Measure Theoretic Probability 1
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Lecture No. 02
Sigma fields and Measurable spaces

Welcome to this lecture. This is the second lecture of week 1. I am sure you have already gone through the introductory lecture video. I have already described the mode of conduct of lectures which will be through some pre-prepared slides. I will be describing and discussing the materials that are already written in the slides. Let us start.

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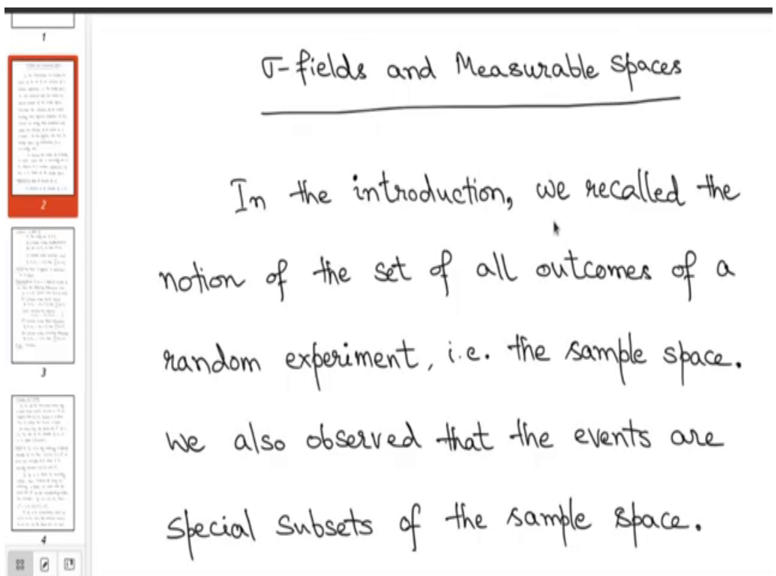

In the first introduction, we recalled the notion of a random experiment, which can be repeated as many times as we need. We observed the outcomes of a specific random phenomenon. And if we list all outcomes if we list them in a set, we will get the sample space.

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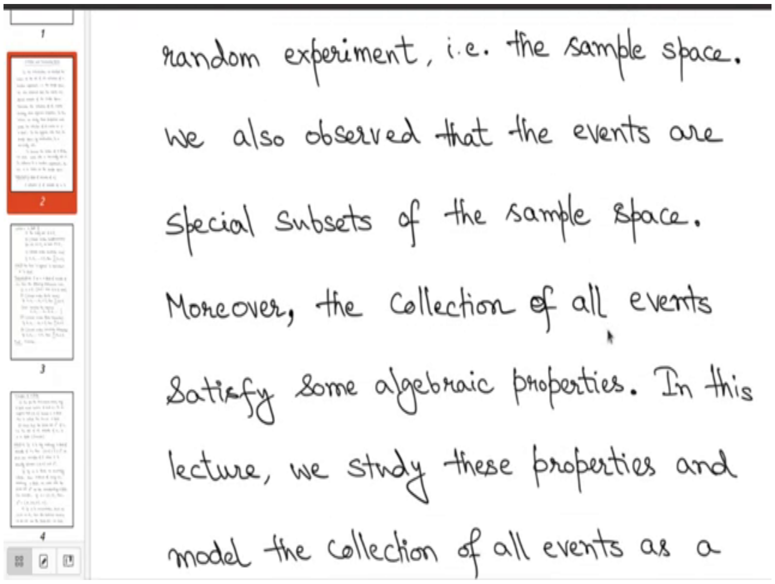

We also observed that special subsets appear as events, and these events satisfy certain algebraic properties. For example, if you take an event and look at its compliment, that is nothing but the compliment of the set in that sample space. So, the complimented set can be interpreted as the nonoccurrence of the original event. So, the complimentary event can be given some physical significance, and these are the types of interpretation that we want to do.

So, this can be more formally done when we discuss the collection of all events related to a specific random experiment. This collection of all events satisfy certain specific algebraic properties like complementation and unions.

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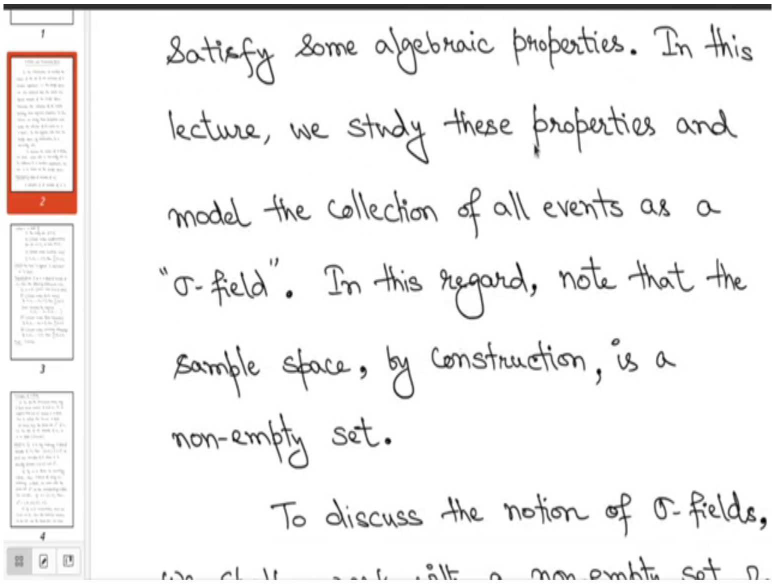

This lecture will study these algebraic properties and model this collection of all events related to a specific random experiment as a σ-field. The strong σ-fields shall be discussed and defined in a minute. In this regard, note that the sample space by construction is a non-empty set. So, whenever you perform a random experiment, you always get an outcome, so, therefore, the list of all outcomes is a non-empty set.

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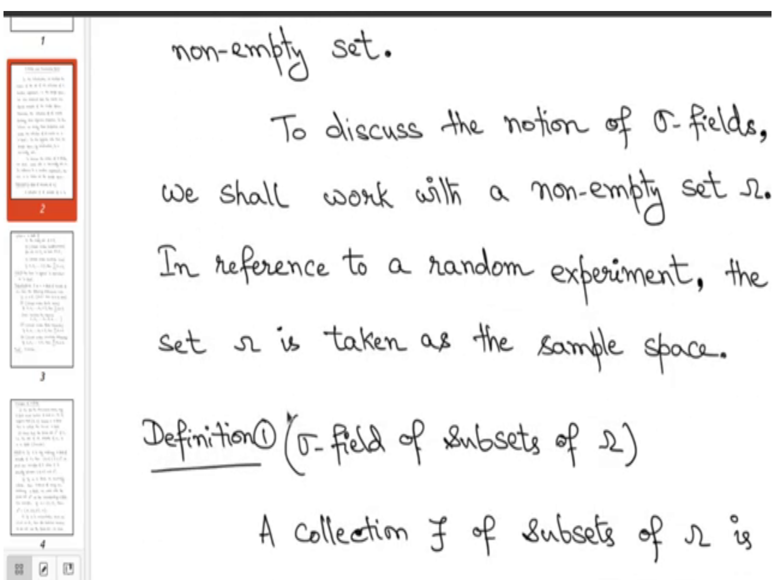

So, more generally, to discuss this mathematical foundation behind the study of probability, what we shall do, we shall work with the notion of σ-fields in quitter general setup, and we shall work with a non-empty set which Ω shall always denote. So, if you find this Ω notation in this course, you always treat it as a non-empty set. About a specific random experiment, Ω is to be taken as the sample space. So, in general, we shall work with some non-empty sets.

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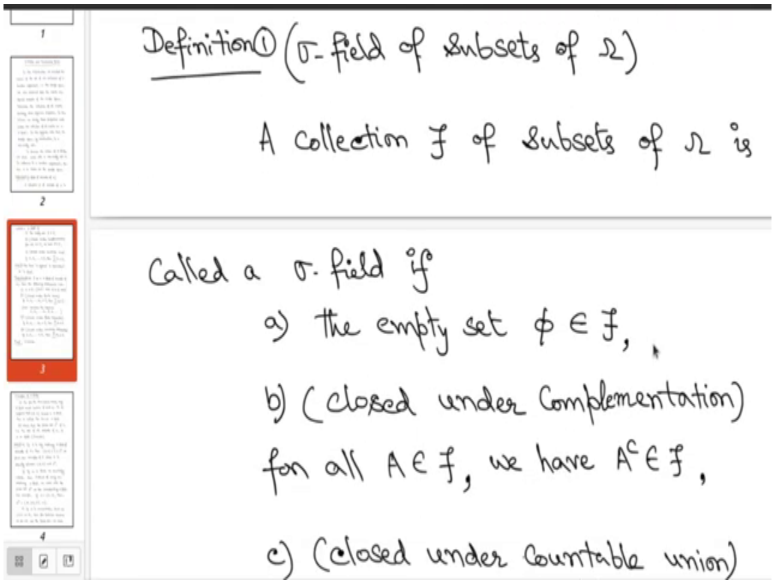

So, let us start the discussion on the definition.

Definition: We shall start with this non-empty set Ω and look at the collection of subsets of Ω. So, $\mathcal{F}$ denotes some collection of subsets. What we want is that this collection of subsets should satisfy certain properties.

(i) So, the first thing is that the empty set ϕ which is a subset as a very specific subset, should be in the list. And this is what we write in the mathematical notation that the empty set ϕ belongs to the list belongs to this σ-field belongs to $\mathcal{F}$. So, this is the notation we write, so the set is a limit in the collection. So, that is why we use this notation.

(ii) So, the second property that we want is closed under complementation. What do I mean? So, for all possible sets in the list, the list need not contain all the possible subsets, but whatever subsets they are there if A is there, we should also have the compliment in the list. So, this is what we mean by closed under complementation.

(iii) And the final property that we want is closed under countable union. What do we want? So, we will take sequences from the list so, A_1, A_2 and so on are subsets. This could be repeated or distinct, but priory this arbitrary collection this arbitrary sequence of setsA_1, A_2 and so on, they are taken from the list from the collection, and then we want to look at its union. So, since we are looking at a sequence, I get a n from 1 to infinity union. This union is a subset, and I want this union to be also on the list.

So, suppose some specific collection of subsets $\mathcal{F}$ satisfies these three properties. In that case, empty set ϕ belongs to the list, closed under complementation, and closed under countable unions. I shall say that this collection is a σ-field $\mathcal{F}$.

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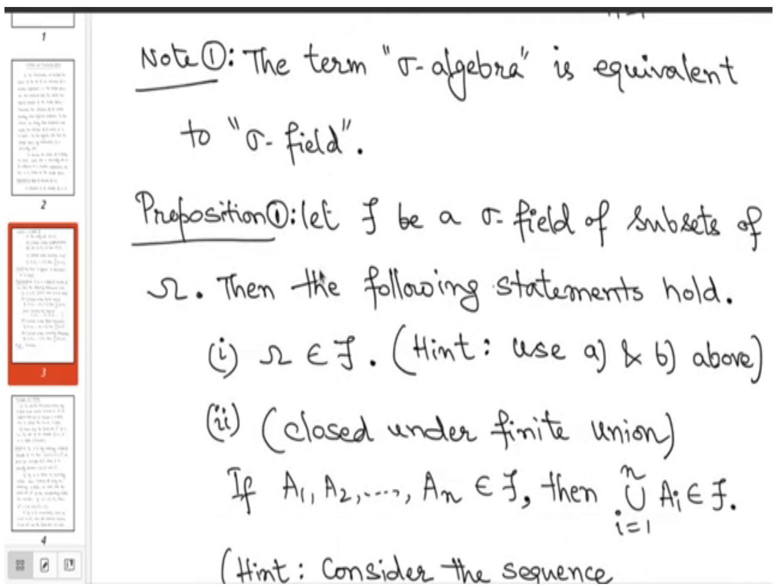

Now, there is an equivalent term which is called a σ-algebra. Throughout the course, we shall use the term σ-field, but σ-algebra is the equivalent terminology. Before going into examples, what do you do? Is that we study some very basic properties which shall help us in construction of examples. So, let us start with these basic sub-properties. So let $\mathcal{F}$ be a σ-field of the subset of non-empty set Ω, then the following statements hold.

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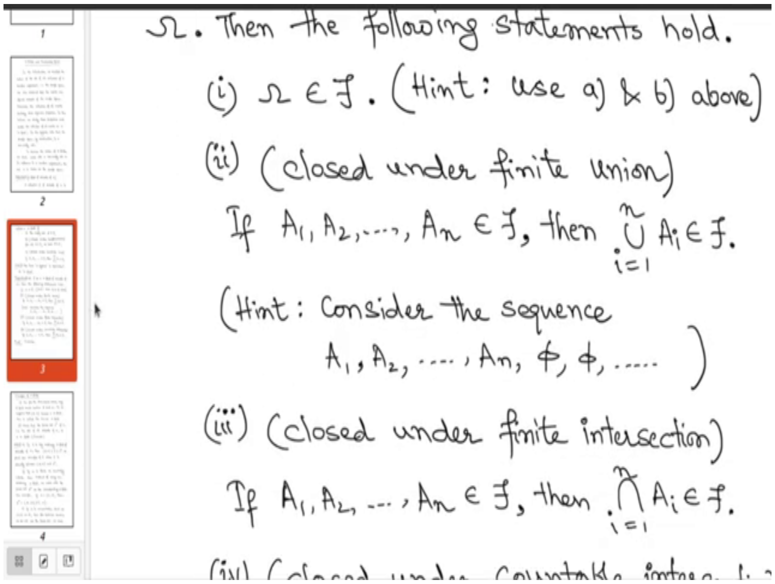

(i) So, the first statement says that the Ω must be in the list. So how do you show this? I have given a hint, so use property a) and b) given in the definition. So, let us go up so if you look at the property a), it says that empty set ϕ in the list empty set ϕ belongs to the σ-field. Then the second property says that it closed under complementation. So, if you now look at the compliment of the empty set ϕ which is in the σ-field, then the compliment of empty set ϕ should be there. But the compliment of empty set ϕ is nothing but the whole set, so the whole set is there. So, this proves the first property. Please try to write this down.

(ii) So, let us look at the second property, which says closed under finite union. So, what do I mean? I take finitely limited sets $A_1, A_2, \ldots, A_n$ from the σ-field and what I want is that the union should be in the σ-field. So, the union here is the finite union of sets. But, in the statement in the definition, what we have taken is a countable union. So, how do you put in that field of so what do we take this sequence $A_1, A_2, \ldots, A_n$ as given sets and then from $(n+1)^{th}$ place onwards, you repeat the empty set ϕ. Remember, empty set ϕ is always in there in the σ-field. So, therefore this sequence that you have constructed just looked at is a sequence of sets coming from the list. These are not an arbitrary subset or an arbitrary subset of Ω, but these are arbitrary subsets from the list. So, this is very important. So, I have taken a sequence from the list from the collection. Then, if you look at this union, this union is nothing but the finite union $\cup_{i=1}^{n} A_i$. So, therefore since this countable union collapses, this finite union must be in the collection $\mathcal{F}$. So, this shows closed under finite union.

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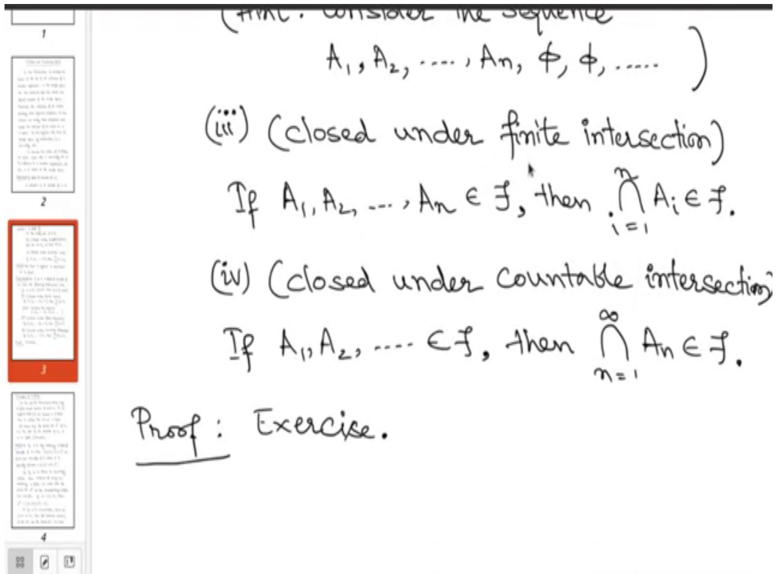

(iii) Similarly, using properties involving countable intersections and finite intersections can be shown. We want to show for countable intersections that if you take the sequence of sets in the σ-field, their countable intersection is also in the σ-field. And you can also reduce it into finitely many intersections. If you take finitely many sets, take their intersection, which should also be in the σ-field. So, to prove property in (iii) and (iv), all you have to do is use some algebraic properties involving complementation and union. Please write this down, and I have left this as an exercise. These are very easy exercises, and they will help you to understand the material better.

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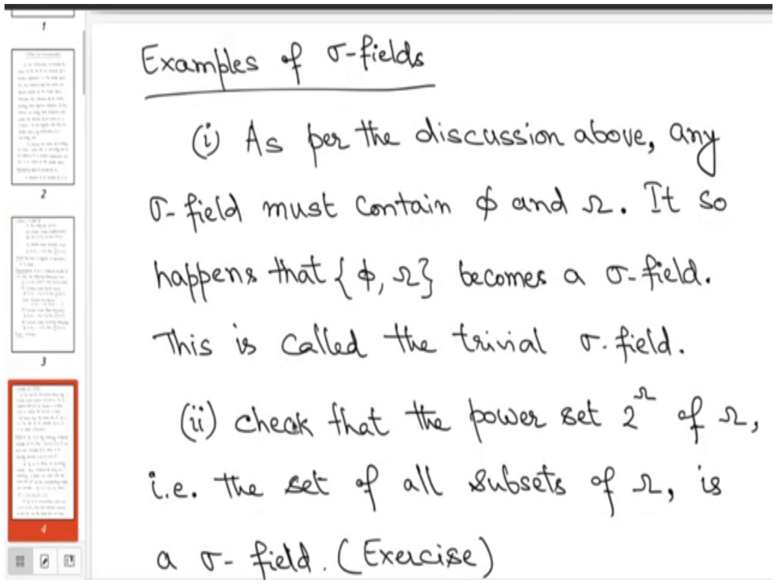

So, now we have some basic properties, and we are ready to construct examples.

(i) As per the discussion above, any σ-field must contain the empty set ϕ and the whole set Ω. So, this we have already seen. It so happens that if you look at this collection of two sets, only the empty set, and the whole set, so this is now a collection this is becoming our first example of a σ-field. So, please check that the empty set ϕ is there. Then if you look at compliments of each set in this collection, empty sets compliment ϕ^c is whole set Ω. Ω^c is the empty set ϕ. So, therefore it is also closed under complementation. Now, if you take sequences out of this so either empty set ϕ or the whole set Ω will be there in the sequence. If could also be repeated so they could also be mixed in whatever situation if it so happens that only the empty set ϕ appears in the sequence, then the union is empty set ϕ otherwise if Ω appears at least once then the union becomes Ω itself. So, therefore it is also closed under countable unions. So, both these properties are true. So this simple example gives us the first example of a σ-field. This is sometimes referred to as a trivial σ-field.

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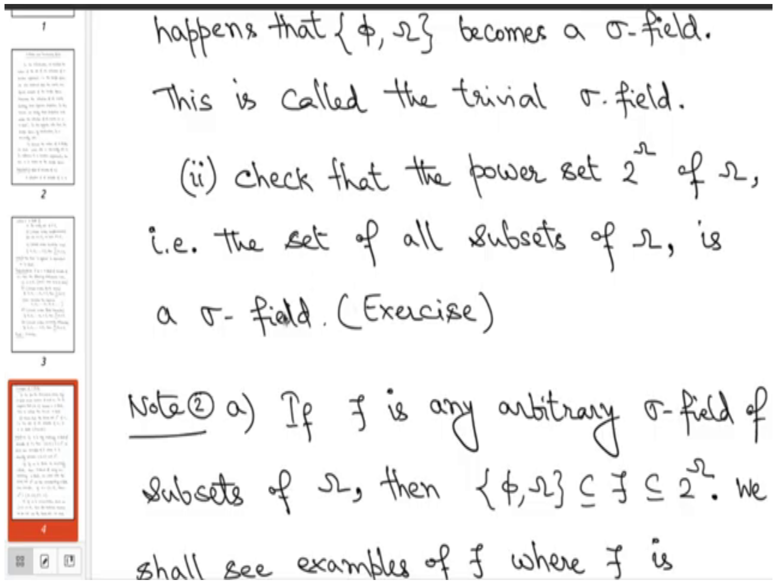

(ii) To the other extent of the things, we can look at all possible subsets called the power set. So, this is the set of all subsets of the Ω. So, this is sometimes denoted as 2^{Ω} to denote that each element in the Ω can have two choices either you take it or not take it. In this sense, you get all possible subsets. So, this is the idea. So, now, if you look at the power set, this has the list of all possible subsets of Ω. Now, this will become a σ-field because no matter what kind of sets operation is performed, you always end up with the subset of Ω, and therefore, it is always closed under all those complementation and countable unions. And also, the intersect is the subset, so it should also be there. So, therefore the power set becomes an example of a σ-field.

Now, we have two examples of σ-field one is a trivial σ-field other is a power set.

What we are interested in are arbitrary σ-fields, which fall between these two things. So, arbitrary σ-fields will always contain the empty set ϕ and the whole set, so this intuition is fine. On the other hand, our arbitrary σ-field is nothing but a collection of subsets, so all subsets are there in the power sets. Therefore, a σ-field is a sub-collection, so we have used this says contains notation.

Note:

(a) So, σ-field $\mathcal{F}$ is a subcollection of totally got Ω the power set. We are going to see the examples of $\mathcal{F}$, which is strictly between these two things. Specifically, if you will work with finite sets or countable infinity sets, then instead of using arbitrary σ-fields, we typically work with the power set.

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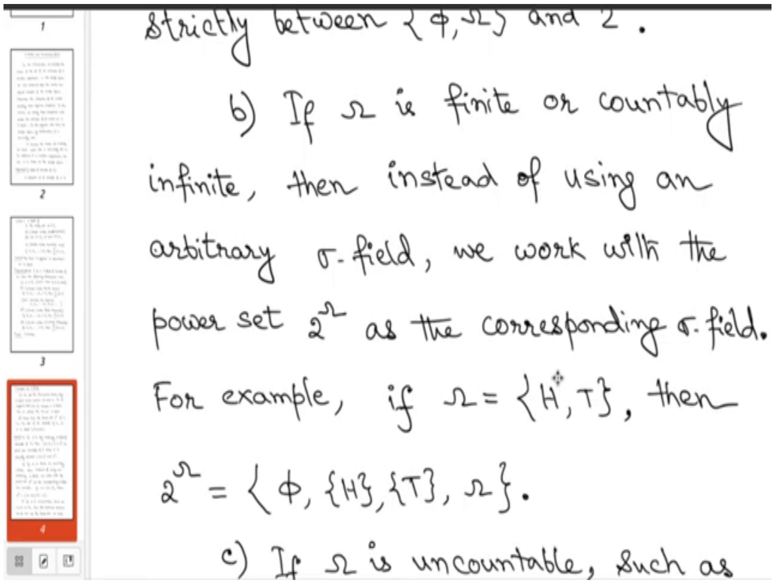

(b) So, if you are working with a coin toss example, you get the outcomes H and T heads or tails, then the sample space is made up of these two elements so, $\Omega = \{H, T\}$. Then, you look for all possible subsets of Ω, the empty set ϕ, two singleton sets $\{H\}$ and $\{T\}$ and then the whole set Ω. This gives you a list of all subsets of Ω, which is a power set, and as per our discussion above, this gives you an explicit example of a σ-field on Ω.

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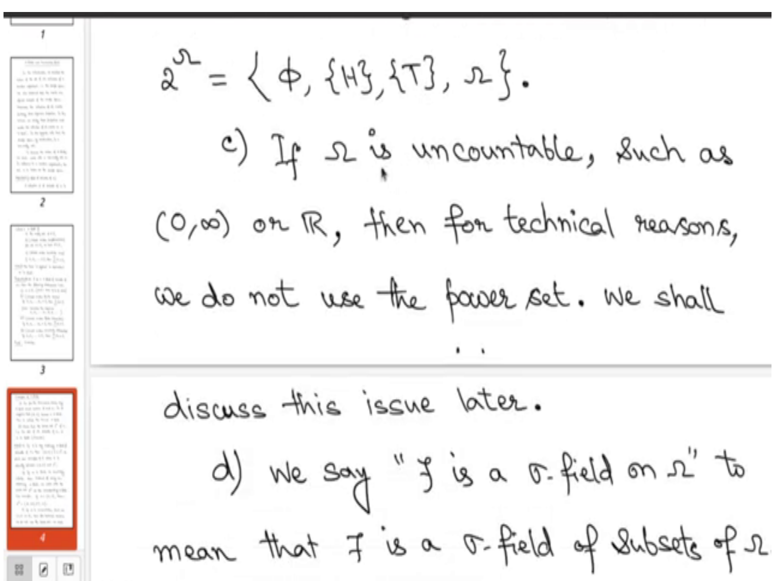

(c) If Ω is an uncountable set and has more elements, then what will happen that we do not use the power set for technical reasons. It will happen that the power set has too many sets to handle, and in principle, we are not going to work with all possible subsets in case of such sets. For example $(0, \infty)$, or $\mathbb{R}$ will have too many elements and too many sets to work with our specific σ-fields in this situation. We shall discuss this issue later on.

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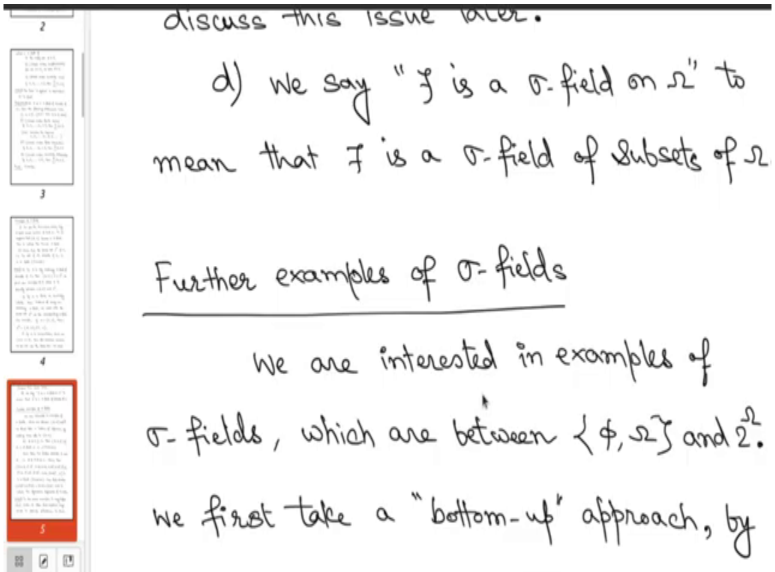

(d) One specific terminology is useful we shall say $\mathcal{F}$ is a σ-field on Ω instead of saying $\mathcal{F}$ is a σ-field of subsets of Ω. Instead of saying this whole thing, we shall only say $\mathcal{F}$ is a σ-field on Ω.

Now let us look at further examples of σ-fields. We are interested in an example that falls between the trivial one and the whole power set. To construct these examples, we first take a bottom-up approach.

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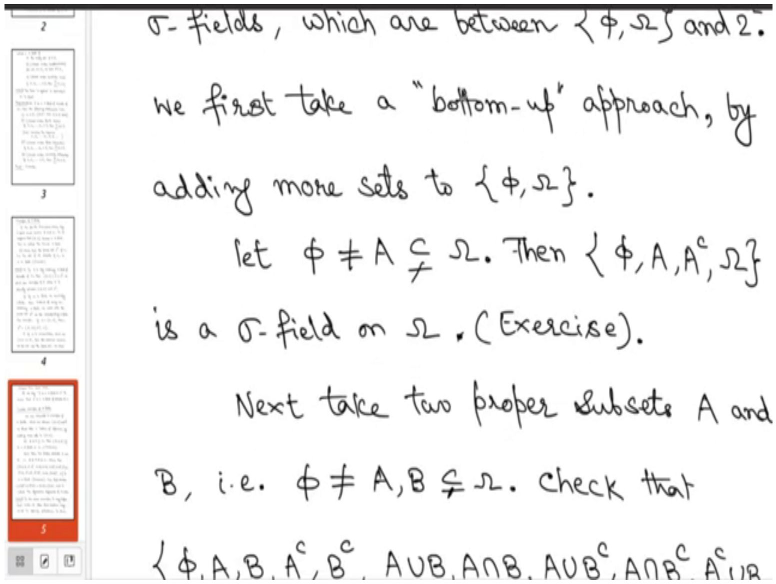

So, what is this bottom-up approach? We look at the trivial σ-field and add sets to this so, we take some non-trivial subset A which is not the empty set ϕ and not the whole set. So, I have taken non-empty set Ω and take a non-trivial set A. Then look at this collection of sets $\{\phi, A, A^c, \Omega\}$. You can now try to check this set of 4 elements construct an example of a σ-field.

So, all you have to verify is that empty set ϕ is there, which is also already seen. Then you have to check that complement of each set is there, and then you have to check that if you create sequences out of these four elements, you always get closed under that. Please check this. So, this will become an example of a σ-field which is slightly more general than the trivial σ-field or the power set.

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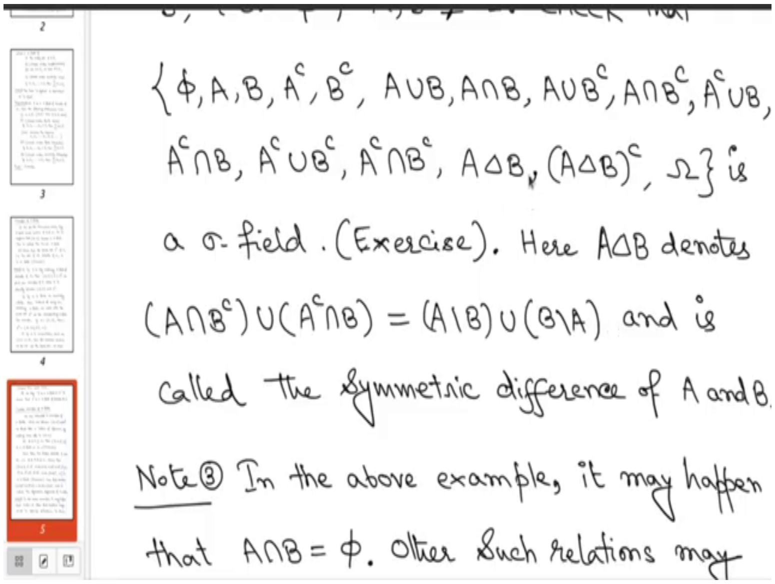

So, now you can ask, I have taken one subset. What will happen if I take two proper subsets.

So, to this, let us see what happens. So, take two proper subsets A and B. So, therefore, A, B are two subsets usually distinct such that they are not the empty set ϕ and not the whole set. Then it will end up with 16 possible sets. So, I have listed them so it will, of course, have A, B, ϕ, thenA^c , B^c and then you shall see all these unions and intersections of this is appearing here. And you also see combinations of complementation and so on.

In particular, I will draw your attention to this set called the symmetric difference of two sets. What is this? This is this notation so that I will write $A \Delta B$. This thing denotes the symmetric difference. So, $A \Delta B$ is exactly the set consisting of elements that fall in one of them. So, they are not at an intersection between A and B. So, it is in A not in B, or the element should be from B but not in A.

So, list all such elements of Ω will construct you the symmetric difference between the sets A and B. So, this symmetric difference will appear in the σ-field, and its complement will also appear. Here I wish also to draw your attention to this notation that $A \cap B^c$ can sometimes be written using this set minus notation. I shall only write $A \setminus B$ to denote all elements in A which are not in B.

So, as if I am subtracting out all elements of B that appear in A. I am subtracting them all, so that is this notation. So, in this list, you will have at max at the most 16 possible sets.

Note: (iii) But, it may so happen that $A \cap B$ is empty.

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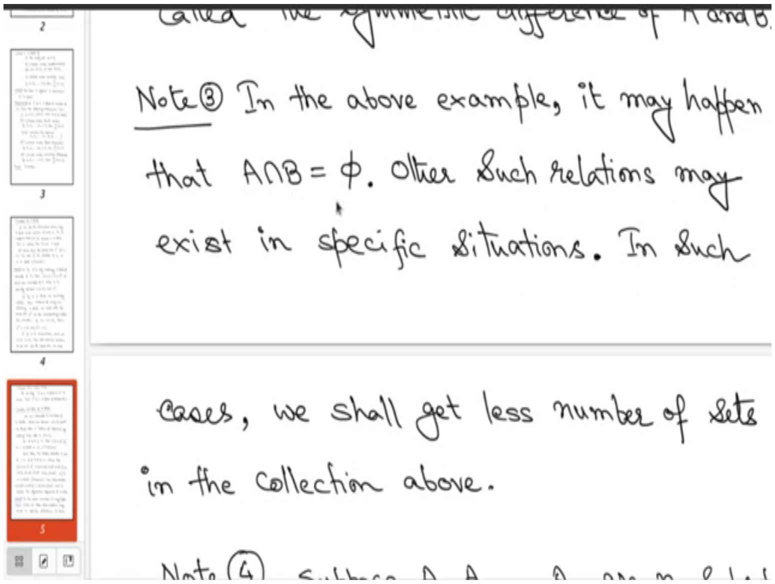

So, let us go up again. So, in this list, $A \cap B$ appears if it so happens that A and B the two proper subsets you have taken are disjoint, then $A \cap B$ becomes empty set ϕ so, which is already listed. So, in principle, while working with specific examples, this list of 16 elements or 16 sets may collapse and give you fewer sets. So, these are the maximum possible sets that you might have. So, we shall get fewer sets in the collection that we have explicitly listed in such cases.

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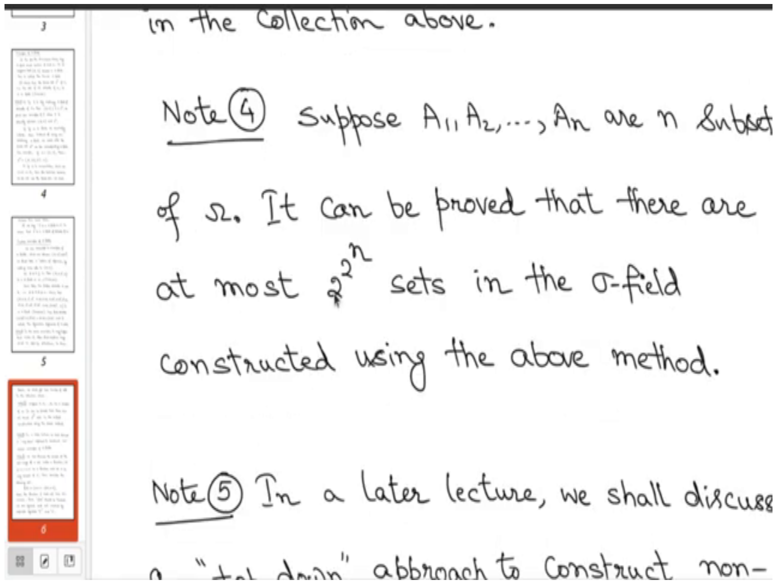

(iv) More generally, you can say this if you have n mini sets $A_1, A_2, \ldots, A_n$ then you shall have n distinct subsets of Ω, and it can be proved that there are at the most 2^{2^n} sets in the σ-field constructed using the above method. So, what do I mean so? If you just go back to the example here, we had constructed the first σ-field using only one set; we had four sets, so that is $2^{2^1} = 4$ sets. Then, you took two sets; as per its statement, you are expected to have $2^{2^2} = 16$ elements, 16 sets. So, this is at the most, and as we have already explained, if you some relations between the sets the list of 2^{2^n} sets may collapse and give you less number of sets at the end, so this will happen in specific examples. But, you shall only have at the most 2^{2^n} sets if you start up with n subsets. It can be showed you take it as a fact.

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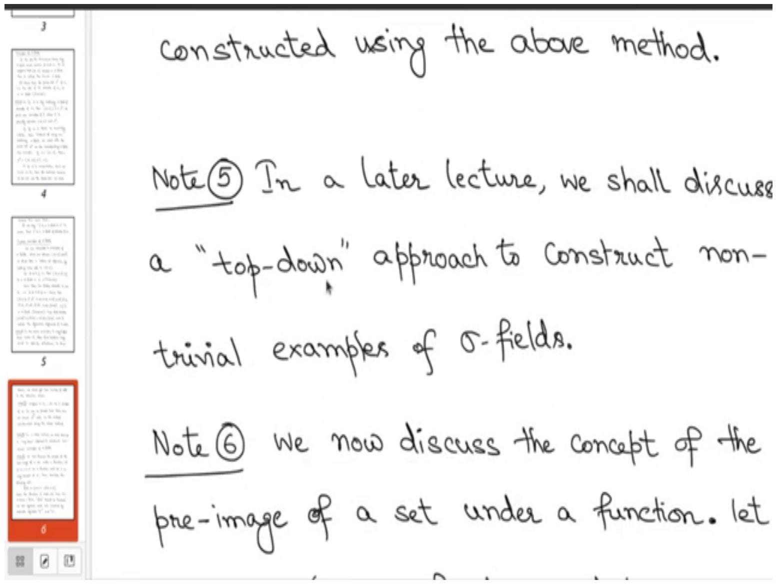

(v) In a later lecture, we shall discuss a top-down approach to construct non-trivial examples. In the above discussion, we have done that we have constructed examples using the bottom-up approach. We have started up with trivial σ-field and then started adding sets. So, from the bottom, we are getting up and getting bigger and bigger examples of σ-fields.

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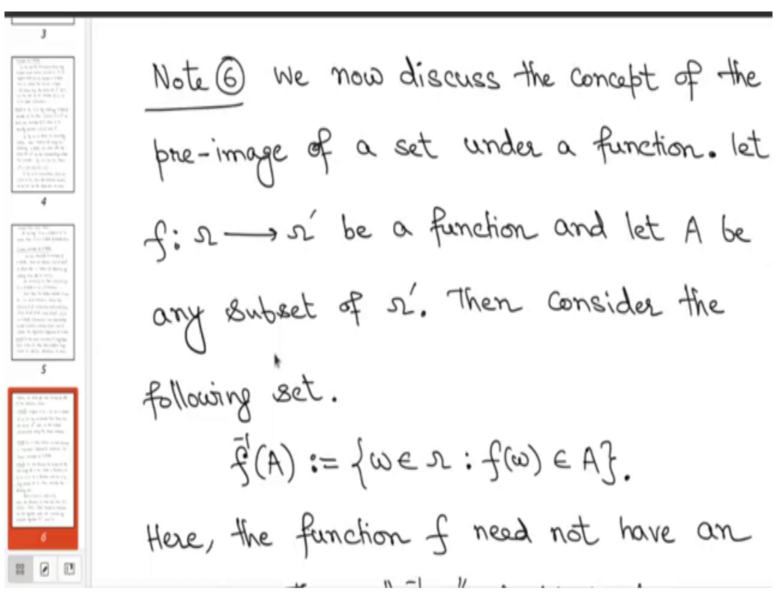

(vi) We now discuss the preimage of a set under a function here take a function $f: \Omega \to \Omega'$. Then if you take specific subsets from the range, let's call A to be a subset, this is an arbitrary subset. You can consider this following subset denoted as $f^{-1}(A)$, which have all elements in the domain such that $f(\Omega)$ falls in the different set A. So, A is a specific subset on the range side, and $f(\Omega)$ should be in that specific subset's range. So, I list all such points on the domain side.

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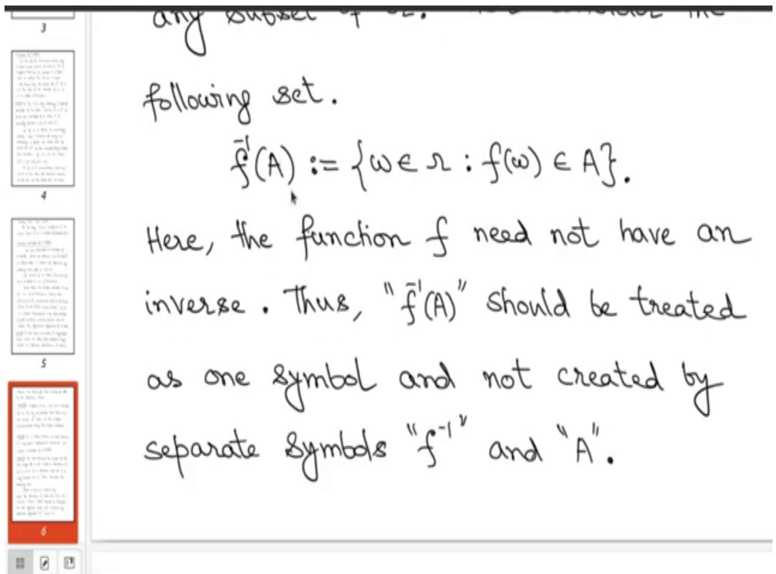

So, this is called the preimage of the set A under f. Here you need not have an inverse. The function f need not have an inverse. Thus $f^{-1}(A)$ should be treated as one symbol and not created by separate symbols f^{-1} and A. So, here this function f need not be an injective function and need not have an inverse.

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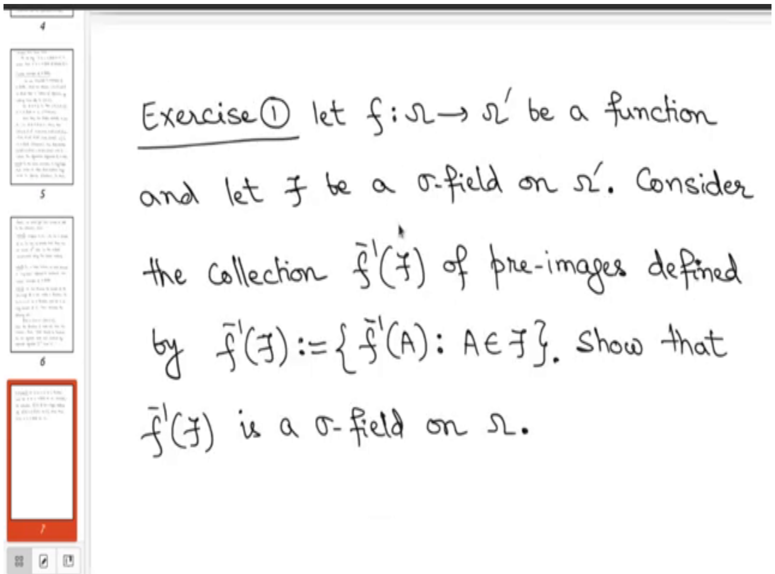

So, using this notation, you can now try to work out with this exercise which says that you can take a σ-field on the range side. So, take this function, take the σ-field on the range side, and then look at preimages for all sets in the range side. So, I take a σ-field on the range side. Take all sets that are in that specific list on the range side. Look at all its pre images, so this is a kind of a list of all preimages. This is the collection of all preimages, then what you can now try to look at is this list of all preimages.

Now, try to show that this is constructing an example of a σ-field on the domain side. So, the preimages are the specific subsets on the domain side and what you are trying to do is show that this construction example of a σ-field.

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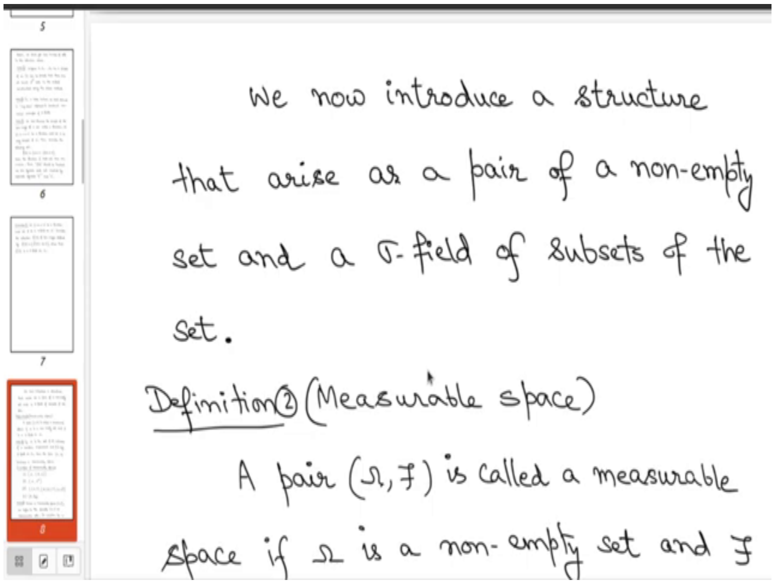

We now discuss a concept very much related to a σ-field.

Definition: So, take a pair consisting of a non-empty set Ω and a σ-field of subsets of this given set. Then if you look at this pair, this constructs a specific structure. This is what we define. We call it a measurable space. A pair $(\Omega, \mathcal{F})$ is called a measurable space. If the Ω the first thing in the notation is a non-empty set Ω and $\mathcal{F}$ is a σ-field on Ω. This is called a measurable space. So, if Ω is a set of all outcomes in a random experiment and $\mathcal{F}$ is any σ-field, then the pair becomes a measurable space.

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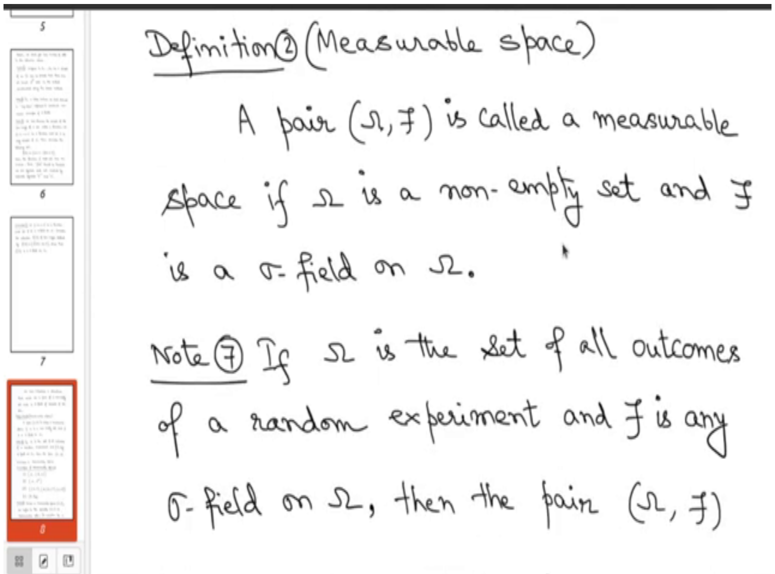

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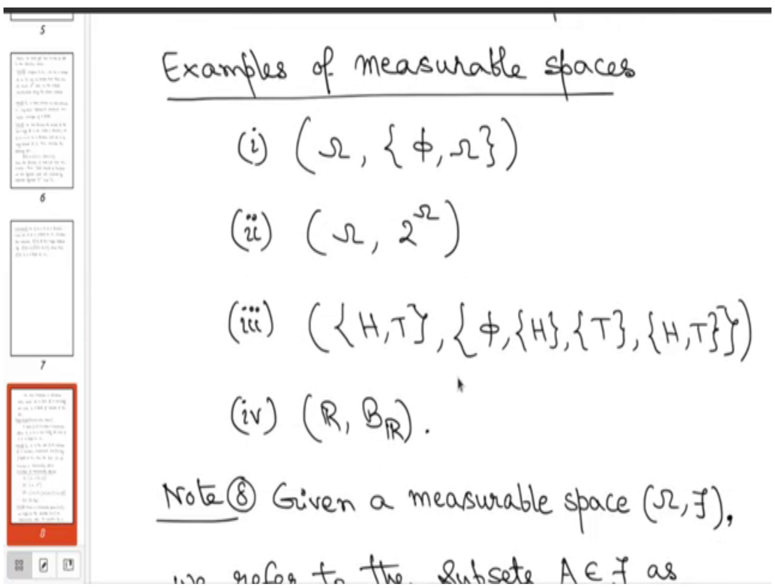

So, this is a more explicit example that you might get from a random experiment.

More specifically, we shall discuss some examples here. We have already discussed that if you take a non-empty set Ω, then the trivial σ-field is always there as an example so, if you write it down in this pair format, this gives you an example of a measurable space. So, Ω together with the trivial σ-field constructs you an example of a measurable space.

Then, looking at Ω together with the power set gives you another example of a measurable space. More explicitly, if you are working with the random experiment of tossing coins, your sample space Ω is composed of two elements heads and tails, and then you can also look at the power set of all subsets. This is an explicit example of a measurable space.

Later on, we shall discuss this notation of σ-fields $B_{\mathbb{R}}$ which is a σ-field on a real line. We shall discuss it in a later lecture. This is kept there just for your understanding that on the real line, we will discuss some specific examples and recall that we have already mentioned that in this uncountable sets situation, we are going to look at specific subsets. This shall be discussed in the later lecture.

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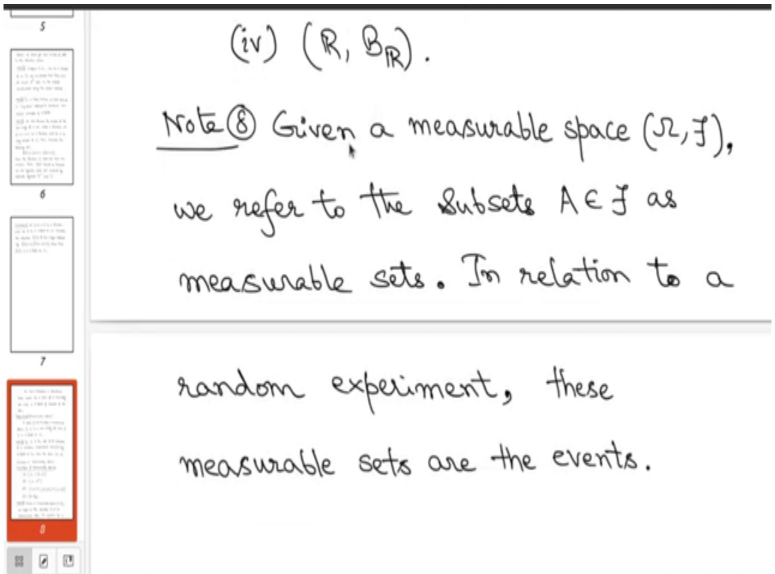

One final comment before we stop is that given a measurable space. We refer to the subsets that you have in the σ-field as measurable sets so σ-field $\mathcal{F}$ is on the non-empty set Ω and σ-field $\mathcal{F}$ need not be containing all possible subsets. So, it need not be the power set; it lists some specific subset some special subsets these subsets given the σ-fields these special subsets we shall refer to as measurable sets.

About a random experiment, these measurable sets are exactly the events. So, if you go up the single head or single tail occurrence, they are specific events about a random experiment of tossing coins. So, we stop here.