of the whole real line should be 1. But then, this is again I am motivating it through the corresponding function values, with the limits at ∞ and $-\infty$. But then what did we do? We specified the values for the empty set and the whole real line, then we specified sizes for the left open right closed interval, and then finally to all possible finite disjoint unions of left and right closed intervals.

So, therefore, what we have actually obtained is that this set function, which you have just defined, is defined on the field $\mathcal{C}$ of finite disjoint union of left open right closed intervals on the real line. So, you now have a set function defined on this field $\mathcal{C}$. So, you will see the usage of this immediately in the next step.

(Refer Slide Time: 20:16)

intervals on $\mathbb{R}$.

Step 2: Since F is non-decreasing, μ_F is a non-negative set function defined on $\mathcal{C}$. Moreover, by definition, μ_F is finitely additive on $\mathcal{C}$.

Step 3: We claim that μ_F is continuous from above at the empty set. We shall

$\mu_F(\phi) := 0,$

$\mu_F((a,b]) := F(b) - F(a), \; \forall -\infty \leq a < b \leq \infty,$

$\mu_F\left(\bigcup_{i=1}^{n} (a_i,b_i]\right) := \sum_{i=1}^{n} \left[F(b_i) - F(a_i)\right]$, if

$(a_i, b_i]$'s are pairwise disjoint,

and $-\infty \leq a_i < b_i \leq \infty, \; \forall i = 1, 2, \ldots, n$

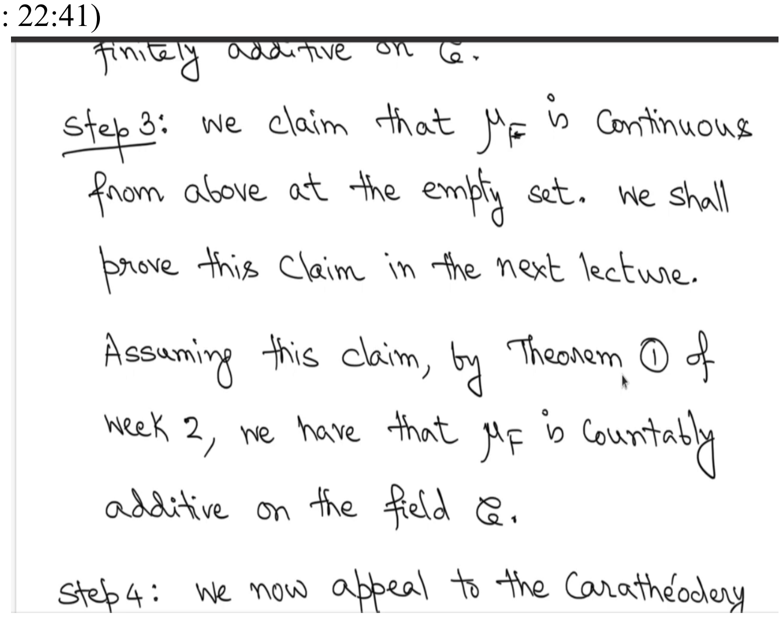
So, observe that since F, the given function f is non-decreasing, μ_F is a non-negative set function defined on this field's $\mathcal{C}$. Why is this? let us go back to the all these values. So, empty sets gets associated the values 0, which is non-negative. the whole real line gets associated the value one which is of course, non-negative, and then look at left and right pose intervals or finite disjoint unions of this.

Since, the function F is non-decreasing, therefore, the function value at b will be at least as big as function value at a. So, therefore, $F(b) \geq F(a)$. And therefore, this quantity $F(b) - F(a)$ is non-negative. And in using that, you can now show that μ_F associates non-negative sizes, to finite disjoint unions of left open right closed intervals. So, therefore, in step two, we immediately observe that the μ_F, the set function thus define is a set function, which takes non negative values.

And it is defined on the fields $\mathcal{C}$. Moreover, by definition, by the construction of new subscript F, it becomes finitely additive. Why? Because if you are going to look at finitely (())(21:36) sets in the field, which are pairwise disjoint, then what are the possibilities. So, first let us take care of the case, when one of the sets is the real line. Then, to get finitely many pairwise disjoint sets, you need to take all the other sets as empty sets.

And therefore, it is easy to check that the finite additivity holds for the choices real line together with any number of empty sets. So, that will be fine. But then, if you are going to use this kind of left open right closed intervals or finite disjoint unions of them by the construction itself, you have put in the finite additivity itself. Because for pairwise disjoint such intervals, you have built in the values to be equal to the individual summation of the sizes of the left open right closed intervals. So, therefore, you can easily observe that μ_F thus defined becomes a finite relative set function on the field.

(Refer Slide Time: 22:41)



Now, you have a non-negative finitely additive set function defined on a field. We are going to claim that μ_F is continuous from above at the empty set. So, we have a certain kind of a continuity for the set function thus defined. but remember that by theorem 1 in week two, we have shown that a finitely additive set function which is continuous from above at the empty set, becomes countably additive on the field.

Therefore, what is happening here is that once you are able to prove this claim, this continuity property for this set function, you will immediately be able to say that μ_F thus defined is a countably additive non-negative set function on the field $\mathcal{C}$. We are going to discuss this proof of the claim in the next lecture, but assume this for now.

(Refer Slide Time: 23:30)

additive on the field $\mathcal{C}$.

Step 4: We now appeal to the Carathéodory Extension Theorem (Theorem 1 of Week 4) to extend μ_F uniquely from the field $\mathcal{C}$ to a probability measure, again denoted by μ_F, on the σ-field $\sigma(\mathcal{C}) = \mathcal{B}_{\mathbb{R}}$. This completes the construction.

Note 4: By construction, the distribution function of μ_F is F itself.

Note 5: Given a probability measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, look at the distribution function F_μ. Then, using the above construction, we can look at the step 1: Motivated by Note ② above, we define, $F(\infty) := 1$, $F(-\infty) := 0$ and

$$\mu_F(\phi) := 0,$$

$$\mu_F\big((a,b]\big) := F(b) - F(a), \ \forall -\infty \leq a < b \leq \infty,$$

$$\mu_F\Big(\bigcup_{i=1}^{n} (a_i, b_i]\Big) := \sum_{i=1}^{n} \big[F(b_i) - F(a_i)\big], \text{ if}$$

$(a_i, b_i]$'s are pairwise disjoint,

So, if you get this that μ_F is countably additive and non negative defined on the fields $\mathcal{C}$, we are now going to appeal to the Caratheodory's extension theorem, which we had discussed just in the previous week, in week four. And therefore, we will be able to extend this set function uniquely from the field's $\mathcal{C}$ to the σ-field (())(23:50) $\mathcal{C}$. And which is nothing but the borel σ-field.

So, therefore, it has a unique extension to the borel σ-field and you are going to get this as a probability measure. So, the extension will be a probability measure. So, for the sake of not complicating the notation, we are going to continue using μ_F to denote the same set function, the extension again we are going to denote it by μ_F. So, by step 4 using the Caratheodory's extension theorem, we can extend this countably additive non-negative set function from the field to the generated σ-field.

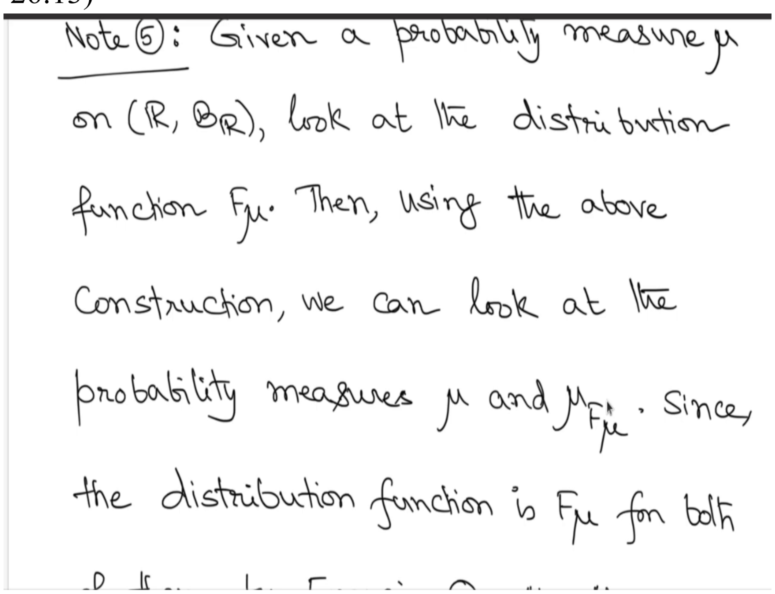
And therefore, we are getting this probability measure on the borel σ-field of the real line. This is completing the construction. However, this is not the end goal. We had started off with the motivation to connect the identifications between the distribution functions and probability measures, both ways.

And then, which we are now going to observe is that this construction is going to give the fact that that probability measure thus constructed, μ_F, the probability measured thus constructed is going to have the distribution function as the given function. So, why is this? So, again go back to the definition of μ_F. And we want to check, what is the distribution function of μ_F.

So, let us go back to the definition. So, look at this left open right closed intervals, $(a, b]$. Say in particular, a can be taken to be $-\infty$ and b to be some real number. And for that, using the notations that we had mentioned earlier, so, we are going to get the interval $(-\infty, b]$. And for that, what is the size of this? So, μ_F associates the size as $F(b) - F(-\infty)$.

But $F(-\infty)$ is 0, so therefore, you just get back the function value at b. But what is the left-hand side? that is $\mu_F(-\infty, b]$. So, therefore, the distribution function of μ_F is the given function F itself that is nice. So, we have that if you start with the distribution function, go to the corresponding probability measure by this construction. And then go back to the corresponding distribution function, you are going to get back the original distribution function.

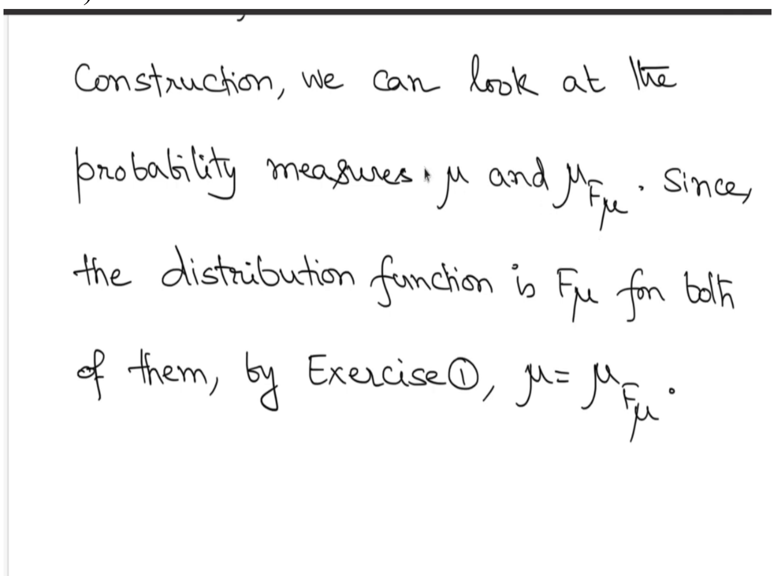
(Refer Slide Time: 26:13)

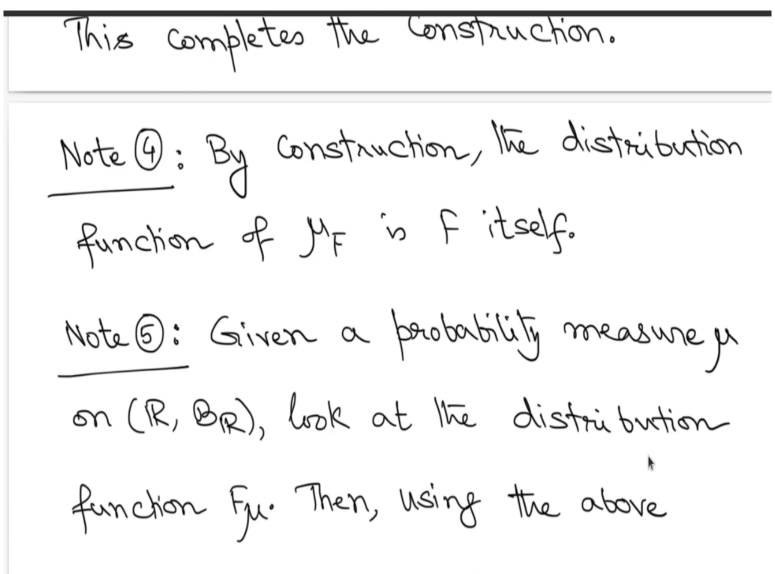


But this is not the end of the story, you can start off with now a probability measure, that is mentioned in note five you now start off with a probability measure again on this measurable space real line together with the borel σ-field. And look at the corresponding distribution function F subscript mu.

This we have discussed extensively in the previous week, but now, you know that this distribution function F_μ is a distribution function as per the definitions in this lecture. It satisfies all those nice properties. Therefore, you can say that you can now construct a probability measure corresponding to F_{μ}. So, call this as $\mu_{F_{\mu}}$ as per the notations of this lecture. So, given this distribution function that you have somehow got, you can now use this construction that we have discussed to construct a probability measure $\mu_{F_{\mu}}$.

(Refer Slide Time: 27:13)





But then observe, that by the previous note $\mu_{F_{\mu}}$ has the distribution function F_{μ} itself. Of course, the given probability measure μ has the distribution function F_{μ}. Therefore, both these probability measures have the same distribution function which is F_{μ}. And hence, using the exercise mentioned earlier that a distribution function uniquely identifies the probability measure, you will immediately claim that μ is nothing but $\mu_{F_{\mu}}$. So, if you therefore, start with a probability measure μ go to the corresponding distribution function and then go to the constructed probability measure, you will get back the original probability measure. So, this is the flip side of it.

So, in note four, we said to start with the distribution function, go to the probability measure, then go to the distribution function, you will get back the original distribution function. In note 5, we are saying start with a probability measure, go to the distribution function, go to the corresponding probability measure and compare final probability measure with the first one, you are going to get back the same thing. This completes the identification between the class of distribution functions and the class of probability measures on the real line.

So, we had skipped one of the steps in the construction that was regarding the accountable activity of the set function μ_F. So, we have to verify that through the continuity property that we had left as a claim. This is what we are going to discuss in the next lecture. And later on in further lectures we are going to discuss the extension of all this construction to higher dimensions. We are going to continue this discussion in the next lecture.