Measure Theoretic Probability 1
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Lecture 20
Caratheodery Extension Theorem

Welcome to this lecture. This is the final lecture of this week. Before we proceed, let us quickly recall what we have done in this week. So, we have defined law or distribution of random variables or random vectors and also defined the corresponding distribution functions. The law was defined as a probability measure, that was the push forward of the probability measure that was on the domain side. And the push forward was done by the measurable structure of the given random variable or random vector.

Now, what we have shown is that given the random variable or random vector, we can define these probability measure on the range side but this identification is not one to one. That means, given probability measure on the range side, you can possibly construct multiple random variables or random vectors with that as the law.

So, we have obtained both sided correspondence, but you have to be careful when you want to go back from the probability measure to the random variable, this identification need not be unique.

So then, we had also gone to these distribution functions and what we have seen is that the properties of distribution functions follows from the law and these properties are familiar properties that we have already seen in basic probability theory. But now we have deduced them from the properties of the probability measure $\mathbb{P} \circ X^{-1}$, the law of the random variable X.

But then, what we have also mentioned that given a collection of functions, let us say on the real line, taking values between 0 and 1, non-decreasing right continuous, with limit at ∞ being 1, limit at $-\infty$ being 0, if you consider such functions, then you can consider, you can construct probability measures on the domain side, that meaning, on the $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

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So, of course, they have their appropriate versions in the higher dimensions and then of course, once you can construct these probability measures, you can of course go back further and construct the corresponding random variables or random vectors.

So, this full correspondence between random variables, corresponding law which have these probability measures and distribution functions will be complete once you have computed or once you have constructed the probability measure corresponding to a given distribution function. By distribution function, I mean a function satisfying those appropriate properties.

So, as a first step towards that direction, we are going to discuss important application of Monotone Class theorem in this lecture. And this relates to extensions of measures from fields. So, let us move on to the slides.

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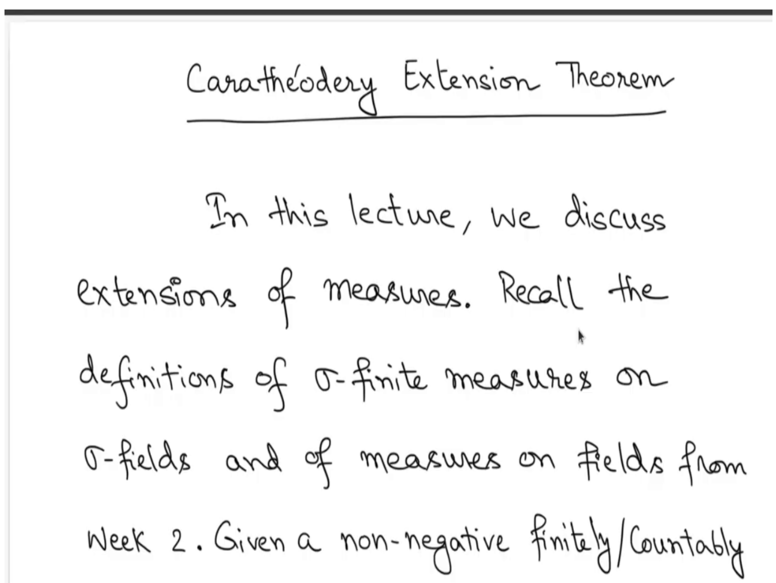

So, in this lecture, we are going to talk about extensions of measures.

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extensions of measures. Recall the definitions of σ-finite measures on σ-fields and of measures on fields from Week 2. Given a non-negative finitely/Countably additive set functions μ on a field $\mathcal{F}$, we want to construct non-negative Countably additive set functions ν on

First, recall that we have defined σ finite measures on σ -fields. What were these measures? These were measures such that you can approximate the whole set from below by sets with finite mass. So, those were σ finite measures.

And we have also defined measures on fields. What were these? These were non-negative, countable additive set functions on fields. So, their countable additivity has to be verified only for the sequence of sets where the sets are pairwise disjoint and their union is already in the field.

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Week 2. Given a non-negative finitely/Countably additive set functions μ on a field $\mathcal{F}$, we want to construct non-negative Countably additive set functions ν on
$\sigma(\mathcal{F})$ such that $\nu\big|_{\mathcal{F}} = \mu$, i.e.

$$\nu(A) = \mu(A) \quad \forall A \in \mathcal{F}.$$

Definition 6 (σ- finite measures on fields)

So, given these non-negative finitely additive all are countable additive set function, on a field, what we would like to do is to construct non-negative countably additive set functions on the generated σ -field, such that these new set function, while you are looking at sets in the field, it will match with the given set function mu.

So, originally the set function μ is defined on the field. You want to construct a set function, countable additive, non-negative, such that this set function can associate values on the σ -field that is generated by the field. So, that is the bigger collection. And on that bigger collection you are defining this new set function, which should match when you try to look at sets from the field. It should match with the given set function that is defined on the field.

So, as a quick example of a field, and the corresponding generated σ -field, always keep in mind the example of the field on the real line given by finite disjoint unions of left open right closed intervals. And also, recall that the generated σ -field is nothing but the Borel σ -field.

So, what we are basically saying is that what we would like to do in this special case is to construct a countable additive non-negative set function on the Borel σ -field on the real line, provided you are given a non-negative set function, finitely countable additive set function on the field of finite disjoint union of left open right closed intervals.

So, that is the type of thing that we are trying to study. So, we are talking about extensions because these set functions are, first of all defined on this collection of sets. So, the original set function is defined on the collection of sets which is a field, $\mathcal{F}$, and then you are keeping those values intact but then also trying to associate values for the other sets that appear in the generated σ -field.

And you are trying to obtain this extension, this extension of the set function to the generated σ -field. And you want that extended function also to be a measure. So, that is basically the idea.

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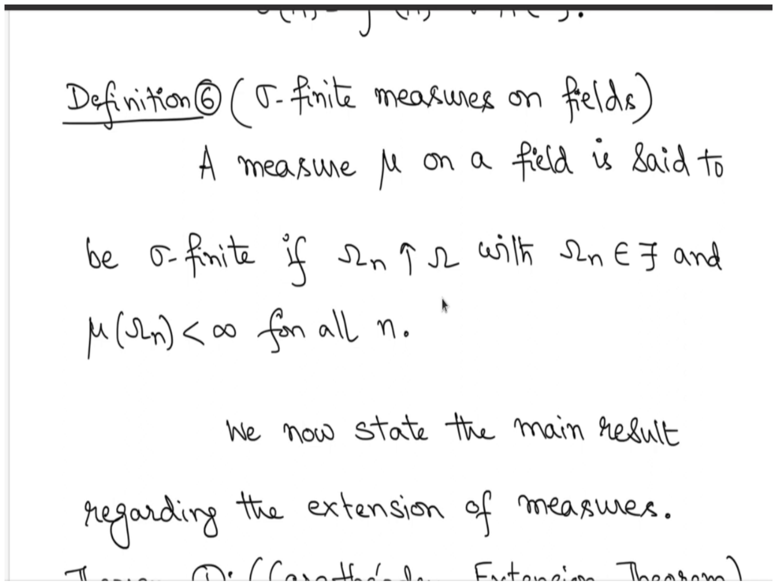

So, in this regard, we want to make this definition. We are calling a measure to be a σ finite measure on a field if you can get approximating sequences of sets from below with finite mass. So, these definitions remains the same as was considered in the case of σ finite measures on σ -fields.

So, what we are taking is, you have a countable additive non-negative set function on a field and you would like to have a approximating sequence of sets, $\{\Omega_n\}$, which approximates the whole set from below. And you would like to have that the sets have finite mass or finite size under the measure mu.

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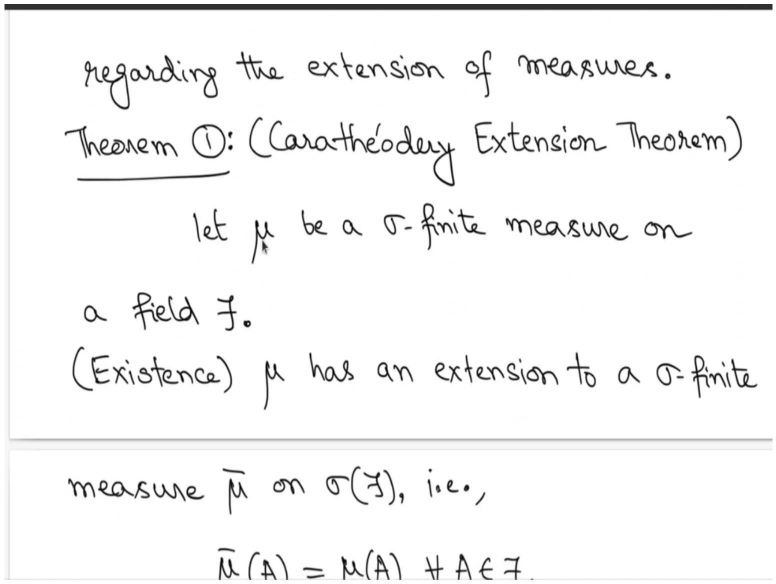

Now, with that at hand, we now state the main result regarding this extension of measures. So, what you are starting off with is a σ - finite measure on a field which we have just defined. So, with that set function at hand, you have this important result called the Carathedery Extension Theorem.

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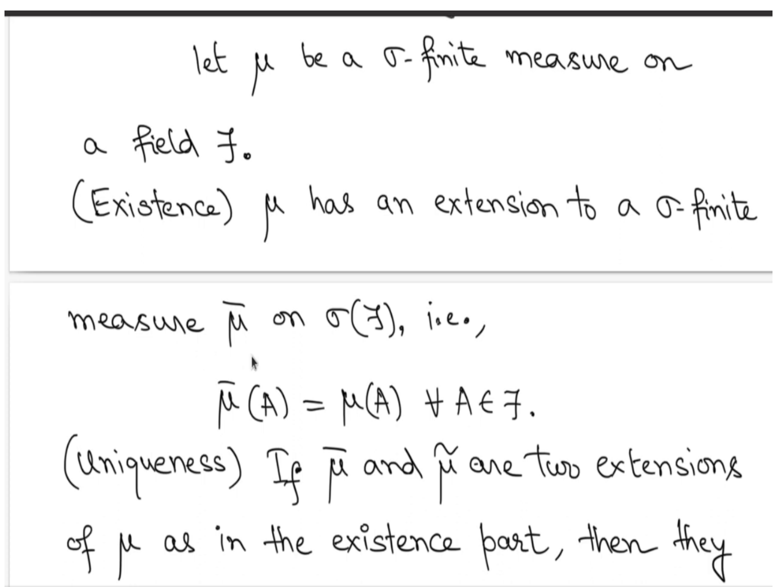

measure $\bar{\mu}$ on $\sigma(\mathcal{F})$, i.e.,

$$\bar{\mu}(A) = \mu(A) \ \forall A \in \mathcal{F}.$$

(Uniqueness) If $\bar{\mu}$ and $\tilde{\mu}$ are two extensions of μ as in the existence part, then they are the same, i.e.,

$$\bar{\mu}(A) = \tilde{\mu}(A) \ \forall A \in \sigma(\mathcal{F}).$$

Note (31): In this Course, we discuss the

Which says that given such a measure, σ finite measure on a field, you can get an extension to a σ finite measure, let us call it $\bar{\mu}$ on the generated σ-field. That means given a set function mu, you can construct a extended set function, you can associate values for the extra sets that appear on the generated σ-field, such that the original values remain intact.

You get new values for the new sets and whatever set function you obtain at the end, this $\bar{\mu}$ will be a σ finite measure. And it also has a uniqueness statement. It says that if there are two such extensions of the set function μ to the generated σ-field, then they must be the same.

So, they, then they must match for all the sets in the generated σ-field. See, here, $\bar{\mu}$ and $\tilde{\mu}$, these extensions are defined on the generated σ-field and therefore what we are saying is that set by set for all the sets in the σ-field, these two extensions must match.

But these two extensions also should match on the field as per the construction but the uniqueness statement says that they should also match over all the extra sets that you get on the generated σ-field. So, that is the uniqueness statement. So, let us see how do we go about proving this.

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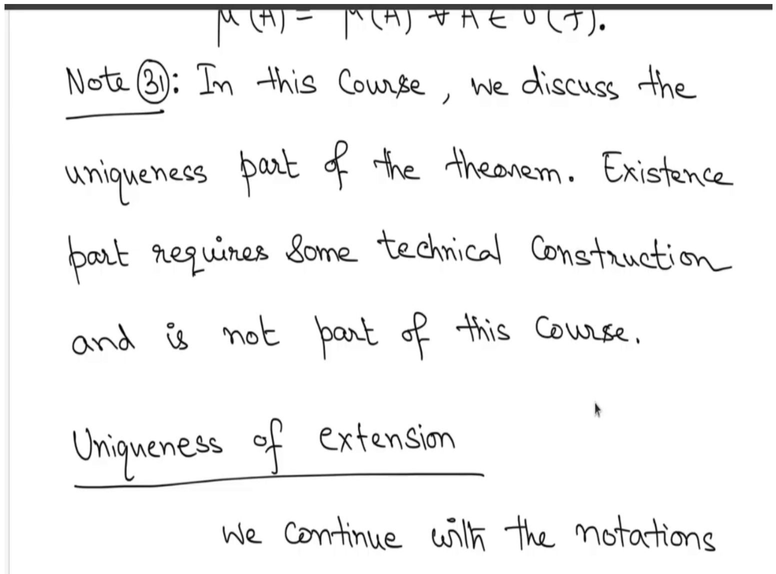

But it is important to note that in this course, we are only going to talk about the uniqueness part of it. The existence parts require many technical constructions which we would like to avoid. And this is not part of this course. So now, what we are going to do is to look at the proof of the uniqueness and we are going to assume that the existence statement holds. So, let us break down the proof into several steps so that it is easier to follow.

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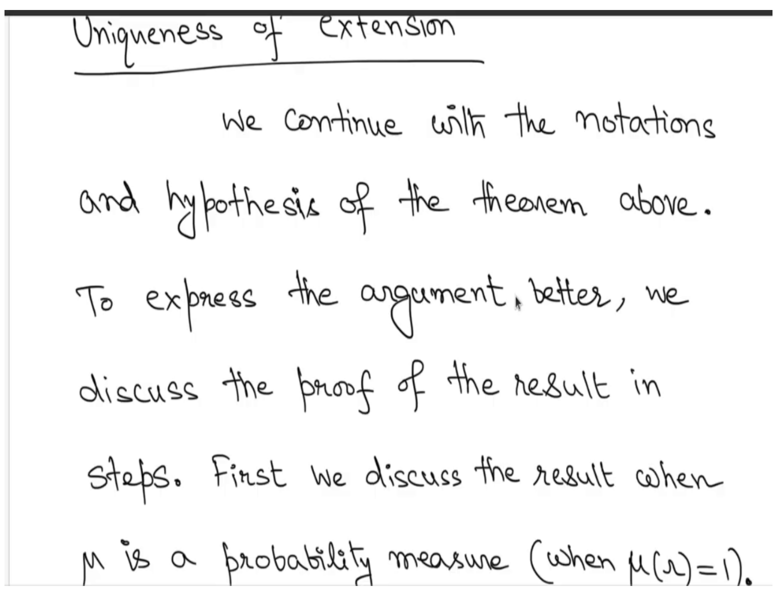

Now, we are going to continue with the notations. That means μ is a σ finite measure on our field and we would like to show that given two extensions $\bar{\mu}$ and $\tilde{\mu}$, they will match on the generated σ -field.

So, as extensions, they already match with the given set function on the field. So, $\bar{\mu}$ and $\tilde{\mu}$ agree on the field but we would like to show that the, for the extra sets that appear on the generated σ -field, that two extensions $\bar{\mu}$ and $\tilde{\mu}$ must match.

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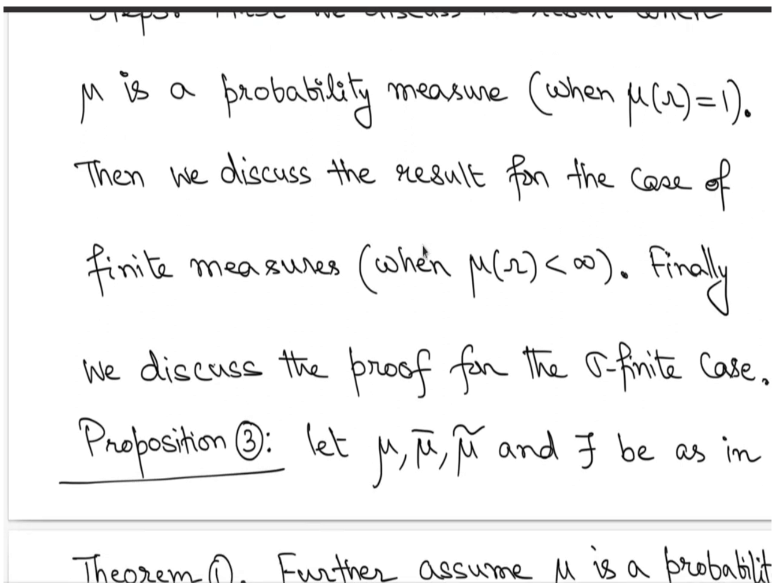

So again, we are going to split this proof into several steps. So, the first case will be when you are talking about a probability measure. That means the given μ associates total mass as 1. So, the whole set has total mass 1. Great. Then, we will extend the case to the case of finite measures and finally to the σ -finite case. So, let us split this.

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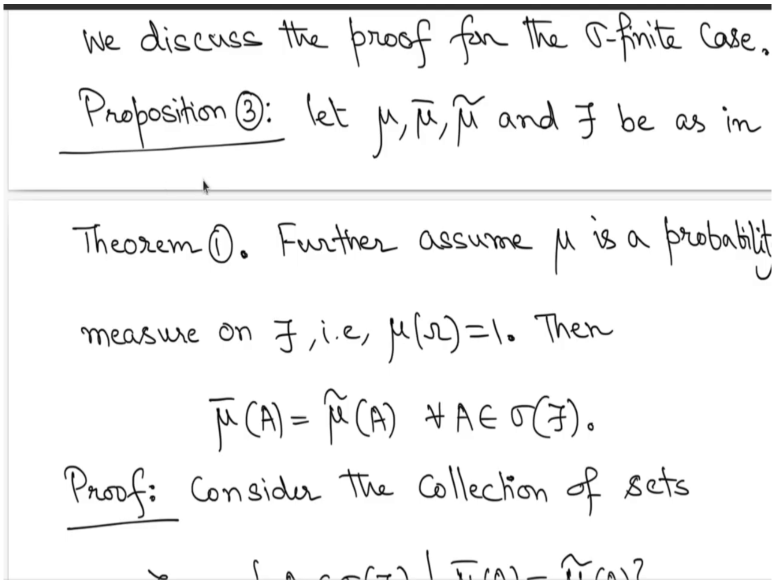

And the first result in this direction therefore is Proposition 3, which is about probability measures. So, continue with the same notations but assume that μ associates total mass 1. So, Ω has mass 1.

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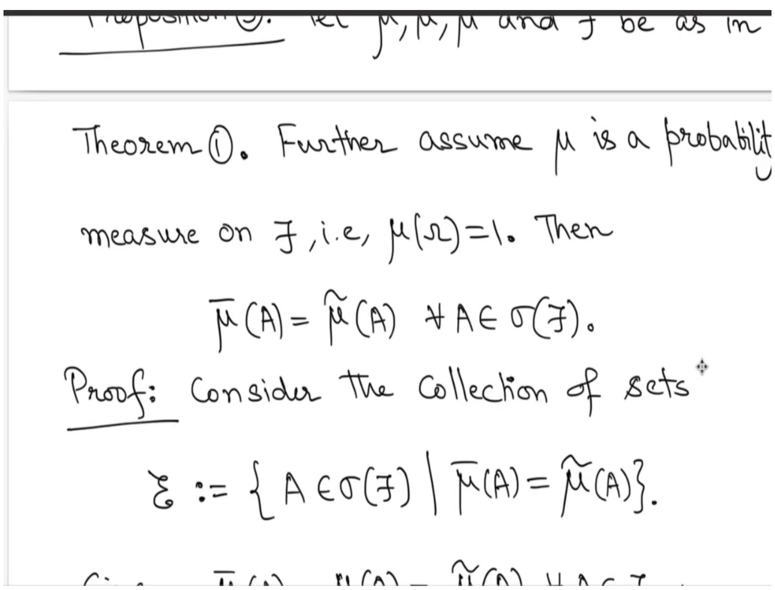

Then we claim that the two extensions must match. So, how do you show this?

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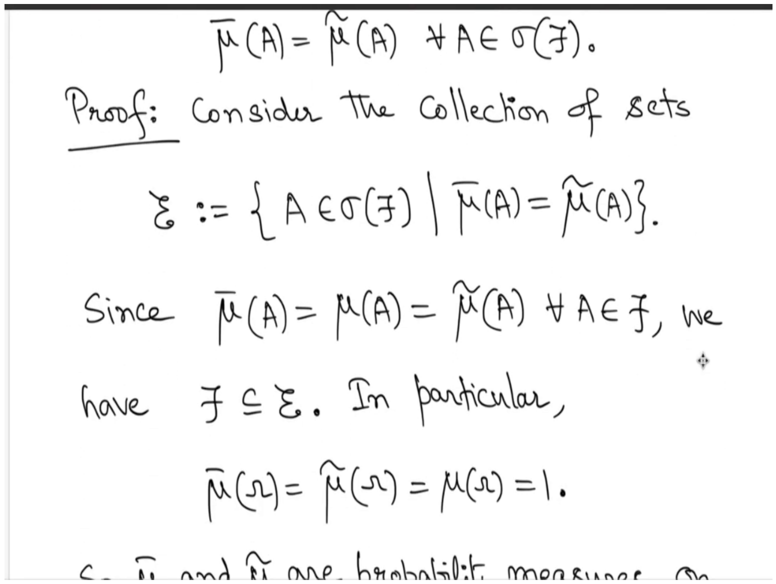

So, to do this, what you do is that you look at this collection of sets, call it $\mathcal{E}$, so this collection of sets is a sub collection of the σ -field generated by the field where the two extensions match. So, when these two extensions match, you put that set in your collection.

So, note that this collection is non empty because you already have the sets from the field agreeing. So, for the sets in the field, you already have these two extensions match, so therefore the, this collection that you are starting off with, it is non-empty. So, all the, the, the field is already containing this $\mathcal{E}$.

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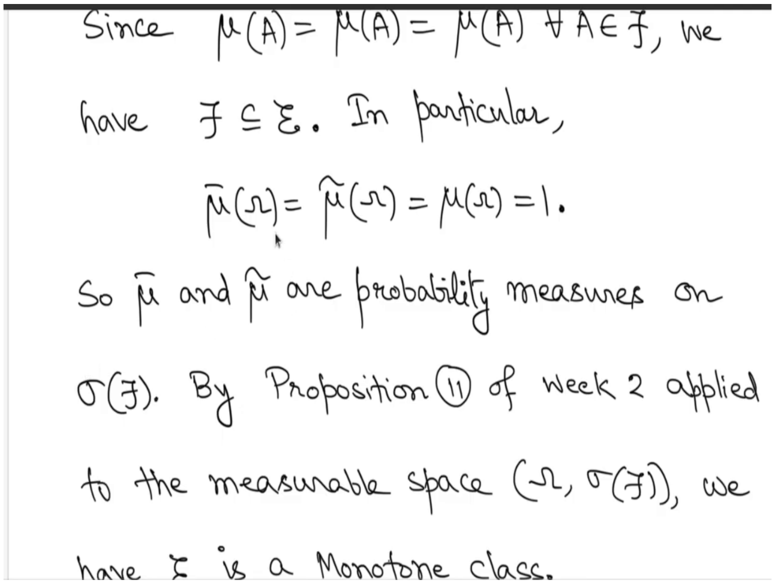

But then, note that the extensions that you were looking for, they are actually also probability measures on the generated σ-field. Why? Because the total mass that that gets associated is exactly equal to 1 because Ω is in field. Therefore, this equality must hold and therefore $\bar{\mu}$ and $\tilde{\mu}$ are probability measures on the generated σ-field.

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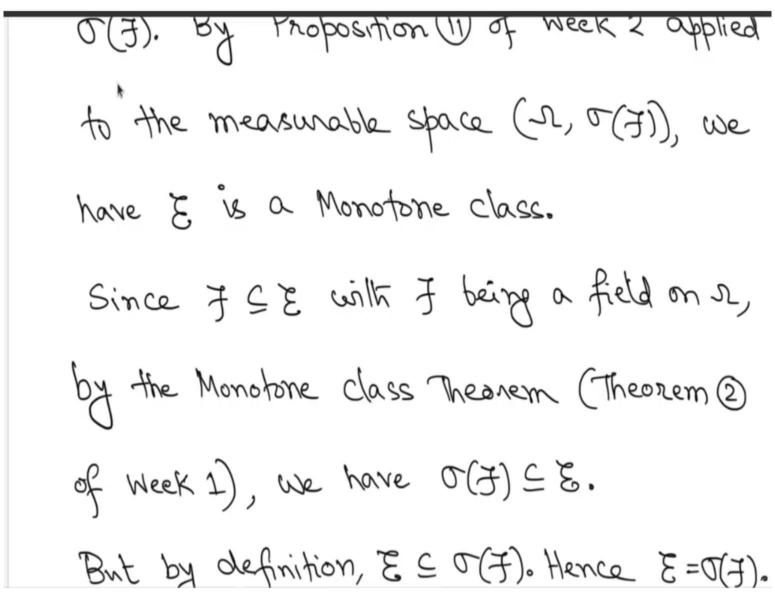

Proof: Consider the collection of sets

$$\mathcal{E} := \{A \in \sigma(\mathcal{F}) \mid \bar{\mu}(A) = \tilde{\mu}(A)\}.$$

Since $\bar{\mu}(A) = \mu(A) = \tilde{\mu}(A) \ \forall A \in \mathcal{F}$, we have $\mathcal{F} \subseteq \mathcal{E}$. In particular,

$$\bar{\mu}(\Omega) = \tilde{\mu}(\Omega) = \mu(\Omega) = 1.$$

So $\bar{\mu}$ and $\tilde{\mu}$ are probability measures on

But now, go back to Week 2, and look at Proposition 11, where you are talking about these sets of equal probability under different probability measures. So, that was $\mathcal{E}$ again in that notation, in Proposition 11, so go back to this.

So, what you had shown was that such sets, this collection of sets will form a Monotone class. So, if you look at that, in this case also, we have these two probability measures $\bar{\mu}$ and $\tilde{\mu}$ on the field generated by the field $\mathcal{F}$. So there, you have this structure that is, where the two measures agree, that collection of sets is a Monotone class.

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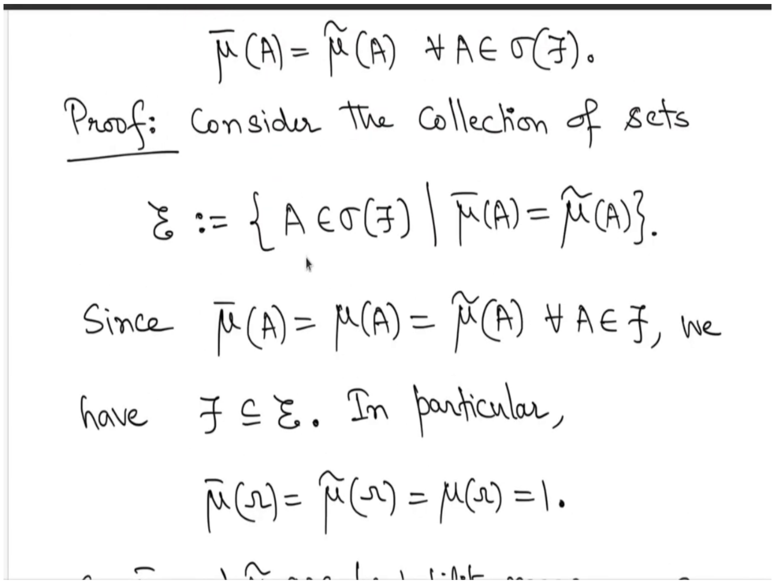
Since $\mathcal{F} \subseteq \mathcal{E}$ with $\mathcal{F}$ being a field on Ω, by the Monotone class Theorem (Theorem 2 of Week 1), we have $\sigma(\mathcal{F}) \subseteq \mathcal{E}$.

But by definition, $\mathcal{E} \subseteq \sigma(\mathcal{F})$. Hence $\mathcal{E} = \sigma(\mathcal{F})$.

Then $\bar{\mu}(A) = \tilde{\mu}(A) \ \forall A \in \sigma(\mathcal{F})$.

This completes the proof.

Note 32: We now make an observation



But then, since this Monotone class contains this field, use the Monotone class theorem and claim that the σ -field generated by the field is contained in this collection. But by definition, your collection was already a sub collection of σ -field generated by the $\mathcal{F}$.

So, let us go back again. So, you had started off with this sub collection where these two probability measures agree. So, therefore $\mathcal{E}$, whatever it is, it is already a sub collection of the σ -field generated by the field.

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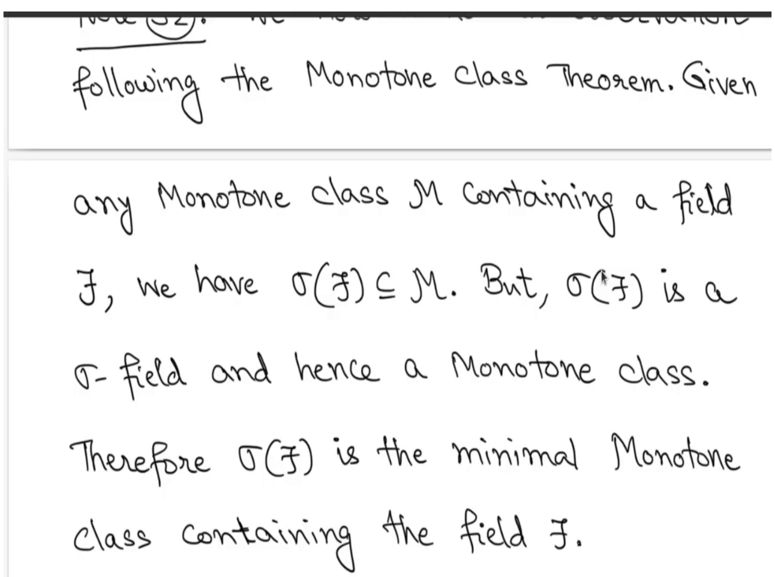

And therefore, you have the exact equality, and which will tell you that for all the sets, these two measures must agree. So, that concretes the proof. Now, before we go forward and extend this result for the case of finite and σ finite measures, it is an important point to note that the Monotone Class Theorem is being applied here and you should note that any Monotone Class containing this field $\mathcal{F}$, contains this σ -field generated by the field F.

So, that was the statement of the Monotone Class Theorem. So, if from Monotone Class, $\mathcal{M}$ contains the field, then the Monotone Class also contains the σ -field generated by the field. So, that was the Monotone Class Theorem.

But now, you also note that σ -field generated by the field is a σ -field and itself it is a Monotone Class. σ -field supports countable unions and countable intersections, in particular it will support countable increasing unions and countable decreasing intersections, therefore σ -field generated by the field is a Monotone Class.

Therefore, what you observe is that among all the Monotone Classes containing a field, this σ -field generated by the field F, that is the Monotone Class also containing the field $\mathcal{F}$ and these Monotone Class σ -field generated by the field $\mathcal{F}$ is contending any arbitrary such a Monotone Class.

That means, σ -field generated by the field $\mathcal{F}$ is the minimal Monotone Class with the property that it contains the field $\mathcal{F}$. So, σ -field generated by the field, contains the field, it is a Monotone Class, fist point.

Second, any general Monotone Class containing the field must contain the σ -field generated by the field. So, therefore the minimum Monotone Class containing the field is the σ -field generated by the field itself. So, this is a important observation.

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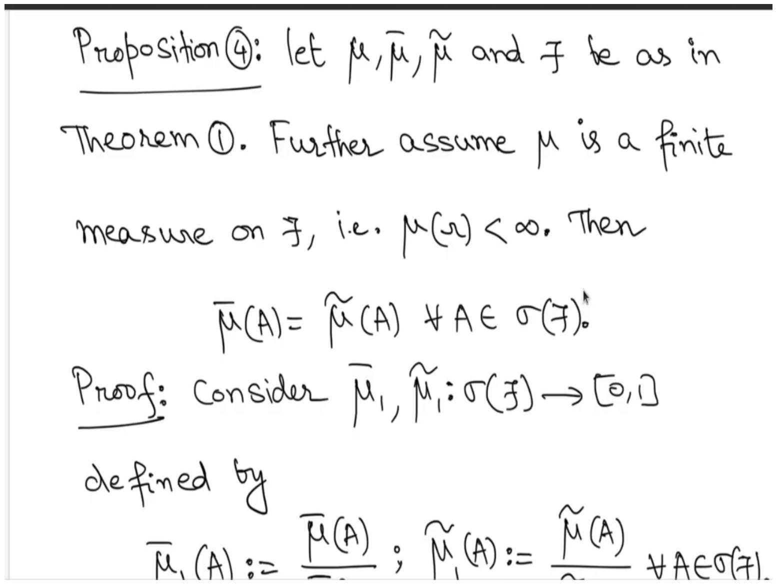

So now, we go to the, other cases, when you try to extend the result from probability measures to the case of finite measures. So again, continue with the same notations that μ is a σ - finite measure on a field and you have two extensions $\bar{\mu}$ and $\tilde{\mu}$ to the generated σ -field. And you assume that μ associate finite mass to Ω.

So, μ(Ω) is finite. So, given that, you want to show that the two extensions agree. So how do you show this? So, consider these two set functions now. What are these set functions?

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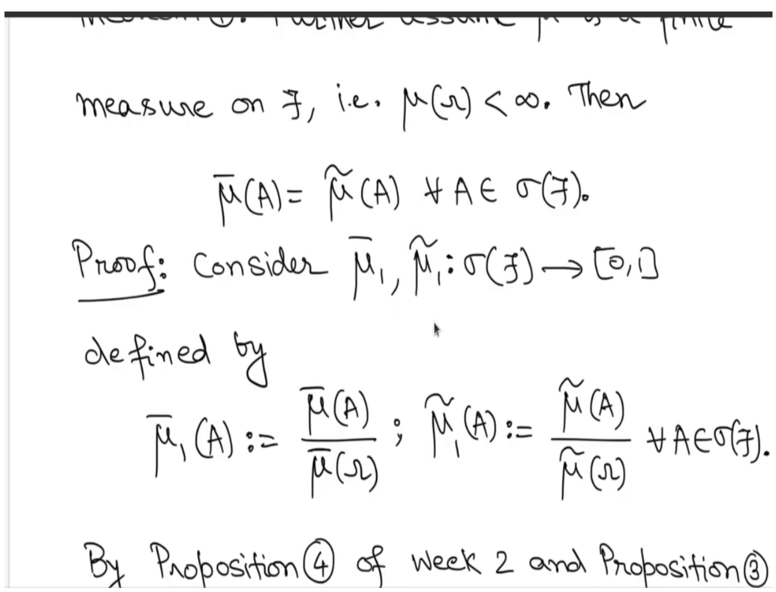

So, I denote them by subscript 1. So, I am given $\bar{\mu}$ and $\tilde{\mu}$ and I am defining two new set functions, the first which is denoted as $\bar{\mu}_1$, the second one is denoted as $\tilde{\mu}_1$. So, these are the two set functions that we define. So how do you define it?

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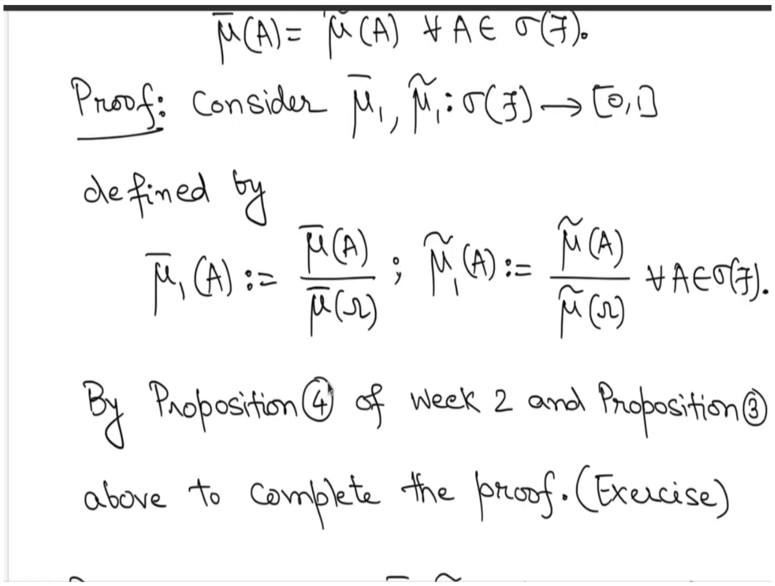

Again, you look at $\bar{\mu}(\Omega)$ and $\tilde{\mu}(\Omega)$. Remember that these two quantities that you are looking at, they will match, $\bar{\mu}(\Omega)$ and $\tilde{\mu}(\Omega)$, they will match and match with $\mu(\Omega)$.

And in all of this, we typically will assume that $\mu(\Omega)$ is not 0. So, it is not the 0 measure that you are looking at. So, again, your underlying hypothesis will always be that $\bar{\mu}(\Omega)$ and $\tilde{\mu}(\Omega)$, they are agreeing by the hypothesis and agreeing with $\mu(\Omega)$ and which is not 0. So, that is some positive quantity and therefore you can divide by that positive quantity. So, what you do, you scale all the measures of the sets, $\bar{\mu}$ by the total mass. So, that is what you do. And remember, by Proposition 4 of Week 2, that these will give you a probability measure. So, you scale a finite measure, divide it by the total mass, you get back a probability measure. So, that was proved in Proposition 4.

So, if you now construct these set functions, $\bar{\mu}_1$ and $\tilde{\mu}_1$, these are now probability measures on the generated σ -field. But then, these two probability measures matches on the field as per the given condition. And hence, by Proposition 3 above, you can apply the uniqueness argument and therefore you can claim that $\bar{\mu}_1$ is equal to $\tilde{\mu}_1$. $\bar{\mu}_1$ is equal to $\tilde{\mu}_1$.

And then, from that you can easily conclude that $\bar{\mu}$ and $\tilde{\mu}$ must match. So, first you prove that after you scale the $\bar{\mu}$ and $\tilde{\mu}$, you get probability measures, then you observe that they, they match, apply the previous results, get the equality of $\bar{\mu}_1$ and $\tilde{\mu}_1$ and from that you conclude the equality of $\bar{\mu}$ and $\tilde{\mu}$. So that is the proof. So, try to write it down. I have already explained the major steps.

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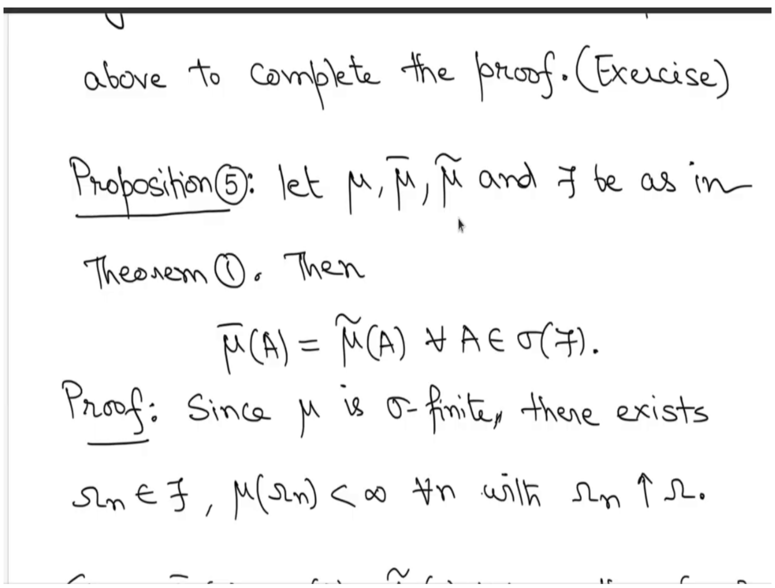

But then, how do you extend it to the case of σ-finite measures as in Theorem 1. So, here, μ is a σ-finite measure on the field and you are looking at $\bar{\mu}$ and $\tilde{\mu}$ which are again, of course two σ- finite measures on the generated σ-field and you would like to claim that they will match throughout the σ-field.

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$\Omega_n \in \mathcal{F}$, $\mu(\Omega_n) < \infty \ \forall n$ with $\Omega_n \uparrow \Omega$.

Since $\bar{\mu}(A) = \mu(A) = \tilde{\mu}(A) \ \forall A \in \mathcal{F}$, then $\{\Omega_n\}_n$ is also an increasing sequence in $\sigma(\mathcal{F})$ with $\Omega_n \uparrow \Omega$, $\bar{\mu}(\Omega_n) = \tilde{\mu}(\Omega_n) < \infty$. Therefore, $\bar{\mu}$ and $\tilde{\mu}$ are σ-finite measures on $\sigma(\mathcal{F})$.

So how do you show this? You scale it down to finite measures case by using σ-finiteness of the given measure. So, what do you? Since μ is σ-finite, you will get this sequence of sets Ω_n increasing to Ω and Ω_n has finite mass according to the measure mu.

Now, observe that Ω_n also will be giving you an increasing sequence of sets in the generated σ-field with the fact that $\bar{\mu}(\Omega)$ and $\tilde{\mu}(\Omega_n)$ are finite. Because Ω_n's are in the field and therefore, $\bar{\mu}$ and $\tilde{\mu}$ must match on the, agree with the values with μ on these sets. So therefore, you get this increasing sequence Ω_n again in the generated σ-field and there you have that $\bar{\mu}$ and $\tilde{\mu}$ associates finite mass to this Ω_n and therefore $\bar{\mu}$ and $\tilde{\mu}$ are also σ-finite measures on the generated σ-field. So, these are of course mentioned in the statement but we just clarified this issue as part of the flow.

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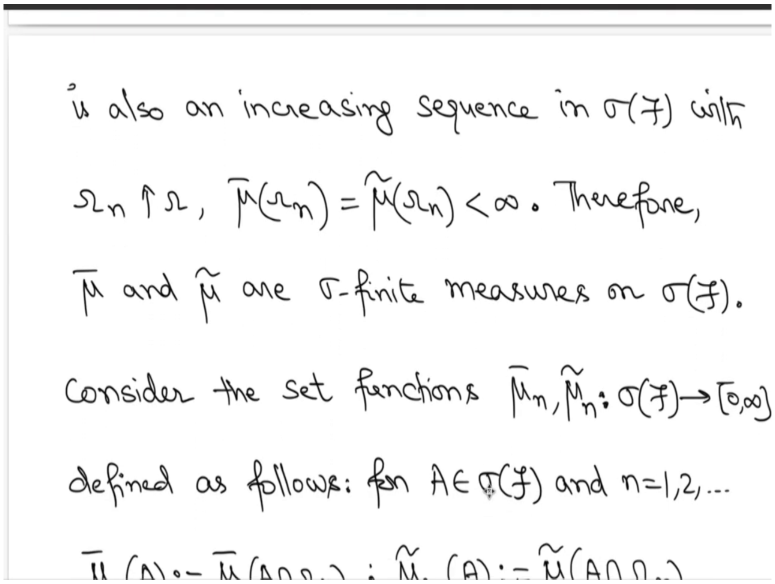

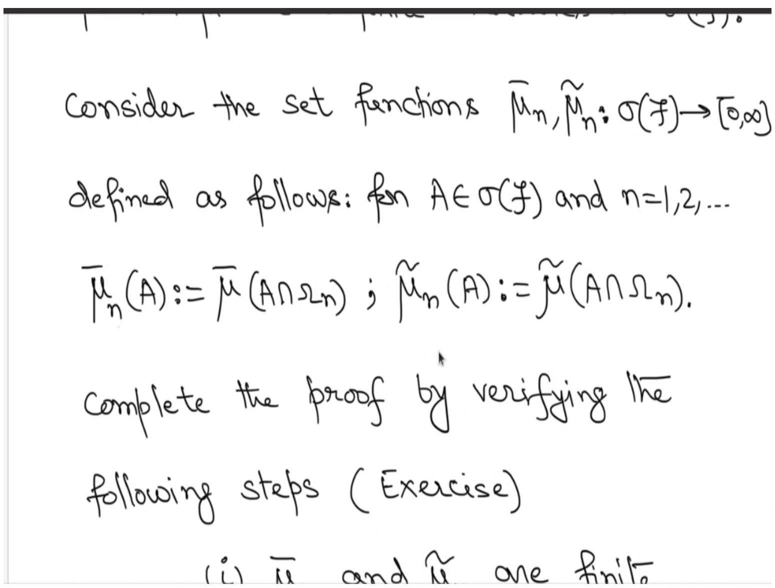

But then you would like to say that these two matches. These $\bar{\mu}$ and $\tilde{\mu}$ matches. So, how do you show this? You now consider a restricted version of the set functions $\bar{\mu}$ and $\tilde{\mu}$. Call them $\bar{\mu}_n$, $\tilde{\mu}_n$. So, what is, what is this?

So, given this n, which is sum natural number 1, 2, 3 and so on, what you define is this $\bar{\mu}_n$ for any arbitrary set in the generated σ -field, you look at $(A \cap \Omega_n)$. So, since $(A \cap \Omega_n)$ is a subset of Ω_n and Ω_n has finite mass, $(A \cap \Omega_n)$ also has finite mass under $\bar{\mu}$.

Similarly, $(A \cap \Omega_n)$ has finite mass under $\tilde{\mu}$. So therefore, when you define $\bar{\mu}_n$ and $\tilde{\mu}_n$, they will be finite measures on the generated σ -field.

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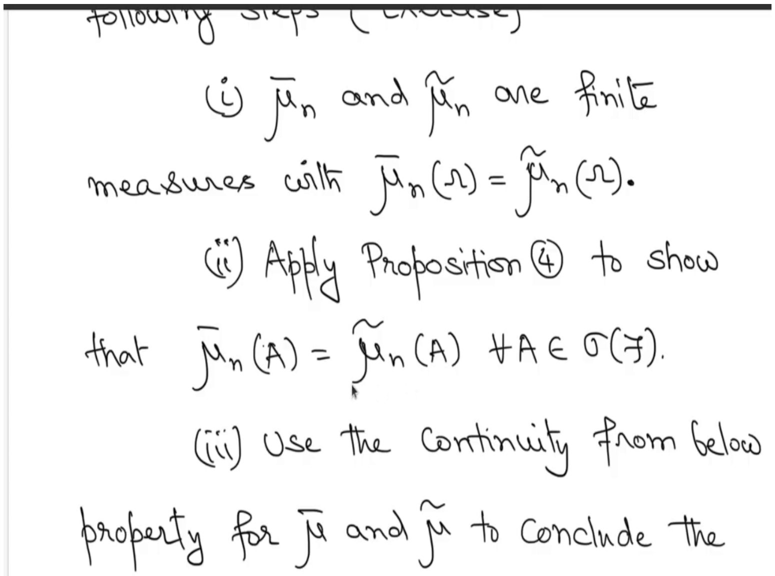

So now, go through these steps. So, the first step, that you are going to show that $\bar{\mu}$ and $\tilde{\mu}_n$ defined as some kind of a restrictions of the original given set functions $\bar{\mu}$ and $\tilde{\mu}$, these restriction find of functions, these are give you finite measures on the generated σ -field with the total mass agreeing. Once you have that, you apply the finite measures case to claim that these two finite measures must match on the generated σ -field.

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measures with $\bar{\mu}_n(\Omega) = \tilde{\mu}_n(\Omega)$.

(ii) Apply Proposition 4 to show that $\bar{\mu}_n(A) = \tilde{\mu}_n(A) \; \forall A \in \sigma(\mathcal{F})$.

(iii) Use the continuity from below property for $\bar{\mu}$ and $\tilde{\mu}$ to conclude the proof.

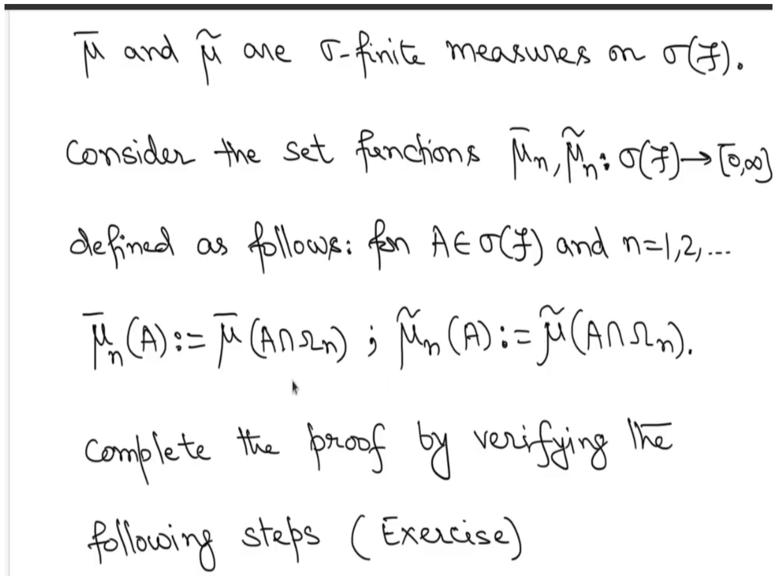
Note 32: Using the Carathéodory Extension

(ii) Apply Proposition 4 to show that $\bar{\mu}_n(A) = \tilde{\mu}_n(A) \; \forall A \in \sigma(\mathcal{F})$.

(iii) Use the continuity from below property for $\bar{\mu}$ and $\tilde{\mu}$ to conclude the proof.

Note 32: Using the Carathéodory Extension

Theorem, we shall discuss the construction



Once you have that, go the third step. You can now use the continuity from below property for $\bar{\mu}$ and $\tilde{\mu}$, to claim that from the quality of $\bar{\mu}_n$ and $\tilde{\mu}_n$, you can show the equality for $\bar{\mu}$ and $\tilde{\mu}$.

How? As long as $\bar{\mu}_n$ and $\tilde{\mu}_n$ agree for all the sets in the generated σ -field, you are saying that $\bar{\mu}(A \cap \Omega_n)$ matches with $\tilde{\mu}(A \cap \Omega_n)$. So, I am making use of that equality.

Bu then use the continuity from below of $\bar{\mu}$ and $\tilde{\mu}$, and use the fact that Ω increases to the whole set, so therefore $(A \cap \Omega_n)$ increases to the set A, $(A \cap \Omega_n)$ increases to the set A.

So therefore, use the fact that $\bar{\mu}(A \cap \Omega_n)$ and $\tilde{\mu}(A \cap \Omega_n)$ agree and let n go to infinity, then the limits will be the same and that will give you $\bar{\mu}(A)$ on one hand and $\tilde{\mu}(A)$ on another hand. So, you are using continuity from below to go from these finite measures case to the σ -finite measures case. So, I have already explained the steps. Please try to write it down and conclude the proof.

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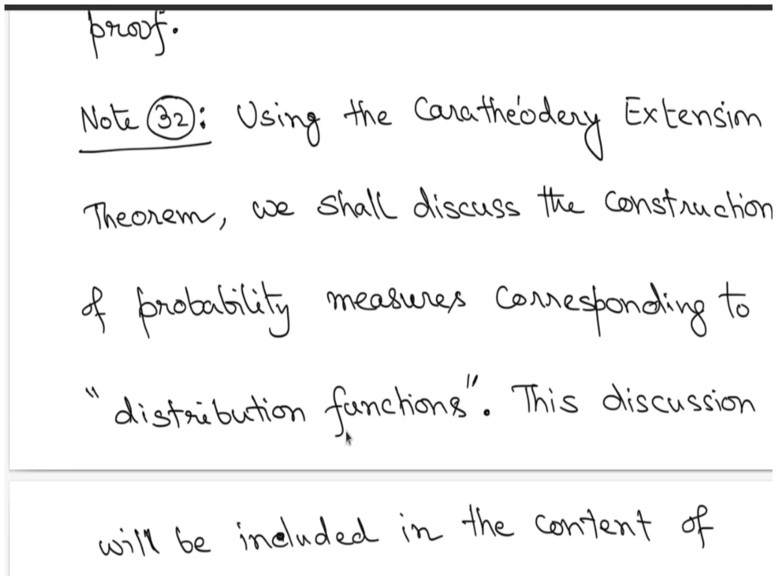

So now, a few comments before we stop. So, using this Caratheodery Extension Theorem, we are going to discuss the construction of probability measures corresponding to distribution functions.

So here, "distribution functions" is under quotes and what do we mean by these type of functions? We mean that these are functions defined on the real line taking values between 0 and 1, are non-decreasing and right continuous, have limits at ∞ and $-\infty$, the values being 1 and 0.

For such functions, you are going to construct probability measures and one of the important steps, steps will be the Caratheodery Extension Theorem as we have discussed here.

Then that will finish that correspondence between distribution functions and the corresponding probability measures. And that will complete the correspondence between random variables of vectors, the corresponding laws, the corresponding probability measures and the distribution functions.

So again, all of these results that we stating, initially will be stated in dimension 1 for simplicity, but appropriate versions can be proved for the d dimensional case as we will discuss in the next week.

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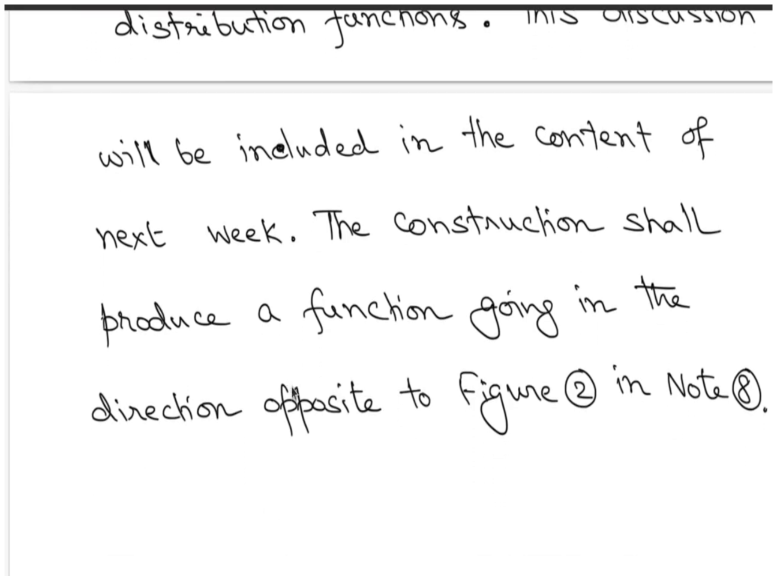

And important thing to note is that corresponding to distribution functions, you are constructing probability measures. And this construction is actually a part of a more general construction where you can construct more general types of measures which will help us talk about absolutely continuous random variables. These, we are going to see later on.

So first, when we start the discussions in the first week, we are going to concentrate on the connection between distribution functions and probability measures. So, this discussion will be done in the next week. We stop here.