Measure Theoretic Probability 1

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Lecture - 13

Algebraic Properties of Measurable Functions

Welcome to this lecture before we proceed let us quickly recall what we have been doing in this week. So, we have looked at certain classes of functions defined on the sample space. And we considered the measurability properties. We defined it as the functions for which the pre-images of all measurable sets from the range side those pre-images should lie in the domain side σ-field. And in particular, we are interested in the functions which take values in the space of real numbers or extended real numbers or in higher dimensional Euclidean spaces.

So, there we are putting the appropriate Borel σ-fields on the range side and looking at all Boreal sets on the range side looking at their pre-images if the pre-images fall inside the domain side σ -field then we are calling it as Borel measurable functions. So, we have so far discussed three types of examples of such functions first were constant functions, second indicator functions, and third continuous functions.

So, continuous functions we consider on the real line and also on higher dimensional Euclidean spaces and we considered the ranges as either the real line or some $\mathbb{R}^d$. So, with such appropriate assumptions we have mentioned that such functions are also Borel measurable. So, we have such types of examples at hand. So, with this, we are now going to proceed and look at certain nice properties of these functions.

And as you shall see that these properties also allow us to construct more examples from the existing examples. So, in this lecture, we are going to discuss the algebraic properties of these functions. So, basically, we are going to look at algebraic operations on these functions like addition and subtraction. So, let us move ahead with the slides.

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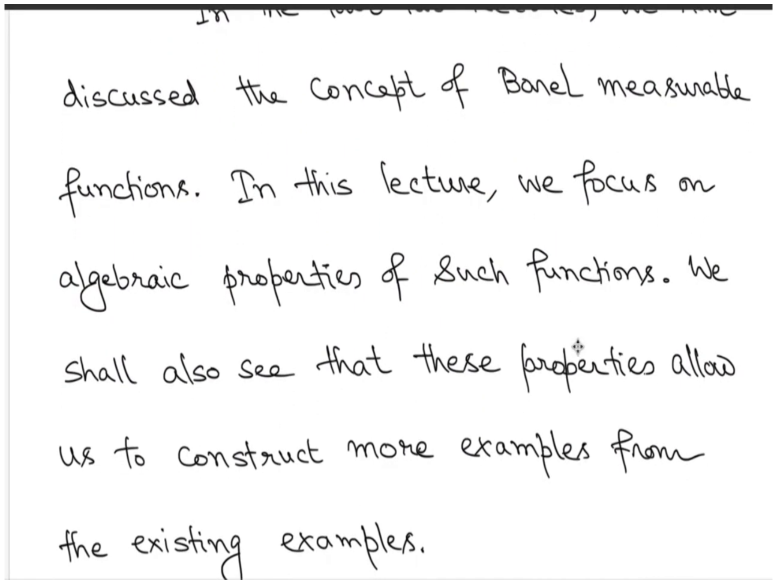

So, in the last two lectures we have discussed this concept of Borel measurable functions.

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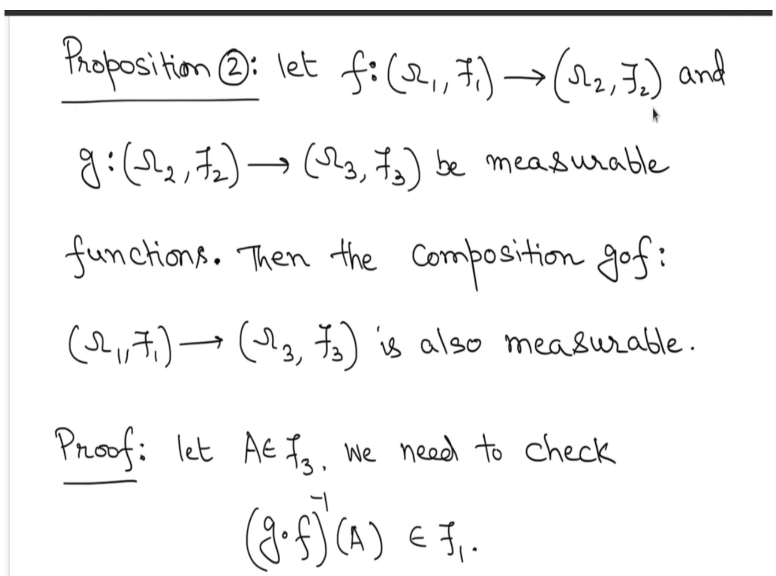

In this lecture we are going to focus on the algebraic properties of these functions. And as mentioned earlier we are going to see that these properties are going to help us construct more examples. So, the first property that we are going to look at is described in this proposition. So, what we are saying is that I am given two measurable functions, first one goes from Ω_1 to Ω_2 and then the second function goes from Ω_2 to Ω_3.

So, if you are just given these two functions f and g. So, you can talk about the function $g \circ f$ whose domain is Ω_1 and the composition should land up in Ω_3. So, $f: \Omega_1 \to \Omega_2$ and then $g: \Omega_2 \to \Omega_3$. So, therefore, the composition is fine. And you can talk about the composition from Ω_1 to Ω_3. See here of course you require that the range of f should be contained in domain of g.

And here it is perfectly fine because the range of f will be some subset of Ω_2 and that Ω_2 is the domain of $g \circ f$ is well defined so that is not a problem. Now we are going to claim that this composition will be measurable provided f and g were given to be measurable. So, now when you are looking at f you will have appropriate σ-fields on the domain side and the range side.

And you will require the appropriate measurability assumptions on f. Similarly, for g you will put the appropriate σ-fields on the domain side and the range side. So, it is important that you keep the σ-field on Ω_2 to be the same. So, you keep the same σ-fields. So, then you also assume that g is also measurable. So, given that f and g are measurable we claim that $g \circ f$ is also measurable.

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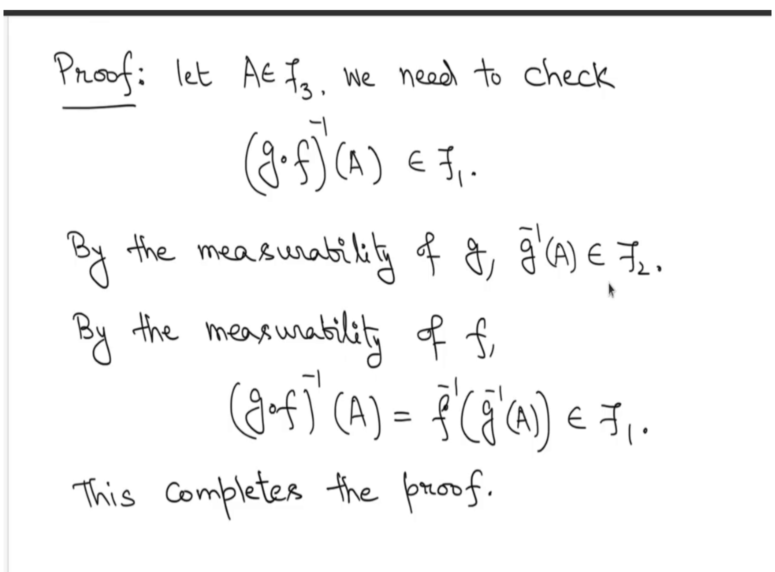

So how do you go about proving this? So again, let us go back to the definition. So, you want to show that for arbitrary sets on the range of $g \circ f$. So, the range is Ω_3. And there the σ-field is $\mathcal{F}_3$. So, you will take arbitrary sets from there and you will like to look at the preimage under the composition map. If the preimages fall inside the domain sides σ-field which is $\mathcal{F}_1$. Then you will claim that $g \circ f$ is measurable.

So how do you check this? So first observe that if you look at the measurability of g you will immediately claim that for any arbitrary measurable set in $\mathcal{F}_3$ which is in Ω_3 you will look at the preimage under g. So, that will fall inside that domain side σ-field of g. So, what is the domain of g that is Ω_2 and that domain side σ-field $\mathcal{F}_2$. So, therefore, the preimage of A under the function g that will land up in $\mathcal{F}_2$ this is by the measurability of g.

But then you use the measurability of f and look at the preimage of the set $g^{-1}(A)$ under the function f. So, $g^{-1}(A) \in \mathcal{F}_2$ and use the measurability of f. So, $f : \Omega_1 \rightarrow \Omega_2$So, now, you will take any set from the range side which is in $\mathcal{F}_2$. So, you can look at this set $g^{-1}(A)$. So, that said if you look at the preimage under f so that should land up in $\mathcal{F}_1$ as per the measurability of f.

But this is a easy statement to check that $g \circ f$ that function the composition function if you look at the preimage of the set A under this function it can be written as this preimage meaning it is the preimage of $g^{-1}(A)$ under the function f. So, if you show this equality you will immediately claim using the measurability of g and f that the preimage of A under $g \circ f$ is in $\mathcal{F}_1$.

And therefore, since you have proved it for any arbitrary set A in $\mathcal{F}_3$ therefore you can claim that $g \circ f$ is measurable. So, this is a very very important property and you will, you are going to see the usage of these right now.

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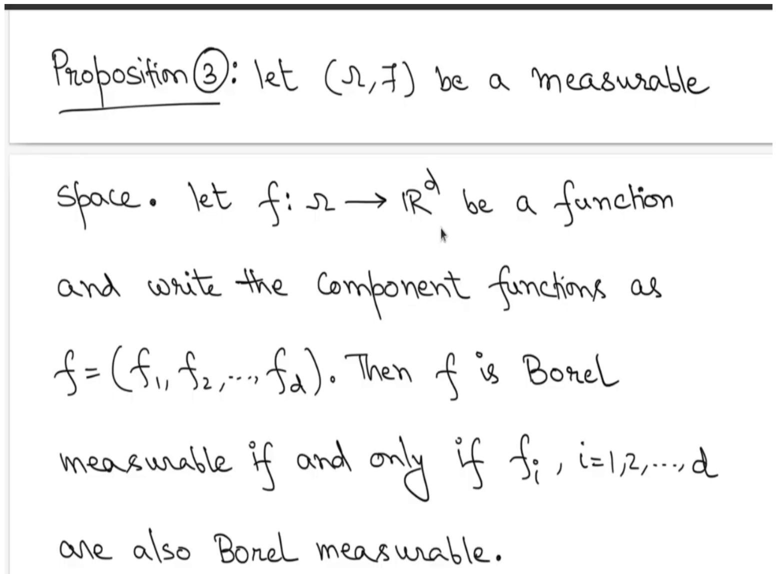

So now what is this? So, we are going to apply this composition operation now. So, what is this statement? So, I am starting off with now one measurable space $(\Omega, \mathcal{F})$. And look at some arbitrary function defined on Ω. So, sticking values in some Euclidean space $\mathbb{R}^d$. So, now you can write the component functions as $\left(f_1, f_2, \ldots, f_d\right)$. So, what are these? These are all the functions which take component wise values.

So how are they defined? So, you take f of some sample points ω so $f(\omega)$ will be a vector in $\mathbb{R}^d$. And the first component of that will be given by the function f_1, second component will be given by the function f_2. So, all of these functions $f_1, f_2, \ldots, f_d$ these functions f_i are functions from Ω to $\mathbb{R}$. So, these are taking values in the real line. So, you can identify these component functions as long as you are given the function f.

Now, here is the statement. So, take f to be Borel measurable that means you are looking at appropriate σ-fields on the domain side $\mathcal{F}$ and appropriate σ-fields on the range side. So, that will be the Borel σ-fields. Then with respect to these σ-fields you talk about the measurability of the function f.

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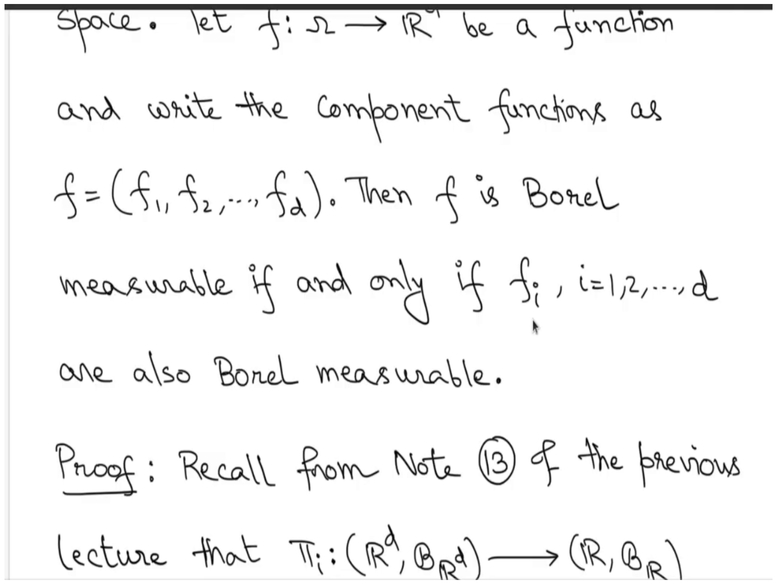

Now, if this is born measurable the statement says that the component functions f_i are also Borel measurable. Now remember that this component functions f_i are from Ω to $\mathbb{R}$. So, therefore, you are going to use the Borel σ-field of the real line on the range side. And you are claiming that if f is measurable then f_i is are also measurable. But this statement says something more that if you know that the component functions f_i are Borel measurable.

So, you have d many component functions which are Borel measurables from Ω to $\mathbb{R}$. Then the original function f is also Borel measurable. So, this is an if and only if statement. So, how do you prove this?

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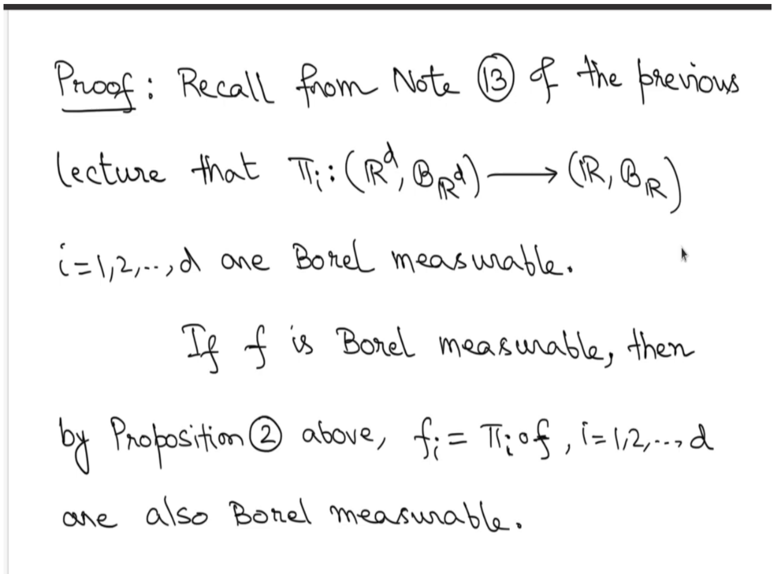

So, the proof goes by checking the measurability conditions on generating sets on the range side. So, this was mentioned earlier in note 13 that we can check this just the measurability conditions or the preimage of sets coming from the generating class from the range side. So, what to do so start with thethese projection maps coordinate projection maps that were mentioned earlier. So, these were also continuous functions.

And these turned out to be Borel measurable by the appropriate condition that we just mentioned by looking at the preimages of the generating sets. Now, we are going to use those two things together. So, one is about this preimages of generating sets. So, that is one thing that we are going to use the second thing that we are going to use are this coordinate projection maps π_i. So, how do you proceed?

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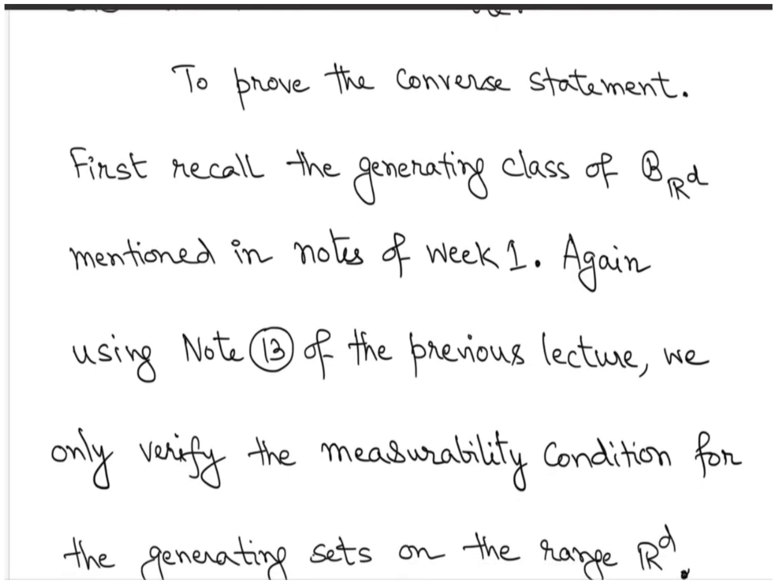

So, start with Borel measurability of the function f which is from Ω taking values in $\mathbb{R}^d$. So, then what you can look at are these component functions can be obtained as a composition of the projection maps with the actual function f. So, this is $\pi_i \circ f$ so you can easily check this. So, take a sample point ω in the domain. So, then if you look at $\pi_i \circ f$ evaluated at that sample point ω you will exactly get back the i-th component function.

So, that is this relation here. So, now as soon as you have the Borel measurability of f put it together with the Borel measurability of the functions π_i. And then by the composition operation that was discussed in proportion 2 above. You immediately claimed that the component functions f_i 's are also Borel measurable. So, you will see the proof is very simple as long as you know all these facts.

So now for the converse we are going to use this thing about the preimage of generating sets. So, now we are assuming that all the coordinate functions f_i's are Borel measurable. So, they are from Ω to ℝ. So, you are using the Borel measurability there with the real line the Borel σ-field on the real line on the range side. But then you want to claim the Borel measurability of the function f which is taking values in $\mathbb{R}^d$. So, we will be using the Borel σ-field of $\mathbb{R}^d$. So, to check the Borel measurability of the function f you need to use the generating sets of Borel σ -field on $\mathbb{R}^d$ which is the range.

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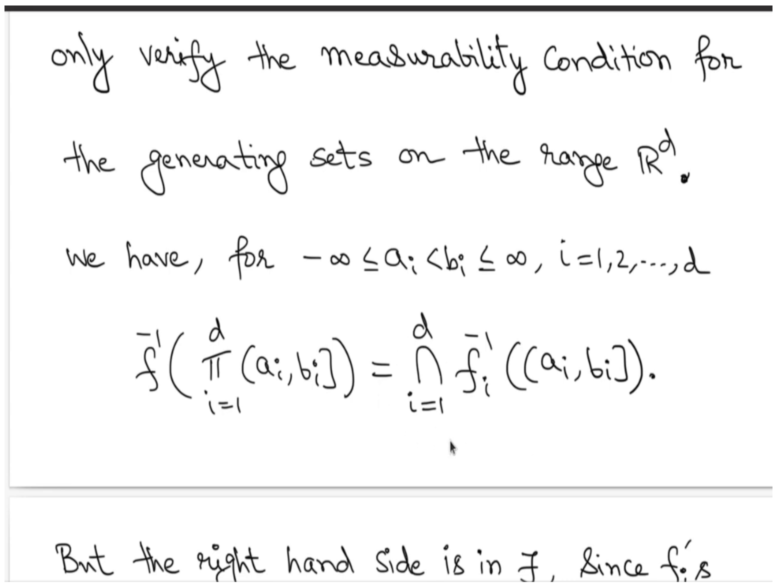

So, then you use that and what are the generating sets there. So, recall from our discussions in week one that for this you have to use this product of this type of left open right closed intervals. So, you will choose this numbers a_i, b_i $a_i < b_i$. And look at this interval (a_i, b_i), take the product of them d for product. So, these are all the generating sets on the $\mathbb{R}^d$. So, the Borel σ -field of $\mathbb{R}^d$.

So therefore, if you look at the preimage of these sets and if you just verify the appropriate condition for these sets then you are done. But then it is easy to check that the preimage under f that can be written as this condition. So, what are these you are saying that these points that are in this preimage. So, for these points the i-th component lies in a_i to b_i. So, that is what the left hand side means. So, if you put a sample point on the left-hand side then $f(\omega)$ lies in this product.

So therefore, individually component-wise the ith coordinate lies in ai to bi that left open right closed interval. And that is what exactly is stated on the range side. So, check this and therefore what you were able to write is that the left-hand side is equal to the range side where on the right-hand side you have the preimages involving the component functions. So, now in this component functions if you assume the measurability as given in the hypothesis.

Then these preimages lies in the domain side σ-field which is $\mathcal{F}$. And therefore, the finite intersection is available there. And therefore the right-hand side is falling inside the domain side σ-field.

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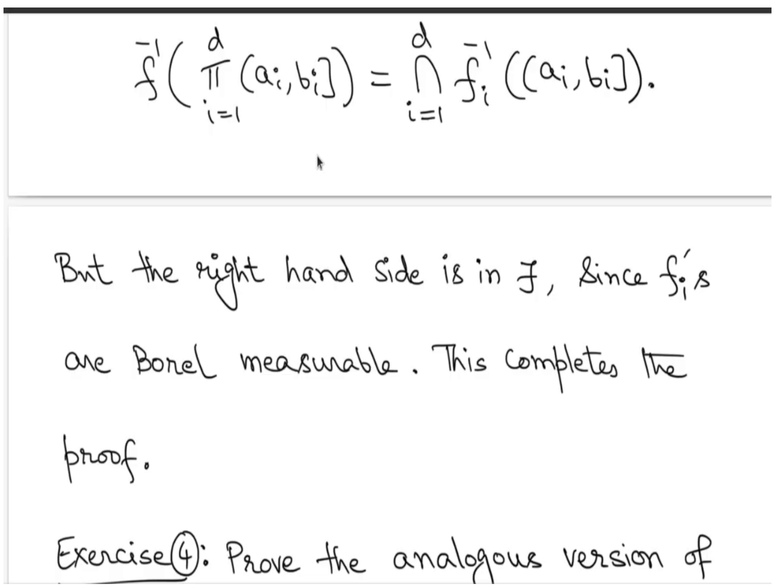

Therefore, using the Borel measurability of the component function f_i's you get the measurability of the function f this completes the proof. So, now what is this identification once more? So, using these composition operations that we discussed.

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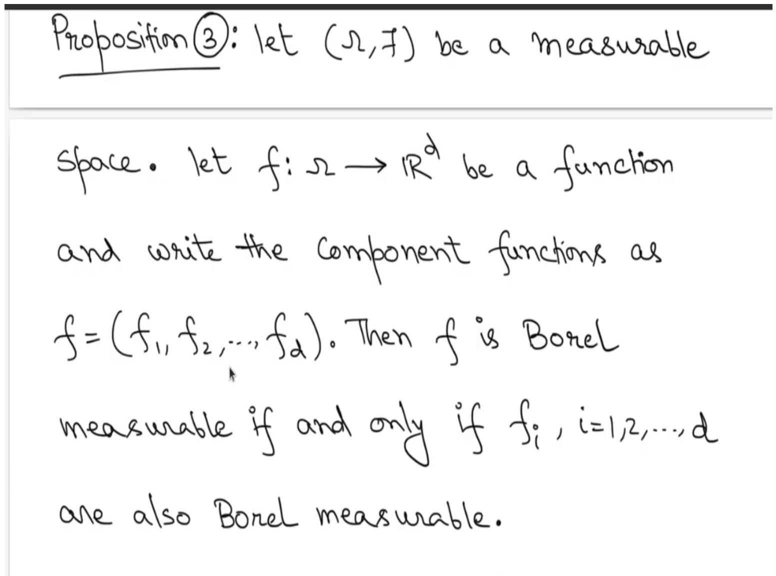

We have been able to show that if you want to talk about $\mathbb{R}^d$ valued measurable functions. It is good enough that you talk about the measurability, the real-valued measurability of the component functions and vice versa. So, that is the identification that as long as you have d many component functions which are measurable put them together you will get a $\mathbb{R}^d$ valued measurable functions.

On the other hand, given a $\mathbb{R}^d$ valued measurable functions the component functions immediately turned out to be Borel measurable. So, with that result at hand, we are now going to proceed towards certain interesting observations.

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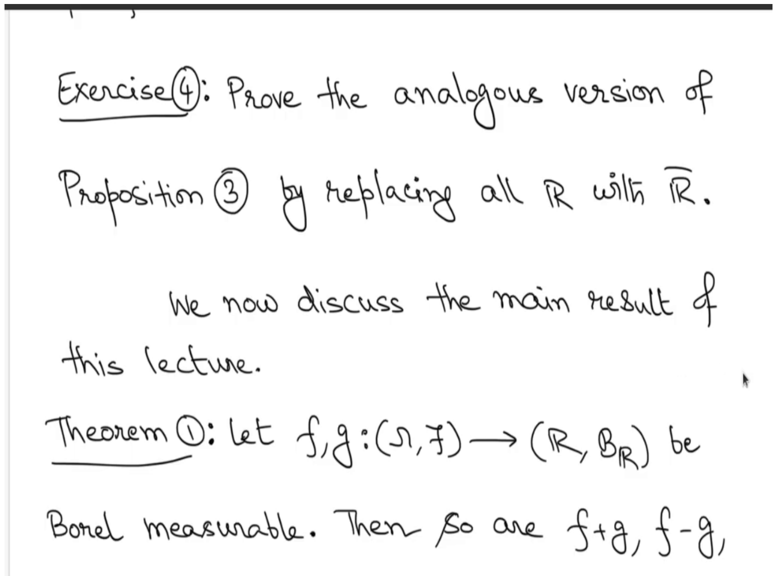

But first in exercise 4 we mentioned that all these results that are discussed for the real valued functions or $\mathbb{R}^d$ valued functions can be extended with extended real line as the range side. But these we are not going to use too much. So, on the range side where we have to use $\overline{\mathbb{R}}^d$ when you are looking at the d- dimensional versions. But you can still use usual ideas to talk about the Borel σ-fields on $\overline{\mathbb{R}}^d$.

So, we have skipped that in the discussions in week 1. But you can still talk about the Borel σ -fields on $\overline{\mathbb{R}}^d$ the default product of the extended real lines by usual method by looking at products of default products of the generating sets left open right close intervals on the extended real line. So, if you look at such set it will generate the appropriate σ-field and you can actually prove the analogous version of this identification of $\overline{\mathbb{R}}^d$ valued measurable functions with the measurability of the component function so you can prove this.

So, this is an optional exercise given to you. So, in all of these results that we are going to state many of these results can be stated either for real-valued functions or extended real-valued functions and according to our convenience we are going to look at any one of these versions. So, now we are going to go forward and discuss the main result of this lecture.

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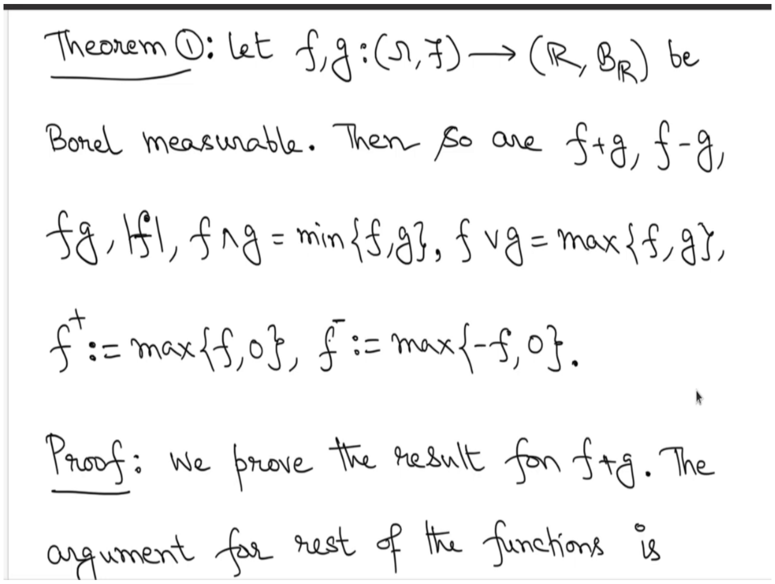

So, what is this? So, look at functions $(f,\ g)$ defined on the same measurable space and taking values in the real line. So, these are all given to be Borel measurable functions. Then what we are going to say that look at other functions like this. So, what are these? So, look at $f\ +\ g$ so how is $f\ +\ g$ defined, so look at a sample point of ω $f\ +\ g$ that function is simply taking the value $f(\omega)\ +\ g(\omega)$.

So, just evaluated the original functions f and g at the sample point add them up you will get the actual function $f\ +\ g$. Similarly, $f\ -\ g$ that function is defined as $f(\omega)\ -\ g(\omega)$. So, likewise, you can go ahead and talk about the all the other functions like fg. So, product of f and g, $|f|$, $\min\{f,\ g\}$, $\max\{f,\ g\}$ such functions. In particular, recall that constant functions are always measurable.

In particular if you take g to be identically 0 function. Then you can look at $\max\{f, 0\}$ and $\max\{-\ f, 0\}$. So, here just a quick note that $-\ 1$ is also a constant function $-\ 1$ is a constant function and it is also Borel measurable. So, now we can talk about the product f with the constant function $-\ 1$. So, that will give you the function $-\ f$. Another way to look at this function $-\ f$ is to look at the subtraction from 0.

So, 0 minus the function f will give you $-\ f$. So, since both the functions 0 the constant function 0 and the f are measurable the function $-\ f$ is also measurable. What we are doing? We are looking at this $\max\{f, 0\}$ or $\max\{-f, 0\}$ and we are going to write as f^+ and f^- these two functions. We will see the usage of these functions in a minute. But what we are saying is that all these functions that we are talking about they also be Borel measurable.

So, provided f and g are Borel measurable you will immediately get the Borel measurability of $f + g$ and so on. So, now let us quickly mentioned the usefulness of f^+ f^-. So, what are these functions? So, when the function f is taking non-negative values then $f(\omega)$ is non-negative. So, the maximum of $f(\omega)$ and 0 will turn out to be $f(\omega)$. So, f^+ will be taking the value $f(\omega)$ for those points where f is already non-negative.

But if f is taking a negative value at some point then $f(\omega)$ is negative so the maximum is 0 here. So, therefore, f^+ will be taking the value 0. So, therefore, you are looking at the graph of the function small as long as it is above the value 0, as long as it is above the x axis in some sense. Then what you are doing is that you are looking at only that part, as long as this goes below the value 0 you ignore those values and put the function constant function 0 there.

So that is what the function f^+ is it is looked it is called as the positive part of the function f. You just look at the points where the function f is taking positive values and that is where you constantly (())(18:54) and obtained this function f^+. On all the points where f is taking negative values you assigned the value 0 in f similarly f^- is called the negative part of f it is the amplitude or the modulus value of how far below f is from 0.

So, as long as f is taking negative values let us say -2 if you compute the value it will turn out to be 2. So, basically, it is giving the modulus of the distance from the 0. So, f^- is saying that how far below the function f is from 0. So, that is the negative part of f. So, let us go ahead and prove this theorem that provided f and g are measurable functions, Borel measurable functions.

Then all these algebraic operations will give you measurable functions. So, with this at hand as long as you have some examples, explicit examples you can use these algebraic operations to construct more examples. So, the proof is pretty simple. And it has the same structure for all these functions. So, we discussed the proof for $f + g$. For all the other functions you can try to work them out.

So how do you prove the measurability of $f + g$? So, remember $f + g$ is a function from Ω to $\mathbb{R}$. So, again, you are using the domain side σ-field $\mathcal{F}$ and the range side σ-field Borel σ-field or $\mathbb{R}$.

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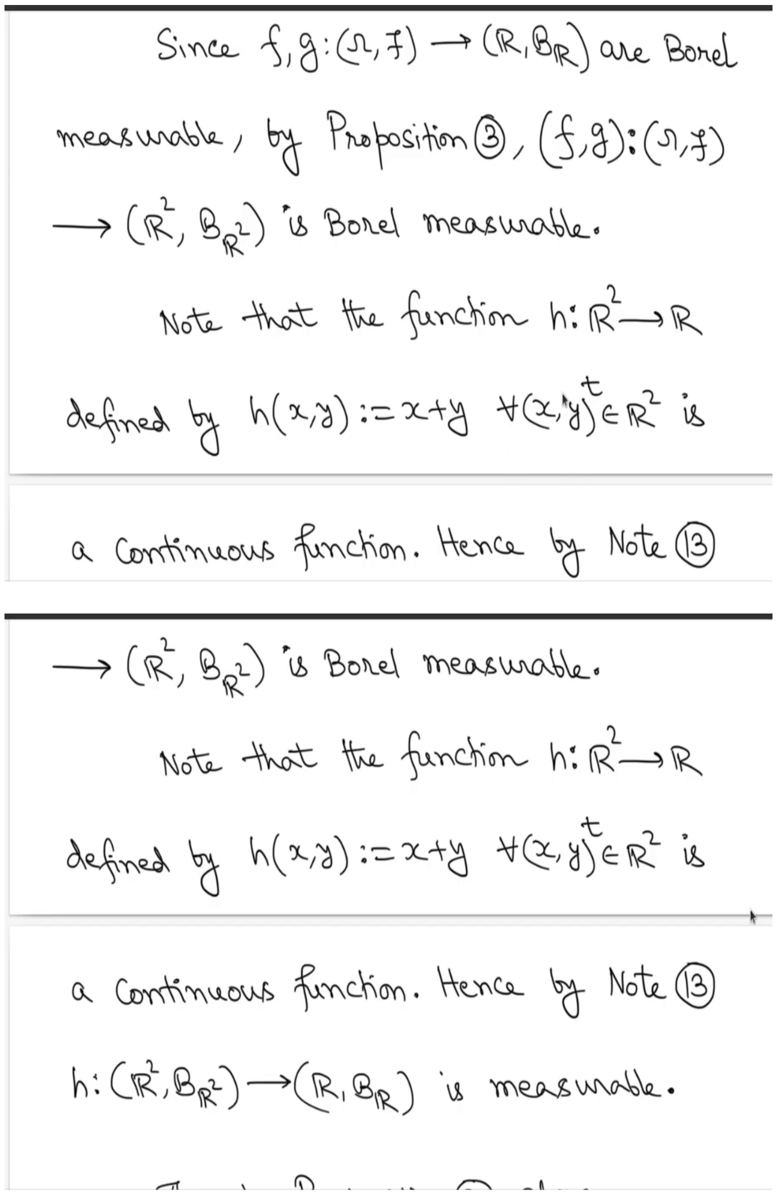

So, how do you prove the measurability? So observe first that as long as f and g are Borel measurable you can talk about this function (f, g). So, this is a two dimensional function taking values in $\mathbb{R}^2$. So, for each point in Ω you look at $(f(\omega), g(\omega))$. So, that is a value in $\mathbb{R}^2$. And by the earlier proposition as long as this component functions f and g are Borel measurable this two dimensional function taking values in $\mathbb{R}^2$ is also Borel measurable.

So, this is immediately where we are applying this proposition. But now note that you have this addition function on $\mathbb{R}^2$ we are calling the function as h. So, h takes points $(x, y) \in \mathbb{R}^2$ and adds them up. So, take the points (x, y) and add up these values x and y. So, that is your function. So, that is our function from $\mathbb{R}^2$ to $\mathbb{R}$, observe that this is a continuous function. You had actually mentioned such functions from $\mathbb{R}^d$ to $\mathbb{R}^n$ any two dimensions.

So, this if they are continuous, they will be Borel measurable. So, in particular here the function h: $\mathbb{R}^2 \to \mathbb{R}$ is also measurable, Borel measurable. So, therefore, what you are going to do is to use these two functions now. So, the first function is made up of this $\mathbb{R}^2$ valued function (f, g) and the second function is this addition function h.

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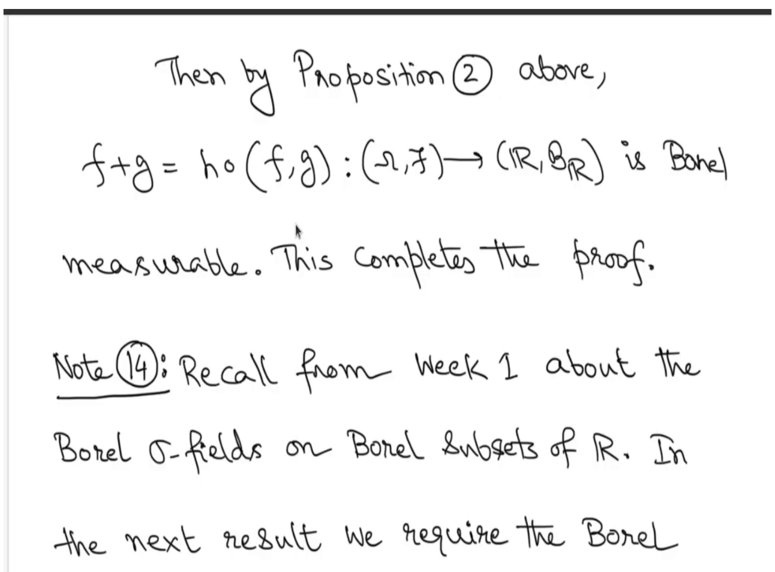

So, put them together so look at the composition of h with the two dimensional $\mathbb{R}^2$ valued function (f, g). So, this composition how does it go. So, (f, g) that function goes from Ω to $\mathbb{R}^2$. And h goes from $\mathbb{R}^2$ to $\mathbb{R}$. So, the composition goes from Ω to $\mathbb{R}$. But if you put the sample points there and compute this composition, what does it give you. It gives you the addition of $f(\omega) + g(\omega)$ for any sample point ω.

So, given any sample point ω if you compute this composition at that sample point ω you will simply get $f(\omega) + g(\omega)$. And that is nothing but the function $f + g$. So, therefore, $f + g$ being the composition of these two measurable functions becomes Borel measurable. So, the proof is pretty simple as long as you use all these algebraic properties that you have already proved.

For other functions like $f - g$ you have to take the function h as going from (x, y) to $x - y$ since that is continuous it will be Borel measurable use the appropriate identification by the compositions and you will get the property. So, once you have done the hard work all these properties now follow pretty easily.

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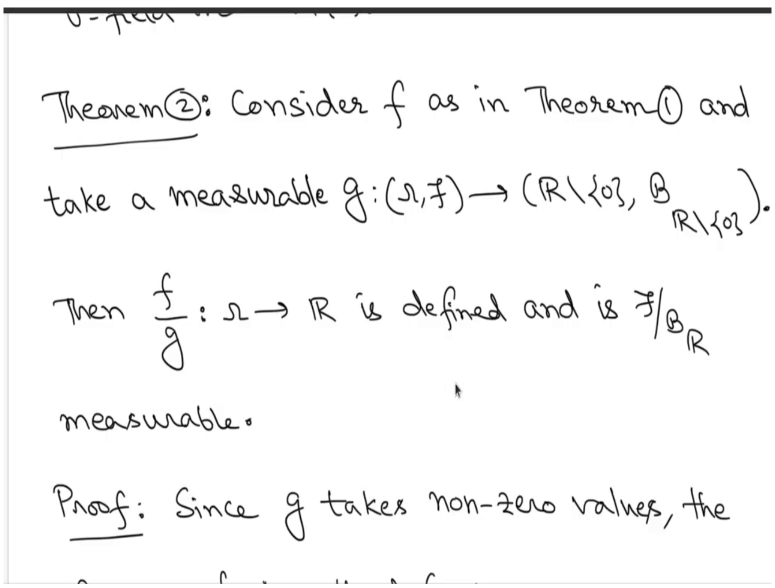

So, now one important factors here that we have so far discussed addition, subtraction, multiplication, and some related operations like taking modulus maximum and minimum. But then one important operation is missing that is the division. So, you are interested in what happens if you divide a measurable function by another measurable function. But here you have to be careful, you cannot divide by any function which is taking value 0.

So first of all, if you are dividing a real number by another real number you have to be careful if the denominator is 0 so that is not defined. So, this is a similar operation when you are dividing a function by another function you have to be careful that the denominator function is not becoming 0 at some point so then it will not be different. So, therefore, we are going to restrict our attention to $\mathbb{R}\backslash\{0\}$.

So, we will exclude the point 0 from the real line. So, that will be considered for our range spaces. So, you have that is what we are going to consider the denominator on. So, denominator will be taking values on $\mathbb{R}\backslash\{0\}$ it will not take the value 0. So, then whenever we are going to talk about the range as $\mathbb{R}\backslash\{0\}$. Then you have to talk about the appropriate σ-fields. And that was discussed earlier in week 1, $\mathbb{R}\backslash\{0\}$ remember this is a Borel subset.

The singleton set $\{0\}$ is a Borel subset so their complement is also in the Borel σ-field. So, it is a Borel subset. For Borel subsets we have defined the Borel σ-fields. So, recall that construction there. And what we are going to talk about $\mathbb{R}$ divisions by functions by measurable functions.

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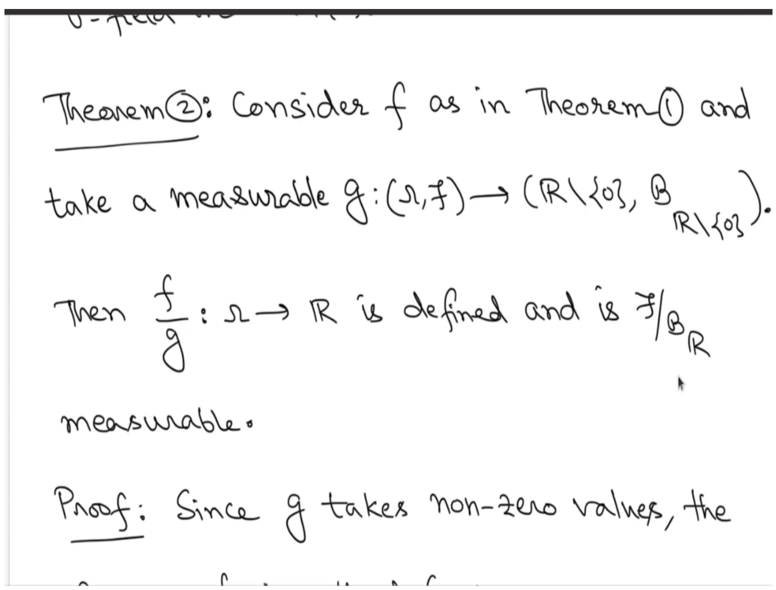

So, continue with f as in real-valued function, real-valued measurable function on the measurable space $(\Omega, \mathcal{F})$. But now the function you are going to divide by the function g that you have to be careful with and that is what we have just mentioned that take the range space as $\mathbb{R}\backslash\{0\}$ so include 0 there. So, here what happens? You have to put the appropriate σ-field here Borel σ-field on $\mathbb{R}\backslash\{0\}$.

Then the claim $\frac{f}{g}$ that function that is well defined and that becomes measurable with respect to the appropriate σ-fields. So, here what happens the division is well defined. So, this is real valued function because f can take the value 0. So, the division can produce the value 0 since the numerator we allowed to be 0. So, how do you show this?

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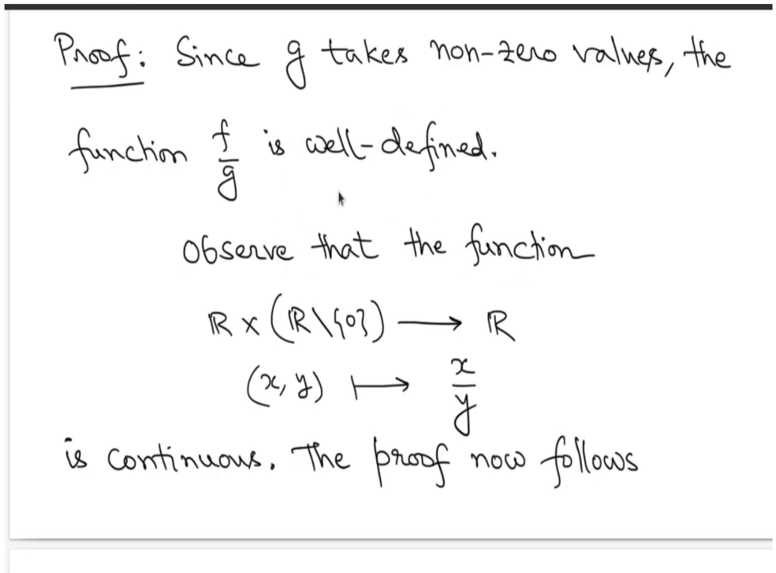

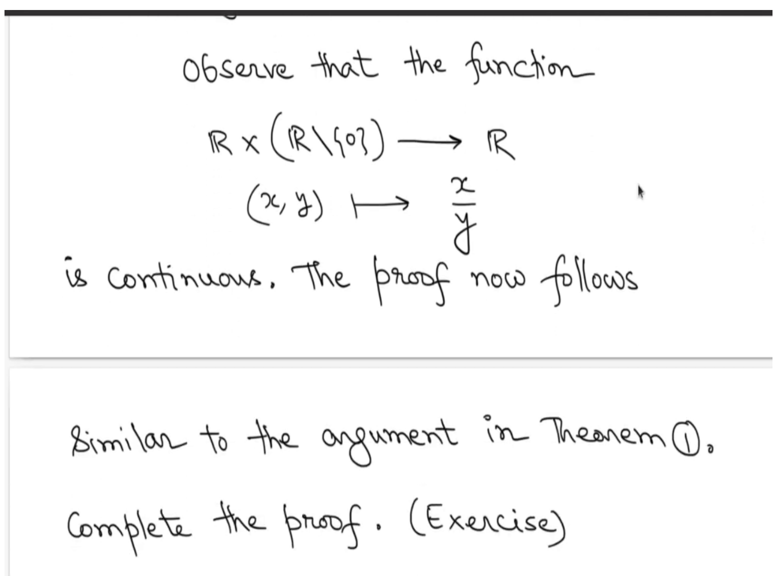

So again, just to repeat. So, this $\frac{f}{g}$ is well defined as long as g takes nonzero values. But then observe that this function the division operation that we have been discussing is can be written as this format $\mathbb{R} \times \mathbb{R}\backslash\{0\}$. So, do you take two real numbers x and y but allow y to take values in $\mathbb{R}\backslash\{0\}$ only. So, do not allow y to take the value 0. Then you can talk about $\frac{x}{y}$. So, $\frac{x}{y}$ will be some real number including 0.

So, this is a continuous function as long as you vary x and y in this space. So, now the proof will follow similar to the earlier argument. Since this function division function that we have been talking about here this is continuous it is also Borel measurable. And hence if you compose with the function (f, g) the two-dimensional function taking values in $\mathbb{R} \times (\mathbb{R}\backslash\{0\})$ then the composition will give you a measurable function Borel measurable function. So, therefore, $\frac{f}{g}$ that is the resultant function will turn out to be measurable with respect to the appropriate σ-fields. So, the proof is pretty simple to work out.

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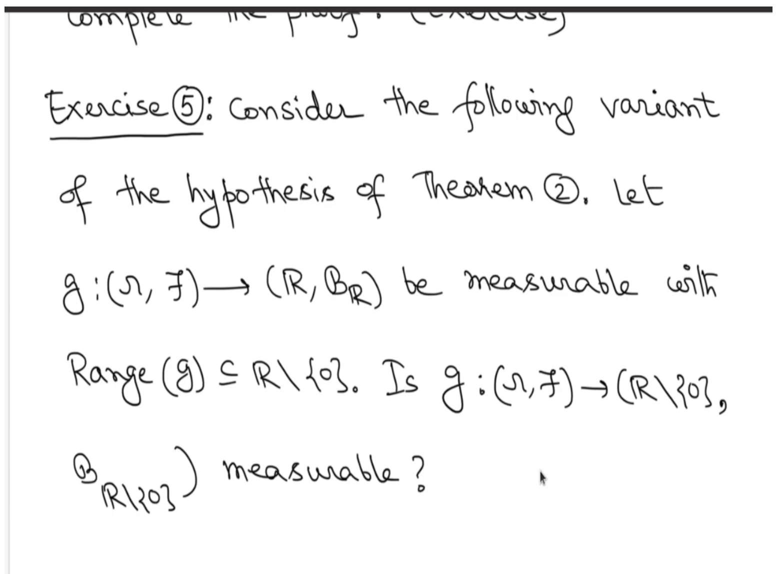

So, our final exercise to finish up the lecture. So, here what we have taken is that we have explicitly stated that the range of g is $\mathbb{R}\backslash\{0\}$ but suppose you look at this following variant of this hypothesis. So, start with g to be measurable but with real values. But then you specify that the ranges $\mathbb{R}\backslash\{0\}$. So, what is the difference with the previous hypothesis? In the previous hypothesis you were only checking measurability for measurable sets coming from $\mathbb{R}\backslash\{0\}$.

But now you are saying g is measurable with Borels in the field on $\mathbb{R}$. So, you have checked it for all Borel subsets of R not $\mathbb{R}\backslash\{0\}$. So, with those measurable sets you have checked the preimages and that turn out to be in the domain set σ-field. So, now with this general condition now you assume that the range of g is contained in $\mathbb{R}\backslash\{0\}$. Question is that whether g is now measurable with respect to the σ-field on $\mathbb{R}\backslash\{0\}$.

Since g is taking values inside $\mathbb{R}\backslash\{0\}$ you can now consider this question that it is a mapping from $\Omega \rightarrow \mathbb{R}\backslash\{0\}$. And you put the appropriate domain side σ-field $\mathcal{F}$ but the range side σ-field now you take it to be the Borel σ-field on $\mathbb{R}\backslash\{0\}$. So, given the fact that g was originally measurable with respect to the Borel σ-field $\mathbb{R}$ with the additional fact that range of g is now $\mathbb{R}\backslash\{0\}$ can you say this, so try to work this out.

So, that is the exercise. So, in this lecture we have looked at these algebraic properties including composition and all these usual arithmetic operations like addition, subtraction, multiplication, and division. So, this will be useful in constructing more examples and we are going to discuss certain continuity or limiting properties of measurable functions in the next lecture. So, we stop here.