Measure Theoretic Probability 1
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Lecture 12
Borel Measurable Functions

Welcome to this lecture let us quickly recall what we have done in the previous lecture. So, in the previous lecture we motivated the need for the analysis of functions that are defined on the sample spaces. Or more generally if you have a non-empty set together with some appropriate σ-field that constitutes a measurable space. We are looking at functions out of those and we want to look at these numerical features which gives us some nice ideas about the underlying measurable space.

So, specifically when you are dealing with a random experiment and looking at the sample space together with the collection of events so that is one measurable space then you are looking at all these outcomes that come out or appear as part of the random experiment and you are interested in these numerical values. So, this is why we started looking at these functions that appear or are defined on the domain space Ω.

So, then with this in hand we then looked at the type of conditions that we want to discuss and we called them measurability conditions and we defined something called measurable functions. So, what was measurable functions so if you take a function from Ω_1 to $\Omega_2$then first fix appropriate σ-fields on the domain Ω_1 and the range Ω_2.

Then with those σ-fields at hand you first look at the sets coming from the range side σ-field. So, those sets which are coming from the range side σ-field you take those and look at their pre-images if their pre-images fall in the domain side σ-field you will call the function to be measurable with respect to those two specific σ-fields.

So, the idea was that if you change the σ-fields then you might get different measurable functions. So, in particular we chose to look at the range when it was the real line and we defined the class of functions called Borel measurable functions. So, on the real line we are working with the Borel σ-field and we had mentioned that if the σ-fields are understood or at the beginning of certain discussions if you fix those σ-fields and do not change them anywhere then you can drop those for the simplicity of notations and simply say that the function is measurable.

You do not have to explicitly mention the σ-fields as long as it is clear from the context. So, that was what we had discussed last time and at the end we had also looked at a couple of examples of these measurable functions. In particular, we talked about this large class $\mathcal{F}$ of examples that appear from the continuous functions on $\mathbb{R}$, real valued continuous functions on $\mathbb{R}$ are Borel measurable functions. So, that gives you a large class of examples to use. So, with that at hand we are now going to start the discussion in this lecture. So, as usual let us move on to the slides.

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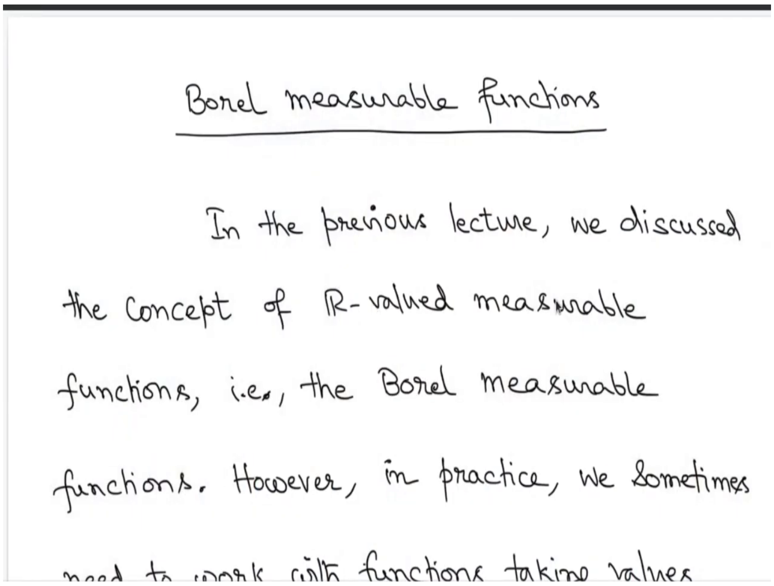

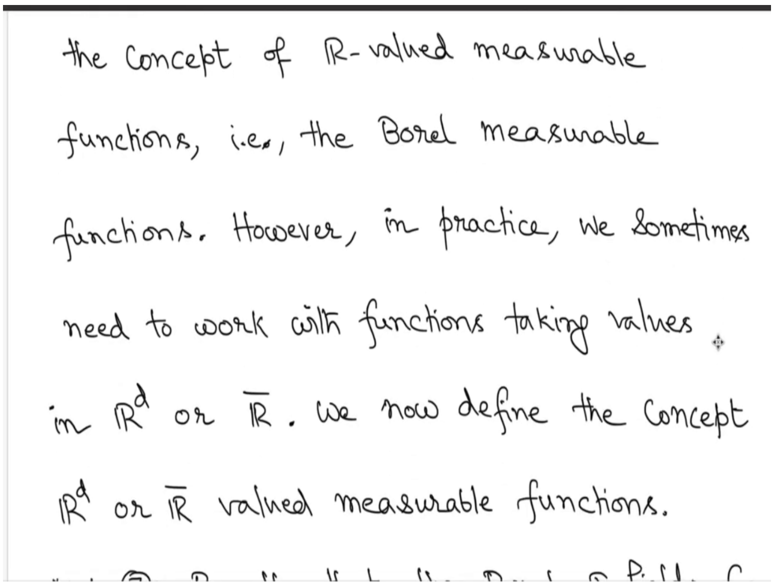

So, again in the previous lecture we discussed the concept of real valued measurable functions, the Borel measurable functions. But, in practice you might need to work with functions that takes values in the vector space $\mathbb{R}^d$ so which are vector-valued functions. Or you might have to allow functions which take $\pm\ \infty$ as their values.

So, you will have to work with the extended real number system. So therefore, if you just work with the real numbers or look at functions taking values in the set of real numbers then this will not be enough for the applications that you might find in practice. So, you have to devise or you have to understand or discuss these functions which could take values in some higher dimensional Euclidean space or the extended real line.

So, we are now going to define the concept of $\mathbb{R}^d$ valued or $\bar{\mathbb{R}}$ the extended real valued measurable functions. So, this will again follow the similar notions all you have to do is to put the appropriate σ-fields on the range side which are these $\mathbb{R}^d$ or the extended real line $\bar{\mathbb{R}}$.

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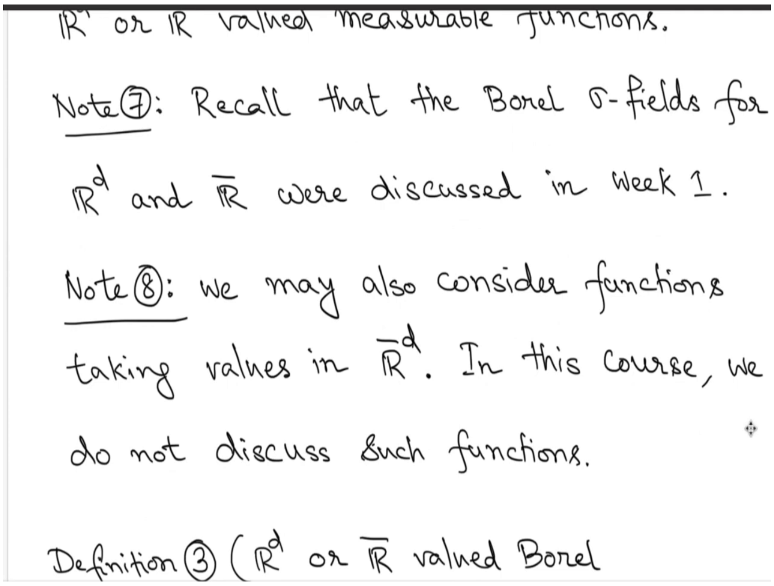

So, recall that the Borel σ-fields for $\mathbb{R}^d$ and $\bar{\mathbb{R}}$ were discussed earlier in week 1 so please check that discussion we had already defined it. Now, we can also consider taking functions which are taking values in $\bar{\mathbb{R}}^d$ so d for product of the extended real line but this in this course we are not going to discuss such functions. We may consider this but these are not going to be used in this course.

So, therefore we are going to ignore this specific type of functions. The understanding obtained from functions checking values in $\mathbb{R}^d$ or the extended real line $\bar{\mathbb{R}}$ is good enough for us.

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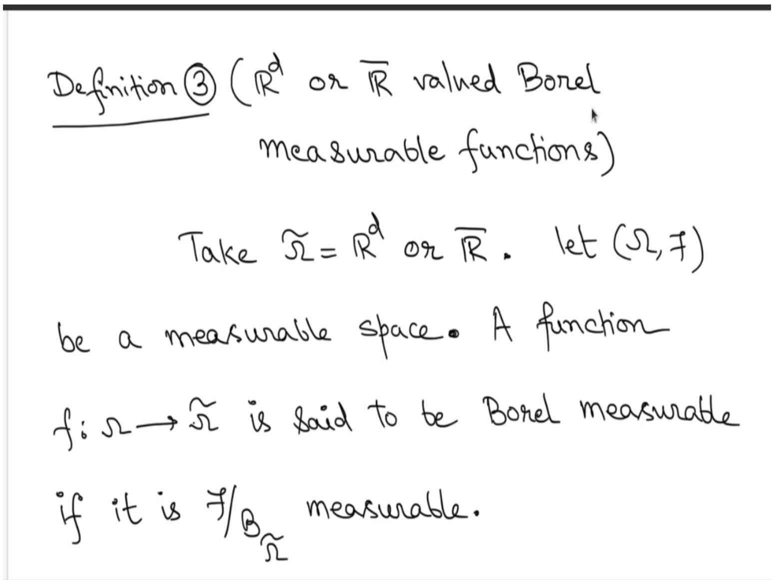

So, with that clarification at hand so, let us look at this definition. So, we are going to look at this $\mathbb{R}^d$ valued or $\bar{\mathbb{R}}$ valued measurable functions and we are again going to call them as Borel measurable functions. So, what are this so take $\tilde{\Omega}$ to be either $\mathbb{R}^d$ or $\bar{\mathbb{R}}$ the extended real line and let $(\Omega, \mathcal{F})$ be the measurable space so that is on your domain side $(\Omega, \mathcal{F})$ is given to you so that is on the domain side.

On the range side you are going to put $\tilde{\Omega}$ so then you are looking at functions with domain Ω and range $\tilde{\Omega}$. So, $\tilde{\Omega}$ has two possible choices one is $\mathbb{R}^d$ or $\bar{\mathbb{R}}$. Now, you are going to call such an arbitrary function from Ω to $\tilde{\Omega}$ to be Borel measurable if it is measurable with respect to these σ-fields on the domain side you are given the σ-field $\mathcal{F}$ and on the range side you have the Borel σ-field.

So, if you are having $\tilde{\Omega}$ as $\mathbb{R}^d$ you are going to put the Borel σ-field of $\mathbb{R}^d$ on the range side. If you are having $\tilde{\Omega}$ as the extended real line $\bar{\mathbb{R}}$ you are going to put the Borel σ-field on the extended real line on the range side. So, with these two appropriate σ-fields at hand $\mathcal{F}$ and the appropriate Borel σ-field if you get the measurability of the function then you are going to call it a Borel measurable function.

So, it will be clear from the context what type of functions you are taking and in what kind of spaces the function is taking values in, it could be real line, it could be $\mathbb{R}^d$ or it could be $\bar{\mathbb{R}}$. So, a clarification is needed with this point.

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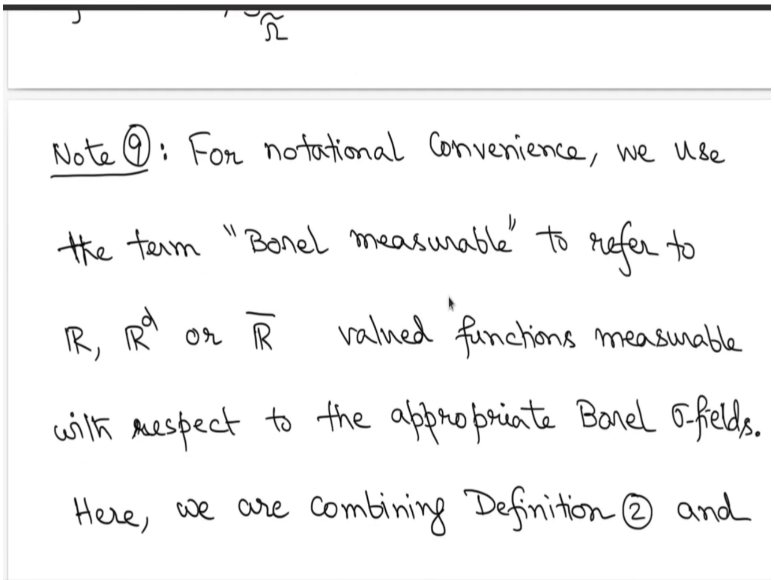

So, for notational convenience, we are going to use this term Borel measurable to refer to functions which may take values in $\mathbb{R}$ or $\mathbb{R}^d$ or $\bar{\mathbb{R}}$. But it will be clear from the context when you fix your function or start the discussion you are going to talk about some specific type of functions and then you will immediately specify the domain and the range.

So, to talk about a function you have to first specify the domain and the range so once you do that it will be clear what space is on the range side. So, it could be real line, it could be $\mathbb{R}^d$, it could be the extended real line $\bar{\mathbb{R}}$. So, in all of these cases if the measurability conditions are satisfied, if you get the function is measurable with respect to the appropriate σ- fields.

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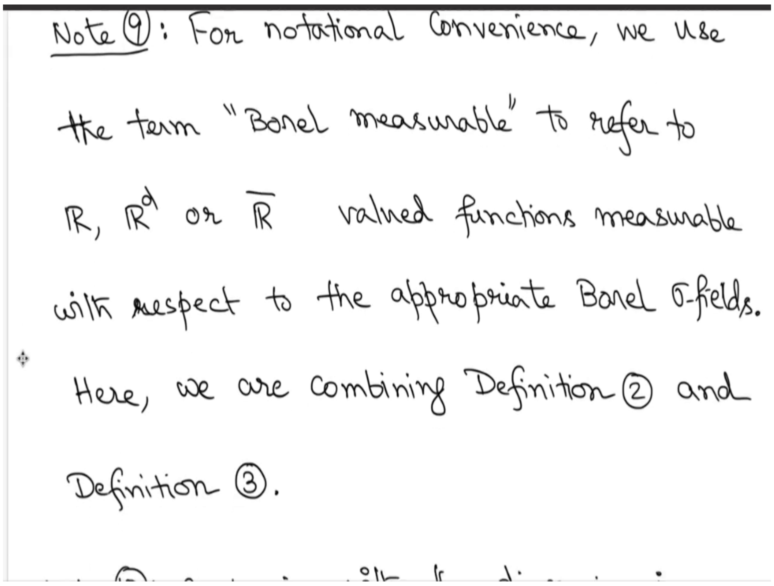

Then you are going to call it a Borel measurable function this is simply for notational convenience. Because, all of these are in some sense related to the real line so for all such functions we are going to use the term Borel measurable, we are not going to restrict it to only the functions which will be taking values in the real line. But, in practice I might be working with only the functions which are taking values in the real line or $\mathbb{R}^d$ or $\bar{\mathbb{R}}$.

So, here what we are doing is that we are this combining this definition 2 that was made in the previous lecture for Borel measurable functions, real valued Borel measurable functions and the definition 3 which we made now. So, we have just combined it and we just call all such functions to be Borel measurable and as explained that we are going to look at all such functions which are taking values in either the real line or $\mathbb{R}^d$ or $\bar{\mathbb{R}}$ but again it will be clear from the context what is your range set.

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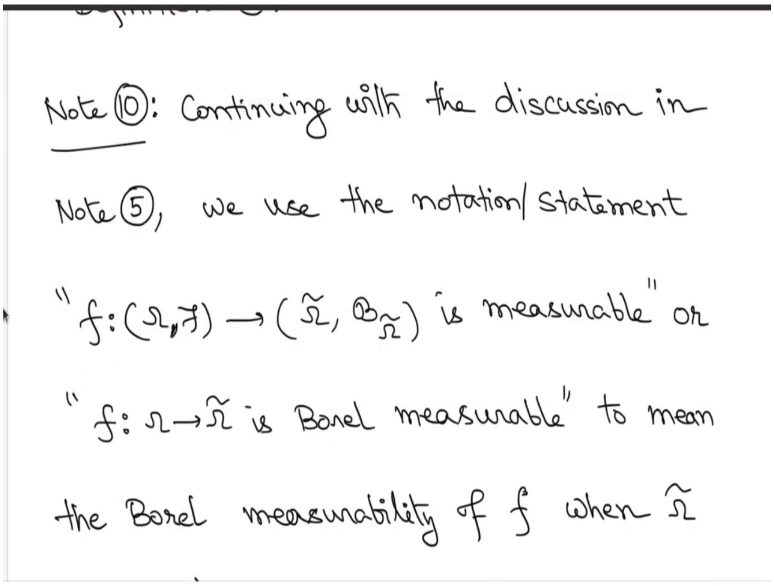

So, now if you continue with the discussions earlier in your note 5 so here we had introduced some certain notations. So, that notations were used for real valued Borel measurable functions. So now, that we have decided to call functions also taking values in $\mathbb{R}^d$ or $\bar{\mathbb{R}}$ as Borel measurable functions then what we are going to do is to use this notation again.

So, we are going to specify the appropriate σ-fields on the domain side and we are going to specify the range together with the appropriate σ-field. So, the function f a prior is just a function from Ω to $\tilde{\Omega}$ but we are putting the σ-fields in to specify its measurable structure.

Meaning to check the measurability of function f you are have to use this σ-field $\mathcal{F}$ and the Borel σ-field on the range side. So, this is the notation or the statement that will be used.

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is $\mathbb{R}$, $\mathbb{R}^d$ or $\overline{\mathbb{R}}$. Since we are interested mostly in Borel measurable functions, we may simply say measurable functions, instead of saying Borel measurable functions. However, in any such discussio the σ-fields should be clearly stated.

Similarly, a shorthand notation we can use just to say that f from Ω to $\tilde{\Omega}$ is modal measurable as long as the σ-fields in question are clear. This is again for simplification of the notation. But then another further simplification once you are only restricting your attention to Borel measurable functions just say that these are measurable functions.

So, if you are working with real-valued functions for example then you can simply say they are measurable functions and it will be clear from the context that you are using the Borel σ -field on the real line and you do not have to specify Borel measurable the term Borel measurable and simply say that it is measurable. So, this is again another simplification but we will try to use the term Borel measurable instead of saying just measurable. But it can be used.

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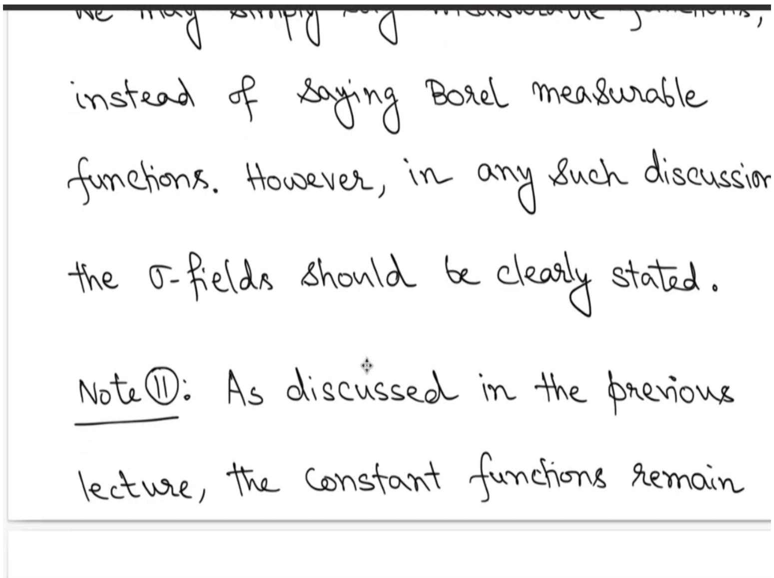

But again, in all such discussions σ-fields in question should be clearly stated so this is very very important. So, with this terminology at hand, you can now ask what are the examples of Borel measurable functions when you are looking at the $\mathbb{R}^d$ or the $\bar{\mathbb{R}}$ as the range space. And it so happens that the constant functions as we have discussed in the previous lecture will again turn out to be Borel measurable functions.

So, why? Because the pre images for the constant function is either in the empty set or the whole set and that is always in the domain side σ-field no matter what σ-field you choose on the domain side. So, with that at hand you always get constant functions to be measurable.

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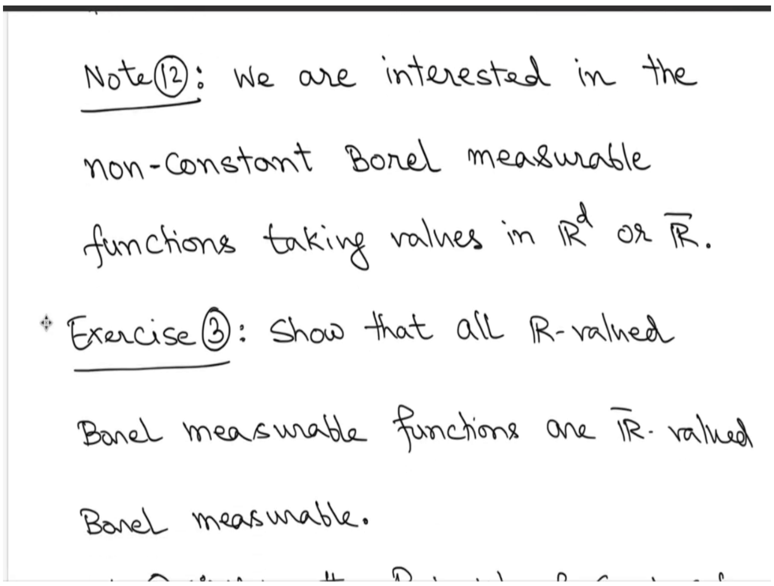

But then we explained or discussed this issue in the last lecture that beyond the constant functions we are interested in the non-constant measurable functions. So now, we are looking at non-constant Borel measurable functions which might now take values in $\mathbb{R}^d$ or $\bar{\mathbb{R}}$. So, with that we want to figure out examples of this.

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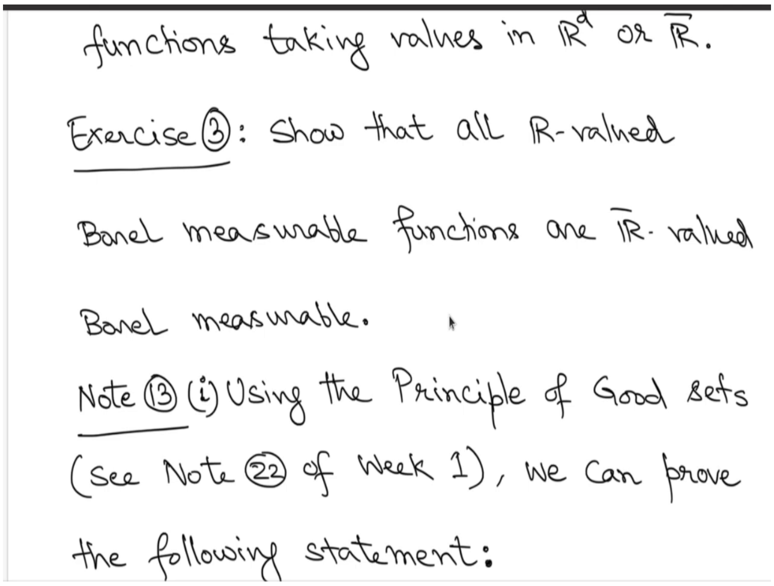

But now, there are a large class of examples suggested by the exercise 3. Show that that all real-valued Borel measurable functions are extended real-valued Borel measurable functions. So, what is the meaning of this so given a function on some domain Ω with appropriate σ -fields and with the range real line with the Borel σ-field on real line if it is given to be measurable function then you can now simply think of that function taking values in the extended real line.

Because, the real line is contained in that extended real line so you can as well think of all those functions as taking values in the extended real line. The function has domain Ω and the function has range the extended real line so I put the given σ-field on the domain side but on the range side with the extended real line I put the Borel σ-field there Borel σ-field of the extended real line.

Now, you can ask okay is this measurable with respect to the Borel σ-field of the extended real line? And the answer is yes as suggested by exercise 3 so please work this out.

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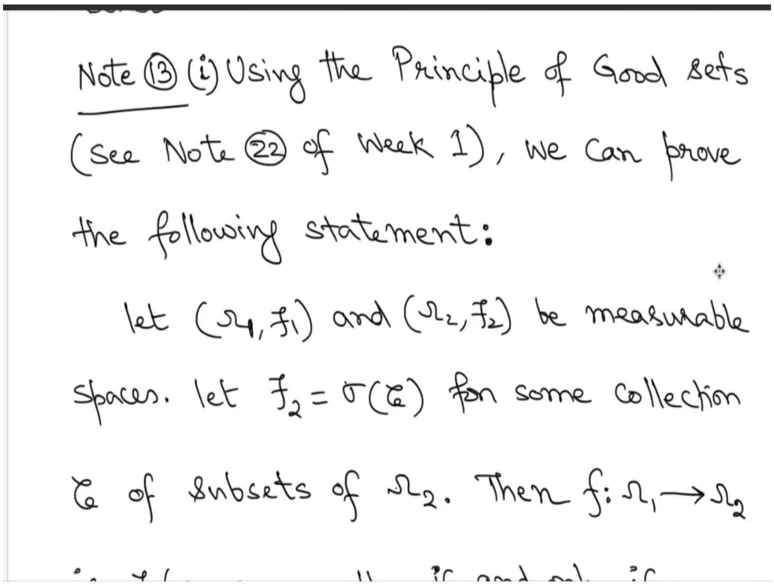

So now, this is a quite as interesting comment about the principle of good sets. So, remember we had discussed this principle of good sets in a brief manner in note 22 of week 1. We said that this general principle that allows us to prove certain types of results. But we said that due to certain complexity of the arguments, we are not going into the actual details and we are going to use it as a fact.

Or whatever facts we can derive from the principle of good sets we will take them as granted. So, we had that is what we had mentioned in note 22 and using such a principle what you can show is that you can prove the following statement. So, take two measurable spaces $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and choose or obtain a class of sets for the range side σ-field.

So, here Ω_1 is the domain, Ω_2 is the range so $\mathcal{F}_2$ is the σ-field on the range side so figure out a generating collection of sets for $\mathcal{F}_2$ call it $\mathcal{C}$. So, then the statement says that the pre images for sets coming from the smaller collection the generating collection take only those generating collections. So, those sets if you look at them take those sets A and look at their pre-images then this pre-images if they lie in the domain side σ-field then it will be measurable.

So, remember, for the measurability as per the definition you are supposed to look at all possible sets coming from $\mathcal{F}_2$ but we are saying that instead of looking at all possible sets coming from the $\mathcal{F}_2$.

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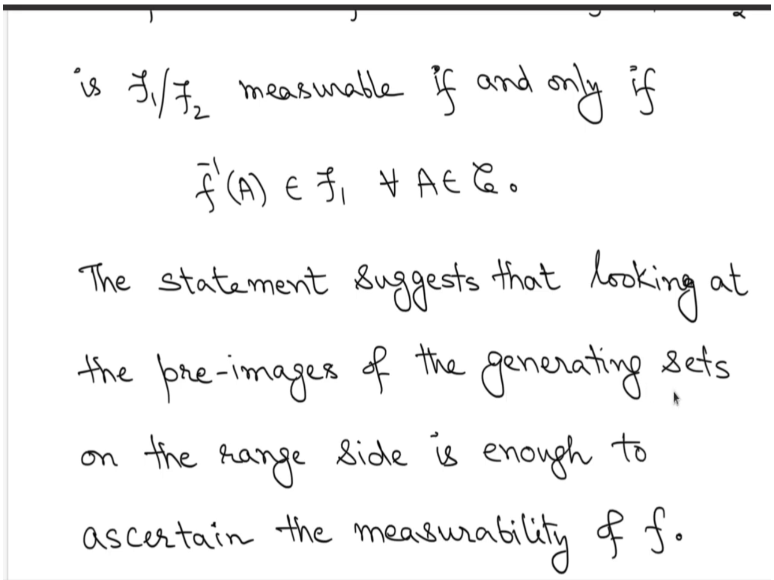

We just choose to look at only the generating sets on the range side. Take only those sets A and look at their pre-images if the pre-images fall in the domain side σ-field then your function will be measurable.

So, this actually is a if and only if condition so that it can be proved by the principle of good sets. So, to check measurability it is enough if you just check the measurability of f these conditions for measurability only for the generating class of sets on the range side.

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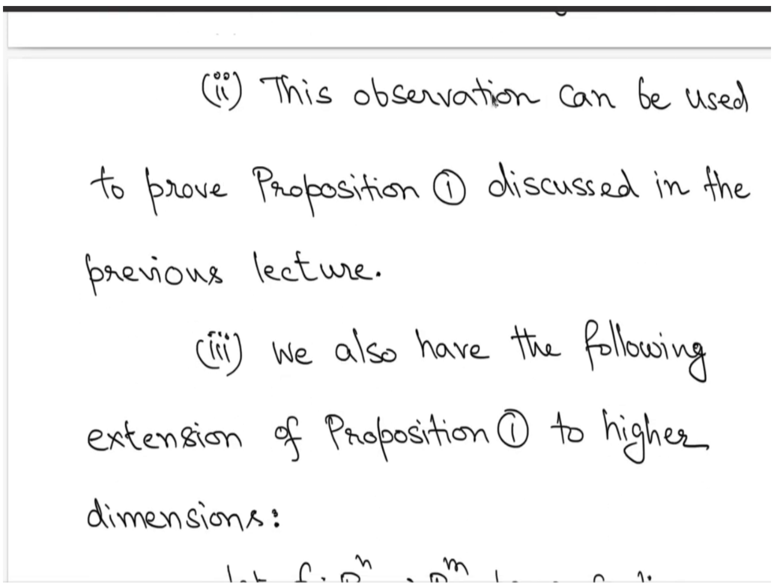

And this observation can be used to prove the proposition one that was discussed in the previous lecture. So, what was in proposition 1, we said that all functions from ℝ to ℝ which are continuous are Borel measurable. So, there we had mentioned one exercise that pre images of open sets are open so if you put it together with this fact that we have just mentioned now you can actually show, you can actually prove the proposition 1. And say that yes, all continuous functions from ℝ to ℝ are Borel measurable. We can also generalize proposition 1 to the following fact.

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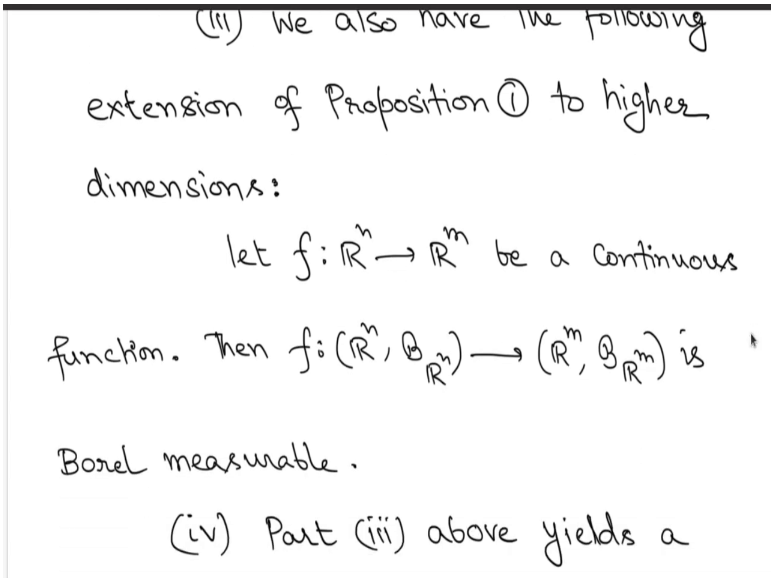

So, this is a statement in higher dimensions so now we are saying take continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ so n and m are two appropriate integers non-negative integers. So, you have a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$ then what you are going to do you are going to put the appropriate Borel σ-fields on the domain side and the range side and claim that these functions are Borel measurable.

So, again you get a large class of examples of $\mathbb{R}^d$ or $\mathbb{R}^n$ valued Borel measurable functions with domain sum $\mathbb{R}^n$. So, this gives you a large class of examples. So, again this fact follows using the principle of good sets and in particular this is using the fact that you only need to check the condition for measurability for the generating sets on the range side.

If you can show that for generating sets on the range side the pre-images are on the domain side σ-field then it is good enough to check measurability. And this part 3 which is talking about this type of continuous functions on $\mathbb{R}^n$ to $\mathbb{R}^m$.

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let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then $f : (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}) \to (\mathbb{R}^m, \mathcal{B}_{\mathbb{R}^m})$ is Borel measurable.

(iv) Part (iii) above yields a large class of examples of Borel measurable functions. In particular, consider the co-ordinate projection

So, if you look at such functions these are going to give a large class of examples.

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(iv) Part (iii) above yields a large class of examples of Borel measurable functions. In particular, consider the co-ordinate projection maps $\pi_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, 2, \ldots, d$ defined by $\pi_i((x_1, x_2, \ldots, x_d)^t) = x_i$ for all $(x_1, \ldots, x_d)^t \in \mathbb{R}^d$.

But, then in particular we are going to look at one very specific continuous function. So, fix some appropriate dimension d look at this $\mathbb{R}^d$ to $\mathbb{R}$ function which I am going to write it as π_i. So, these are the coordinate projection maps so what do they do, so take a vector $(x_1, \ldots, x_d)$ so that is in your domain and map it to the ith component. So, ignore all the other components, map it to the i-th component so that is what π_i does. It just projects to the ith coordinate then this function is a continuous function so this is a continuous function.

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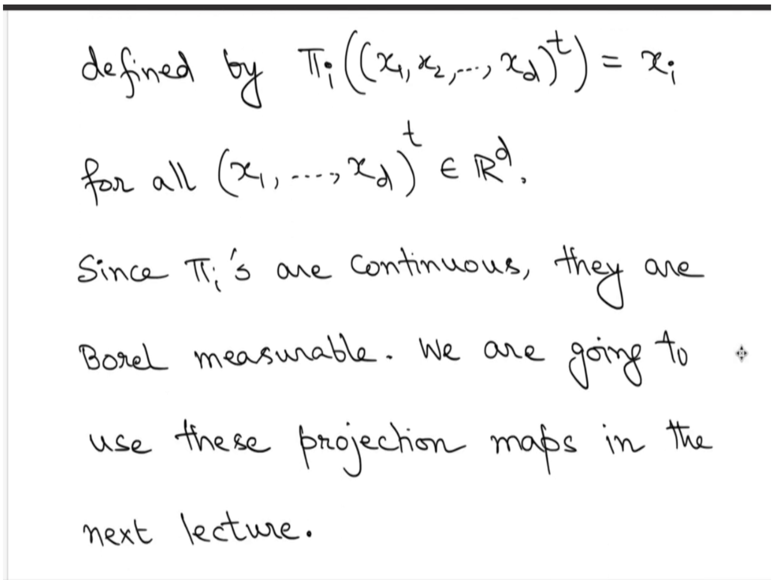

And as discussed above these are Borel measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$ with the domain side σ-field as the Borel σ-field of $\mathbb{R}^d$ and the range side σ-field as the Borel σ-field of $\mathbb{R}$. We are going to use these projection maps in the next lecture so there we are going to talk about the properties of measurable functions and we are going to need it in the next lecture. So, we stop here.