Measure Theoretic Probability 1
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Lecture 11
Measurable Functions

Welcome to this lecture, this is the first lecture of week 3. Before we proceed let us first quickly recall what we have done in week 1 and 2. In week 1 we looked at the collection of events corresponding to a random experiment or more generally certain collection of special subsets of a general non-empty set. And we configured or we looked at certain structures like fields, $\sigma -$ fields and modern classes. We also studied the properties of these classes or collections of sets.

In week 2 we entered this area of measures, we introduced this topic and the idea was that we wanted to understand which events were more likely to occur or which sets are larger in size. To understand this, we assigned these numerical values to these sets we obtained set functions but since our motivation is to look at these sizes of sets or the lightness or appearance of certain events we looked at the special structures which we called as measures.

We studied this algebraic property, continuity properties of measures and in particular we restricted our attention to probability measures because this is what we are interested in. So, these are what we have covered under the materials in weeks 1 and 2. So, as before let us switch over to the lecture slides and start the discussion for the materials in week 3.

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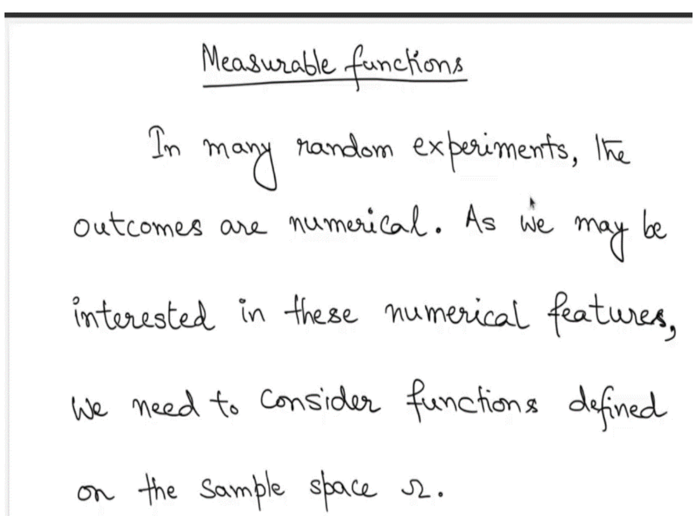

So, this is the motivation to look at certain types of functions on sample spaces. So, in many random experiments, the outcomes are numerical and as we are interested in these numerical outcomes numerical values, and numerical features we need to consider these functions defined on the sample space Ω. So, we are looking at the functions for each point or each sample point in the sample space. So, we are looking at these values assigned to each sample point and we are looking at these collections of functions.

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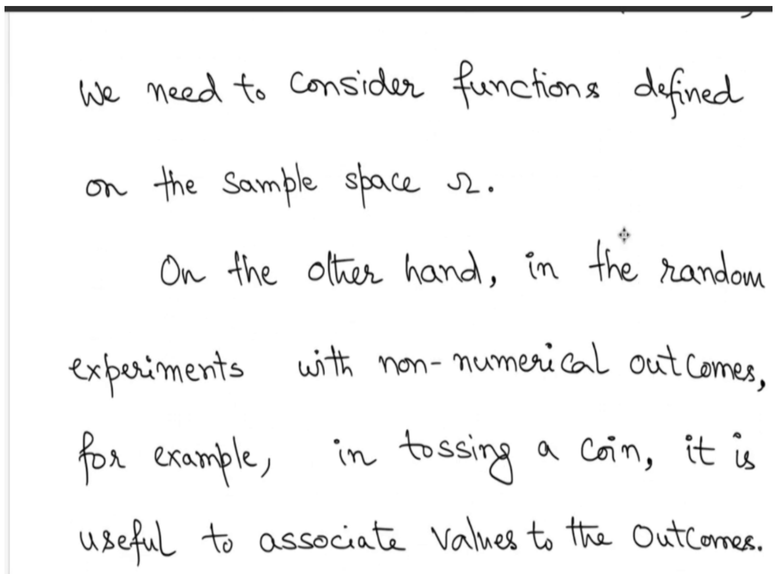

On the other hand, there are certain random experiments where the outcomes are non-numerical. For example, in tossing a coin the outcomes are either a head or a tail. So, now you will ask what is the value but in practice, you associate certain values to these heads or tails you assign certain meanings. For example, when you toss a coin you might be calling heads and if the head appears you win the toss. So, there is this notion of winning a toss and losing a toss so, therefore there is a physical meaning associated with these outcomes which are heads and tails.

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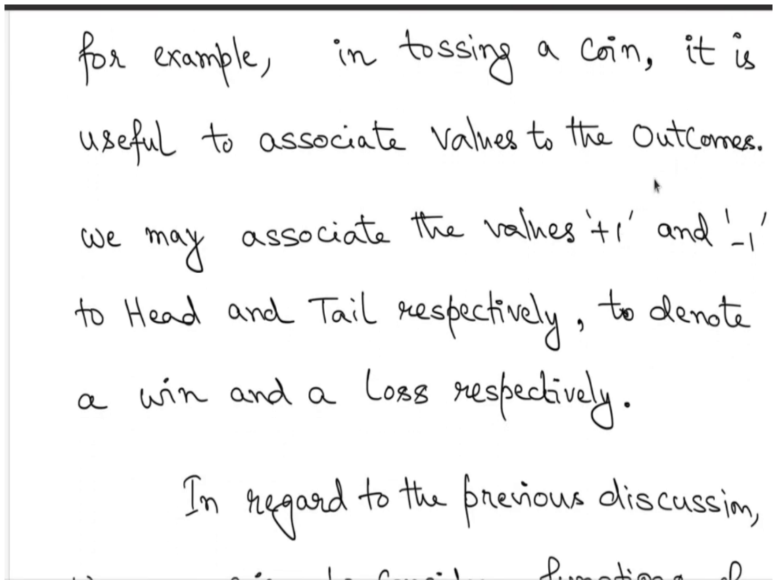

So, just to understand this better it is sometimes useful to associate some values to these non-numerical outcomes like head or tails. So, what you can do you can associate the values $+ 1$ and $- 1$ to head and the tail and you can think of this $+ 1$ and $- 1$ as winning ones or losing ones. Or wining 1 Rupee and losing 1 Rupee. So, you can interpret it in this fashion and therefore now the interpretation is much more quantitative it is terms of certain values. So, this is what we are going to restrict our attention to.

So, no matter what kind of outcomes we have we will typically associate certain meaning to them which will be numerical and we are going to study those values and get some more intricate features or intricate analysis of the corresponding random experiment.

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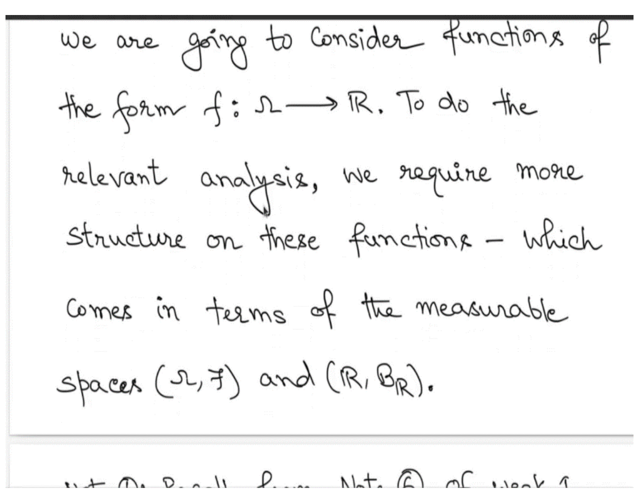

So, in regards to this discussion that we are having what we are going to consider are the functions of the form f with domain sample space Ω and the range space real line $\mathbb{R}$. So, we are looking at real-valued functions on the sample space.

And to do this analysis that we really want to do we require more structure on this function so you cannot just start looking at this arbitrary type of functions which are real-valued and defined on the sample space. You need certain nice regularity structures on these functions. So, this is what we are going to discuss in this lecture.

So, this is in terms of certain measurability structures associated with the domain space and the range space. What we are going to do so since Ω is the sample space of a random experiment it comes with a collection of events which we have already associated with the structure of $\sigma -$field.

So, on the domain side you have the structure of the measurable space so $(\Omega, \mathcal{F})$ so this you can think of as the sample space together with the collection of events. On the other side the range side you have the real values and on the real line we have discussed extensively about this $\sigma -$field which is the Borel $\sigma -$field.

So, what you are going to focus on are certain structures with this informations that you have already collected like these $\sigma -$fields $\mathcal{F}$

on the domain side and the Borel $\sigma -$field on the range side. So, with this in hand we are going to look at certain specific structures of these functions f. We are not going to look at arbitrary functions f.

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Note 1: Recall from Note 6 of week I that for a function $f:\Omega_1 \to \Omega_2$ and $A \subseteq \Omega_2$, the pre-image of A under f is denoted by $f^{-1}(A)$ and is defined by

$$f^{-1}(A) := \{\omega_1 \in \Omega_1 \mid f(\omega_1) \in A\}.$$

Here, f^{-1} does not represent the inverse of f — in fact the inverse function need

So, recall from note 6 of week 1 that for a function which we can consider now a general function mapping from domain space Ω_1 to a range space Ω_2. So, Ω_1 and Ω_2 are some non-empty sets and you look at our arbitrary function define on top of that. Now, if you fix a subset $A \subset \Omega_2$ the range side then you can define the preimage of this set under the function f. This we have defined in note 6 of week 1.

So, what was this let us quickly recall that so we looked at this $f^{-1}(A)$ defined by all points in the domain so the now the domain is Ω_1 so you are choosing all points, all points ω_1 coming from the domain side such that the value associated to those two main points the $f(\omega_1)$ the associated value lands up in the set A. So, $f^{-1}(A)$ that is nothing but the preimage of A which is defined as all points in the domain side such that the values on the function of those points lands up in the set A.

So, remember, f^{-1} here does not represent the inverse of f because a priori when you are taking a function it is not really invertible. So, you do not know whether the function is one-to-one onto you do not know anything like that. So, here you should treat f inverse of A as one symbol instead of treating it as two separate symbols like f^{-1} and A. So, you should think of it as one big symbol signifying this preimage for A.

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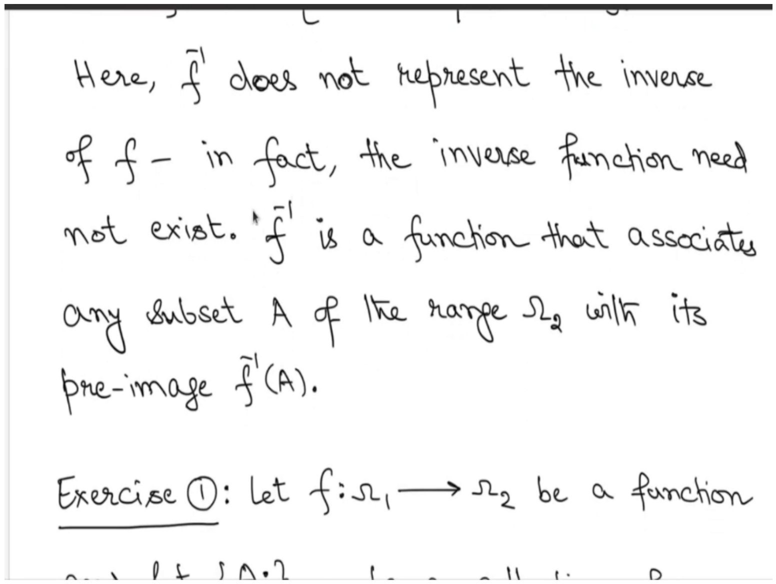

So, now here f inverse you can now think of it as a function that associates any subset on the range side to its preimage. Or this is now connecting or associating sets from the range side to the domain side. So, this you can think of as a function in that sense. But this is not the inverse function of the function f. Now, if the function f has an inverse then of course you can connect the notion of this inverse with the actual inverse of a function. That you can try to work it out.

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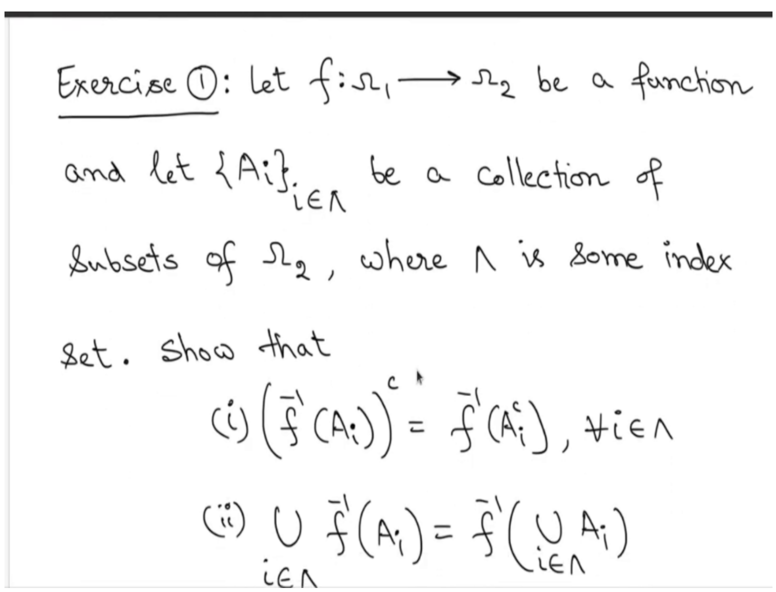

But now what we are interested in are the properties of these preimages. So, the properties of preimages come in this form. So, take this general function which starts in the domain Ω_1 lands up in the range Ω_2 and choose a collection of subsets from the range side. So, A_i, $i \in \Lambda$ so when Λ is some index set so here it is some arbitrary set it could be finite, countable, uncountable, anything. So, it is just some arbitrary index set.

So, we are looking at a collection of subsets of Ω_2 we are not associating any structure of any type. So, function is general, a collection of sets is general. Now, what we are discussing now are properties of the preimages of the sets Ai under the function f. So, what are the properties? So, the first property says no matter which set you take if you look at the compliment of the preimage so, f inverse of Ai that is the set on the domain side if you look at the compliment that can be again identified as a preimage of another set.

And that is turns out to be the preimage of A_i^c. So, this is a nice fact you can try to prove this. Similarly, there are these nice properties that connects the unions and intersections over this indexing sets. So, here we are looking at that same index set lambda this is some arbitrary index set and it simply says that whether you take unions or intersections of the preimages these unions or intersections cam simply go inside.

So, by that timing if you are looking at the unions of all preimages it is nothing but the preimage of the $\bigcup_{i \in \Lambda} A_i$. And intersections you have a similar result. So, try to prove this so these are some very nice useful observations about preimage of a set under the function f.

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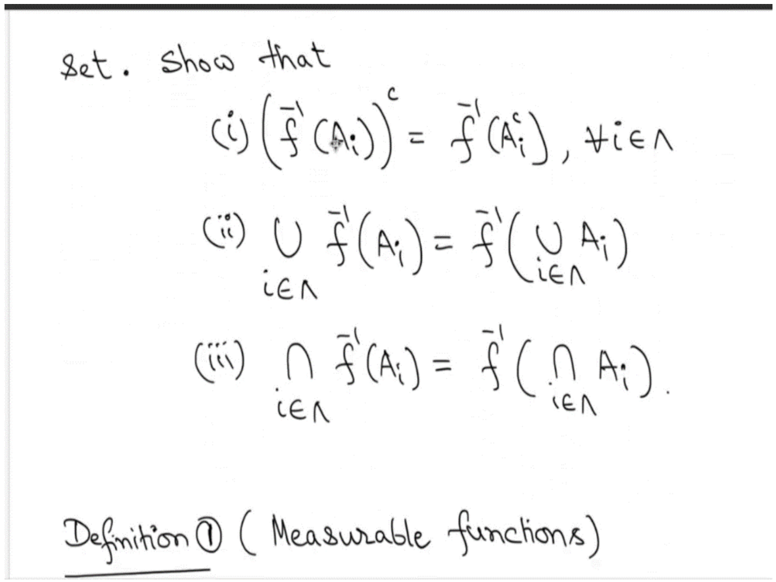

Now, with this in hand we are now focused at the preimages of sets coming from the range side and we are going to look at certain properties which restrict the collections of functions on the sample spaces.

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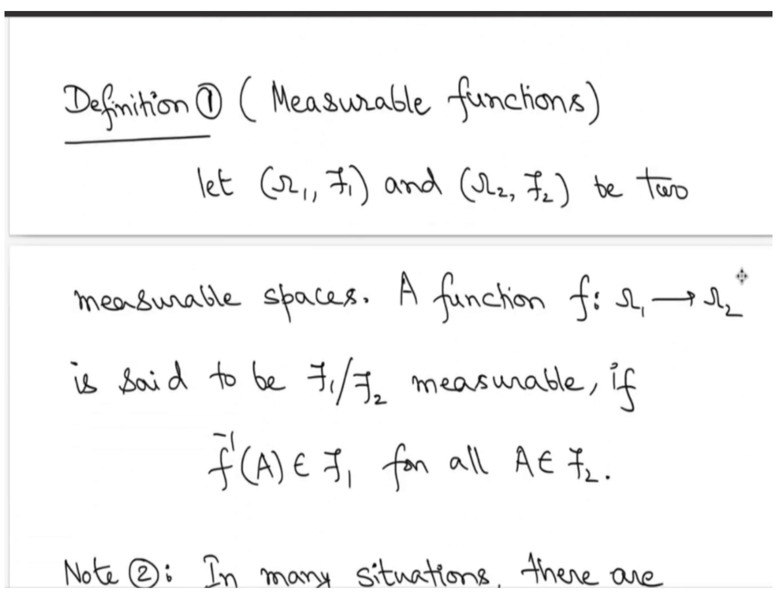

And with this we now arrive at the first definition so, we are going to talk about measurable functions. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces so, I take 2 measurable spaces on domains Ω_1 and Ω_2 and I already need to have σ −fields on top of them. So, I now have $\mathcal{F}_1$

and $\mathcal{F}_2$ the appropriate σ −fields on these domains. Now, I look at a general function f from Ω_1 to Ω_2 so with domain Ω_1 range Ω_2.

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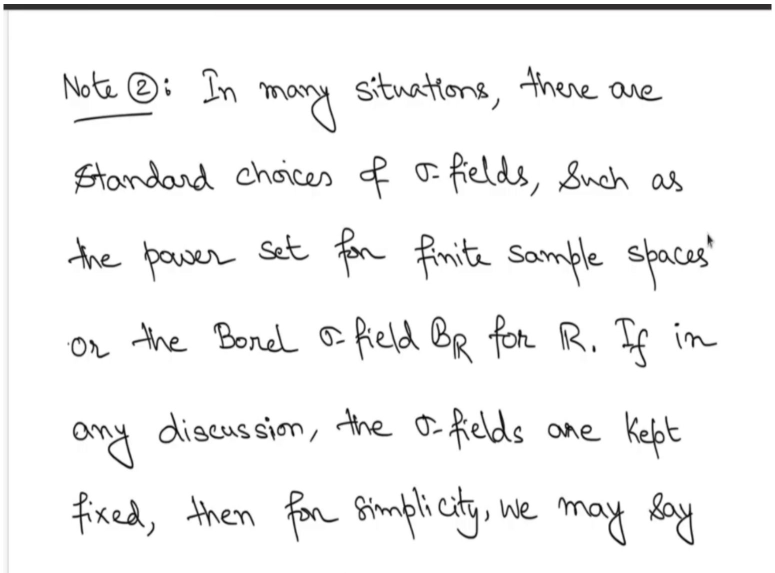

So, we say that this function $\mathcal{F}_1$ over $\mathcal{F}_2$ measurable if for all preimages we land up in the domain side. So, you take all subsets that are on the range side σ −field which is the $\mathcal{F}_2$ so you take all such subsets which are in the range side σ −field look at the preimages those preimages I want them to be on the domain side σ −field.

So, this is the restriction we impose here. And if its so happen for a function f defined on a domain Ω_1 to the range Ω_2 with this associated σ −fields $\mathcal{F}_1$ $\mathcal{F}_2$ if it so happens for such a function then I am going to call it $\mathcal{F}_1$ over $\mathcal{F}_2$ measurable. So, this is the definition, this is the restriction that we impose on a general function. So, if it happens then we introduce this concept of measurability.

Now, in many situations there are standard choices of σ −fields. So, for example you can start with a finite sample space and then you see that the power set is a natural example of a σ −field there. On the real line we have extensively discussed this Borel σ −field these are certain natural choices. So, once you have this natural or standard choice you typically like to work on only these type of σ −fields. So, you will put this natural or standard choice on the domain side and the range side as appropriate.

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or the Borel σ-field $\mathcal{B}_{\mathbb{R}}$ for $\mathbb{R}$. If in any discussion, the σ-fields are kept fixed, then for simplicity, we may say "f is measurable" instead of "f is $\mathcal{F}_1 | \mathcal{F}_2$ measurable".

Note (3): We focus our attention to the case $\Omega_2 = \mathbb{R}$, as motivated above. In the

So, if in any discussion the σ −fields are kept fixed then for simplicity we may simply say that f is measurable instead of saying f is $\mathcal{F}_1 | \mathcal{F}_2$ measurable. So, two points to note, first thing is that there may be standard choices for σ −fields so we will use that but once you have these standard choices the σ −fields are now understood from the context once you have these standard choices the σ −fields are understood from the context.

Then for simplicity of the notation you will simply say that f is measurable instead of explicitly mentioning these σ −fields $\mathcal{F}_1 | \mathcal{F}_2$. So, for example on any of these sides, on the domain side or the range side if the real set of real number appears and you are going to use the Borel σ − field then you can ignore that and do not mention it. Just say that f is measurable this will be understood from the context that you are using the Borel σ −field. So, this is the simplicity of the notation.

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measurable".

Note (3): We focus our attention to the case $\Omega_2 = \mathbb{R}$, as motivated above. In the next definition, this is explained in more concrete detail.

Definition (2) (Borel measurable functions)

let $(\Omega, \mathcal{F})$ be a measurable space

Now, we focus our attention to the specific case when the range side is the real line. So, as discussed above so, when you are looking at certain random experiments with numerical outcomes you exactly get this type of a function so each sample point gest associated to the real number otherwise if the random experiment appears with non numerical outcomes you will associate certain meaning or values to these numerical values and therefore obtain these types of functions. Such that defined on the domain Ω and range side is the real line.

So, as motivated like that we are going to focus our attention to the case of functions which are defined on the domain Ω the sample space, and the range inside is the real line. So, in the next definition this is explained in more concrete detail. So, we are looking at a specific type of measurable functions now.

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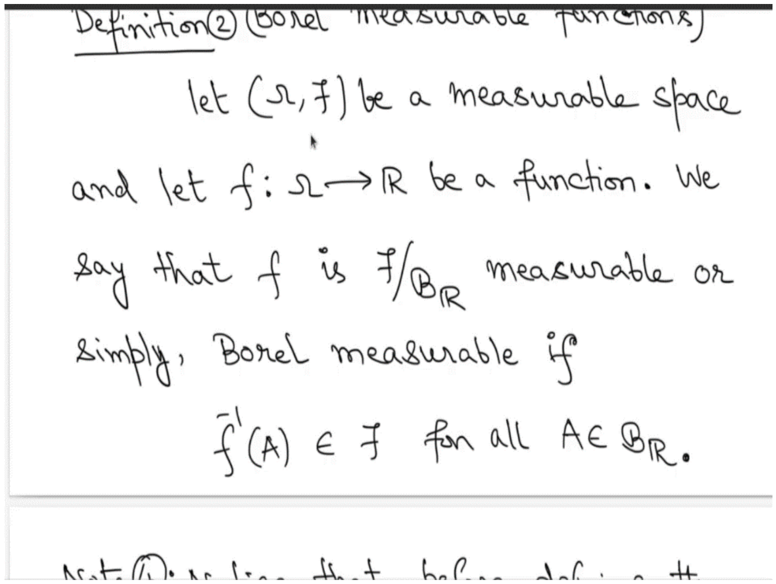

So, what is this? So, now the domain side is fixed that is $(\Omega, \mathcal{F})$ so that is some measurable space that is given to you it may be coming from a random experiment or maybe you can look at some general non-empty set together with some appropriate choice of a σ −field. So, that gives you a measurable space and that you are going to put on that domain side, on the range side as discussed we are going to look at the real line and we are considering the Borel σ −field on top of the real line. So, that will give you the appropriate measurable space on the range side.

Now, look at the measurable functions which comes out from these conventions. From these choices of measurable spaces. We say that the function f which is defined on the domain Ω with values in the real line to be $\mathcal{F}$ over the Borel σ −field measurable or simply Borel measurable if the same condition that you take now all Borel sets coming from the real line so, you are looking at all Borel subsets of the real line which is on the range side you are looking at all subsets looking at their preimages now the preimages are subsets on the domain side and you want those subsets to be in your collection $\mathcal{F}$.

If it so happens you are going to call this function f as $\mathcal{F}|\mathcal{B}_{\mathbb{R}}$ measurable or for simplicity of notation you will simply say its Borel measurable. You are going to talk about Borel measurable functions on domains Ω and that idea is that once the range is real line and you are talking about the Borel σ −field you do not have to mention it explicitly once more it is understood from the context.

So, you are looking at for all sets on the range side all Borel subsets on the range side you are looking at their preimages these preimages must fall in the domain side collection of the σ −field $\mathcal{F}$. So, if this happens you are going to call it a Borel measurable function.

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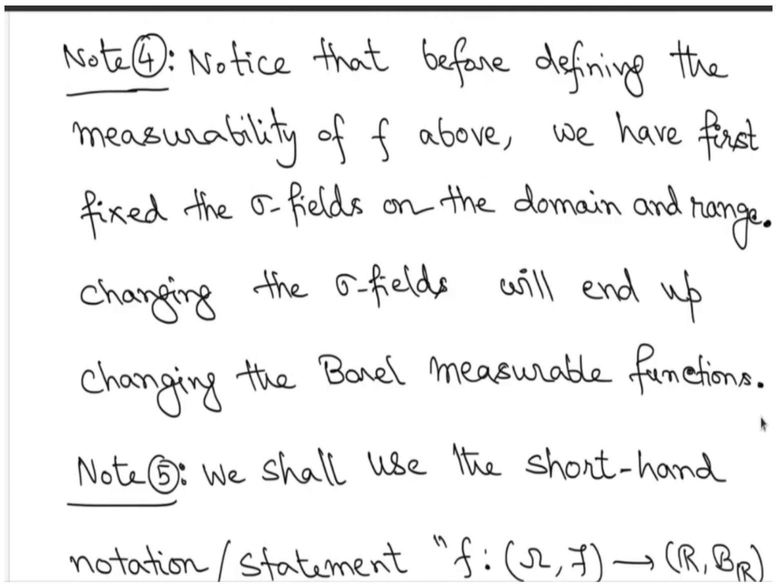

So, notice that before defining these measurability concepts we are fixing the σ −fields. So, this you have mentioned earlier that you have also talked about these standard choices of σ −fields and other things. Now, we have always fixed the σ −fields on the domain and the range first before defining the measurability. What is the reason for this? The reason is this if we change the σ −fields on either the range side or the domain side you will be getting different types of Borel measurable functions.

So, if you make a specific choice of σ −fields on the domain side and the range side you will get some type of the functions which will turn out to be measurable with respect to those σ −fields. But then if you change the σ −fields either on the domain side or the range side you might get other functions which will now turn out to be Boreal measurable all measurable and you might get certain functions which were measurable in the earlier sense but no more in the for the new σ −fields.

So, you have to be careful with these choices of σ −fields once you make these choices before your discussion you always make these choices, keep it fixed and then continue on the discussion. So, we are going to do this.

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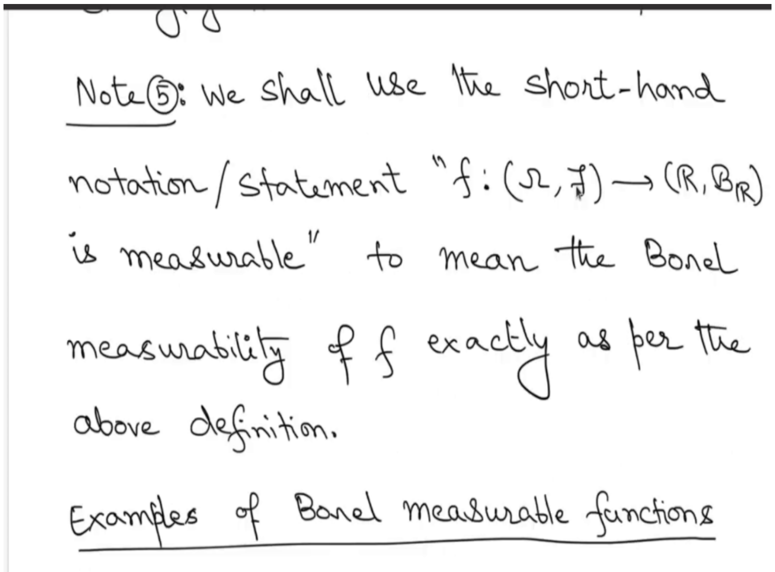

We are also going to use this short hand notation or the statement what does it read, it reads as follows $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. What does it mean? You are explicitly stating the domain and the range which is the domain is the Ω, range is R but you are also putting in the σ −fields, appropriate σ −fields on the domain side and the range side. So, just keep it clear the function, the given function is purely a function defined on the domain so for each point in the domain Ω you are associating a real number.

So, that is how the function is defined. But whenever you are talking about the measurability you have to mention the appropriate σ −fields so what you do whenever you are talking about this measurability you explicitly state that so that this notation is now explaining these choices. So, you simply write f, from Ω f it as if it coming from that appropriate measurable structure on the domain side and lands up on the measurable space on the range side. So, that is $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

So, that is how the measurability should be explained. So, if you write this then it will mean the Borel measurability of f is exactly as per the above definition. Meaning you explicitly specify the σ −fields. If the σ −fields are understood then you can remove it, you do not have to mention it but it is always good to keep track of the σ −fields on the domain side and the range side. And to do that we are going to use this term $f : (\Omega,\ \mathcal{F}) \rightarrow (\mathbb{R},\ \mathcal{B}_{\mathbb{R}})$. That notation, that function is measureable. So, we are going to write this statement.

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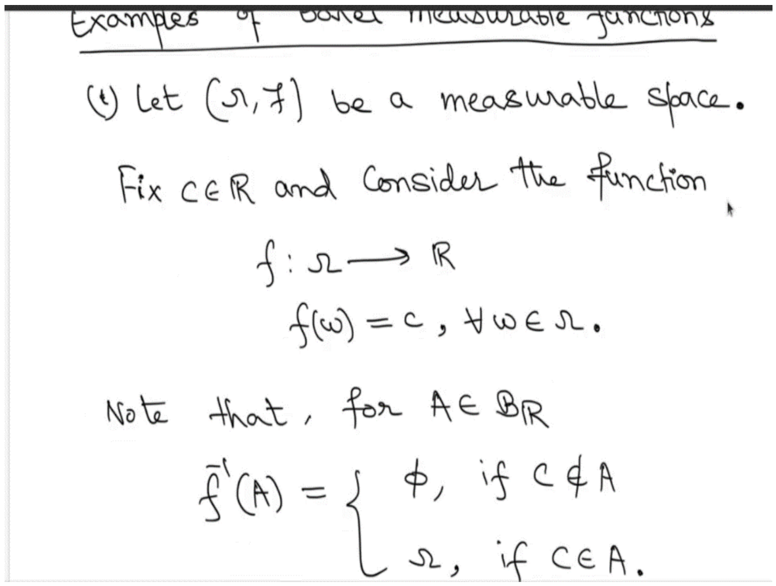

With this introduction of measurable functions, we are now ready to talk about examples of Borel measurable functions. So, lets start with the simplest example that you can think of. So, you take a measurable space $(\Omega,\ \mathcal{F})$ and fix a constant. So, the simplest function that you can think of is a constant function.

So, what you do so for all points in the domain Ω you associate the value that constant value small c. So, that gives you a constant function on the domain. Now, you are going to ask if this constant function is a measurable function or a Borel measurable function.

So, on the domain side you have a, you have already been given some arbitrary σ −field, arbitrary fixed $\mathcal{F}$ so that is a σ −field that is on the domain side, on the range side, on the range side you have the real line which is of course having the Borel σ −field and with these two specific choices of the σ −fields you are going to talk about the measurability of the function.

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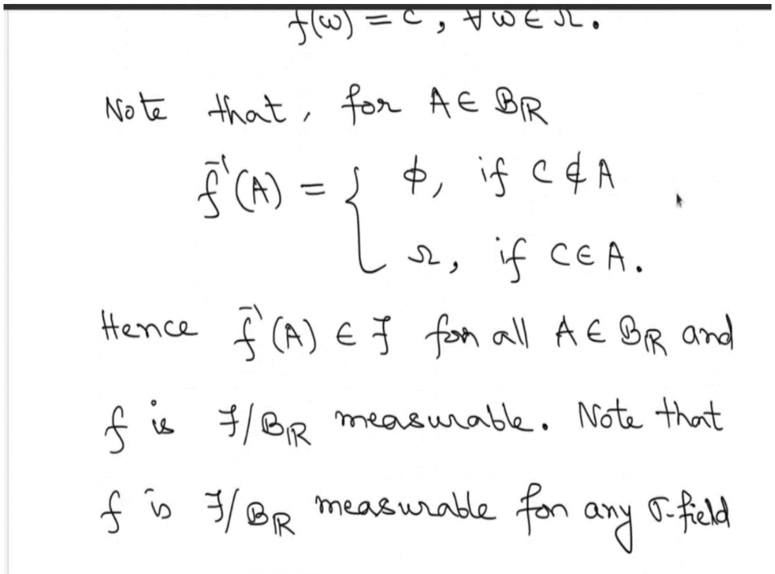

But now what you need to do you are going to choose subsets, Borel subsets from the range side, so here you are going to choose this set A and you are going to look at the preimages of those sets. So, it turns out that when the function f is just a constant function taking the constant value c then the $f^{-1}(A)$ that can be only two possible things, what is this, so it depends on the choice whether the constant value is in the set A or not.

If the constant value is not in the set A, then the preimage turns out to be an empty set. If the constant value is in A then the preimage turns out to be the whole set. Why is this? So, again go back to the definition what is $f^{-1}(A)$ you are going to look at all possible points in the domain such that the $f(\omega)$ that value lands up in the set A. But for this specific function the constant function $f(\omega)$ no matter what ω is, is just in the constant c so therefore if your constant values c is in the set A then you are going to get all points in the domain side as its preimage.

Because, all those points are landing up exactly in c so therefore $f(\omega) \in A$. So, therefore you get the whole set. But if the point c is not in the set A then the point is that $f(\omega)$ is never in A for all possible ω in the domain side. So therefore, you end up with the empty set. So, now you get these two possible cases of the preimages. So, no matter what A you are choosing you always end up with these two choices, either the empty set or the whole set.

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$$f^{-1}(A) = \begin{cases} \phi, & \text{if } c \notin A \\ \Omega, & \text{if } c \in A. \end{cases}$$

Hence $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}_{\mathbb{R}}$ and f is $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurable. Note that f is $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurable for any σ-field $\mathcal{F}$ on Ω. (Why? Exercise)

But then if you take any arbitrary σ −field it will always contain this empty set or the whole set that is by construction by definition and hence the preimages are always in the σ −field no matter which Borel subset you choose from the range side.

So, for all possible Boreal subsets if you look at the preimages they always fall in at the domain set σ −field. And therefore, it turns out to be Boreal measurable. So, this is therefore Borel measurable or $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurable for any choice of σ −field. So, this is happening because the empty set and the whole set is always in the σ −field. So, that is what I have already explained please try to write it down I have let this as an exercise. So, please check that no matter which σ −field you choose constant functions always remain Borel measurable.

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(ii) let $(\Omega, \mathcal{F})$ be a measurable space and let B be a Subset of Ω. Recall the indicator function 1_B defined by

$$1_B : \Omega \longrightarrow \mathbb{R}$$

$$1_B(\omega) = \begin{cases} 0, & \text{if } \omega \notin B \\ 1, & \text{if } \omega \in B. \end{cases}$$

Then, for $A \in \mathcal{B}_{\mathbb{R}}$

$$1^{-1}(A) = \begin{cases} \phi, & \text{if } 0, 1 \notin A \end{cases}$$

So, now you are interested in some more general functions. So, we have already introduced this indicator function of sets. And that is what we are going to look at next. So, take a measurable space and choose some subset on the domain side. So, B is a subset on the domain side. So, let us take some arbitrary subset it need not be in the σ −field $\mathcal{F}$.

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Then, for $A \in \mathcal{B}_{\mathbb{R}}$

$$1_B^{-1}(A) = \begin{cases} \phi, & \text{if } 0, 1 \notin A \\ B, & \text{if } 0 \notin A, 1 \in A \\ B^c, & \text{if } 0 \in A, 1 \notin A \\ \Omega, & \text{if } 0, 1 \in A. \end{cases}$$

(Exercise)

check that 1_B is $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurable if and only if $B \in \mathcal{F}$. (Exercise)

So now look at the indicator function, so how is defined? So, for all point in the domain all you need to check whether is the point is in the set B or not. If it is there you associate the value 1 otherwise you associate the value 0, it is simply a 0, 1 valued function. That is the function. Now, you want to check that given this choice of $\mathcal{F}$ σ $-$field on the domain side you are going to check whether this function, this indicator function is measurable or not.

So, again what you need to do you are going to choose these specific, these Boreal subsets from the range side so take this set capital A and you are going to look at the preimages. And you are going to check whether the preimages fall in your σ $-$field on the domain side $\mathcal{F}$. So now what are the possible cases so you can now check as done for the constant function case that you can split it into four possible cases.

So, for the constant function case you had two choices but now you have to look at these two values 0 and 1 and with these possible choices if 0 and 1 belongs to A or $1 \notin A$ and so on. So, you have these four possible choices you will get these different sets. If neither a 0 or 1 is in A you are going to get empty set as a preimage and so on. Please check this, this is left as an exercise for you.

But, then what do want you wanted this indicator B to be measurable. Now, remember these four possible options are all these things that will appear as the preimages. Now remember, that a σ $-$field $\mathcal{F}$ always contains the empty set and the whole set. So, you do not need to worry about these two things. But a priori you do not know whether the sets B or B^c is in the collection $\mathcal{F}$.

So, to check that 1_B is measurable what you need to check the set B is in the collection $\mathcal{F}$. So, I am saying that if we start with the assumption that $B \in \mathcal{F}$ then you can show that indicator B is $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$ measurable. Why? Because, if B is in $\mathcal{F}$ then B^c is also in $\mathcal{F}$ and therefore all possible preimages for Borel sets will land up in your domain side σ $-$field $\mathcal{F}$.

So, therefore 1_B will become Boreal measurable. On the other hand, if you start up with assuming Borel measurability of the functions indicator B then you must have these preimages B or B^c in your σ $-$field and that simply reduces to the condition B belongs to $\mathcal{F}$. So, please try to write it down. I have already explained the ideas. So therefore, for 1_B to be measurable where B is a subset on the domain side the set B must be in the σ $-$field.

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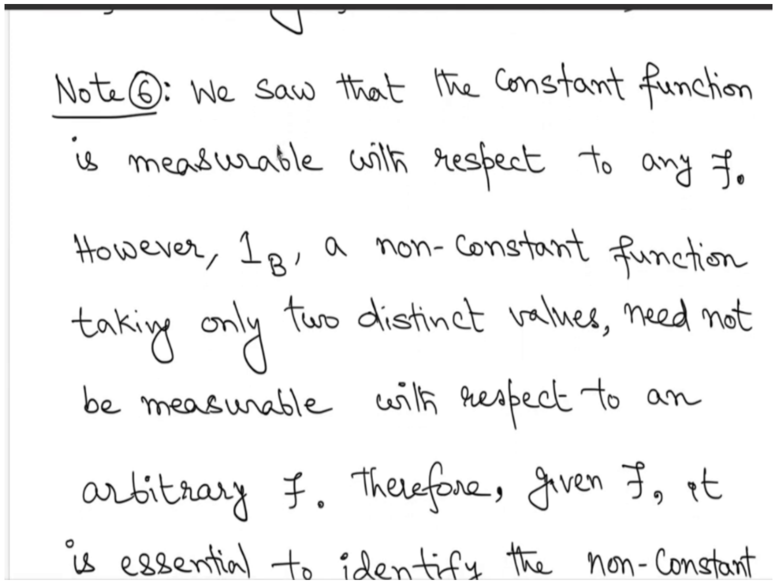

So, what is the interpretation now? So, we saw that the constant function is measurable with respect to any σ −field that you choose on the domain side but when you went to a non-constant function this 1_B takes two values. So, whenever you shifted away from the constant function immediately there came some restrictions on the measurability conditions. So, for this to be measurable this to be Borel measurable you only need to choose those σ −fields where the set B is included.

If you do not have the set B any of σ −field on the domain side then 1_B will not be measureable. So that is what we have discussed. So therefore, this already put some restrictions on the σ − fields or on the other side on the measurable functions. So therefore, first you always guess constant function as measurable functions but whenever you move on to specific choices of σ − fields then for non constant functions you have to be very very careful.

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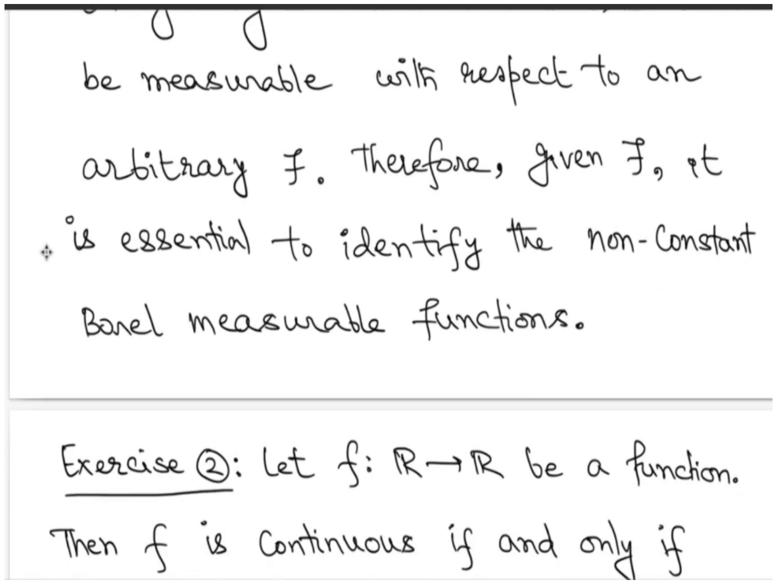

And it is very important to identify which are the non-constant measurable functions given the situation. So, given the appropriate choices of the σ −fields that you start off with, you always should try to identify whether there are any non-constant Boreal measurable functions.

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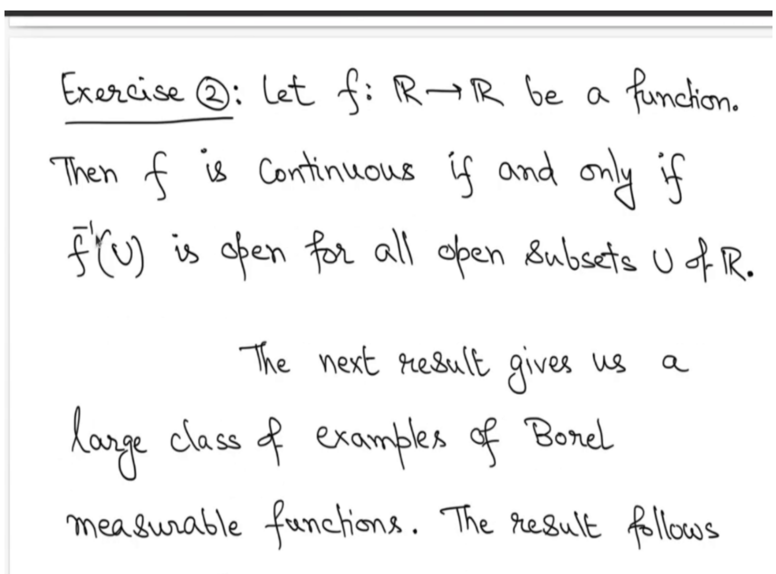

Now, with these two examples at hands we are not (())(28:52) satisfied we would like to obtain more examples. So, now in what follows we are going to discuss a large class of examples. What do we do? So, we consider functions with range R but the domain is also taken to be R. So, we are taking R to R functions.

And we are going to talk about Borel measurability of such functions. Meaning you are going to put Borel σ −field both on the range side as well as the domain side. Because both domain and the range are the real lines. We are going to make this choice and we are going to talk about the Borel measurability here.

Now, you can check that if you take a continuous function then you can look at the preimages of all open subsets and it will turn out to be open. So, if you start up with a continuous function it will imply that preimages of open sets are open. On the other hand, if you assume this condition that the preimages of all open sets are open then you can show that the function is continuous. So, this is an if and only if condition. This is a statement that comes from the real analysis please try to work this out.

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large class of examples of Borel measurable functions. The result follows using the "Principle of Good Sets" discussed in Note 22 of week 1 and Exercise 2. The proof of Proposition 1 is not included in the course.

Now, using this exercise we get this nice collection or large class of examples for Borel measurable functions but with a domain real line. So, this result follows using the principle of good sets which we have discussed in note 22 of week 1. And combined with this exercise that we have mentioned. About identification of continuous functions with these preimage conditions for open subsets. So, if you use the continuous functions if you look at the continuous functions on the real line then.

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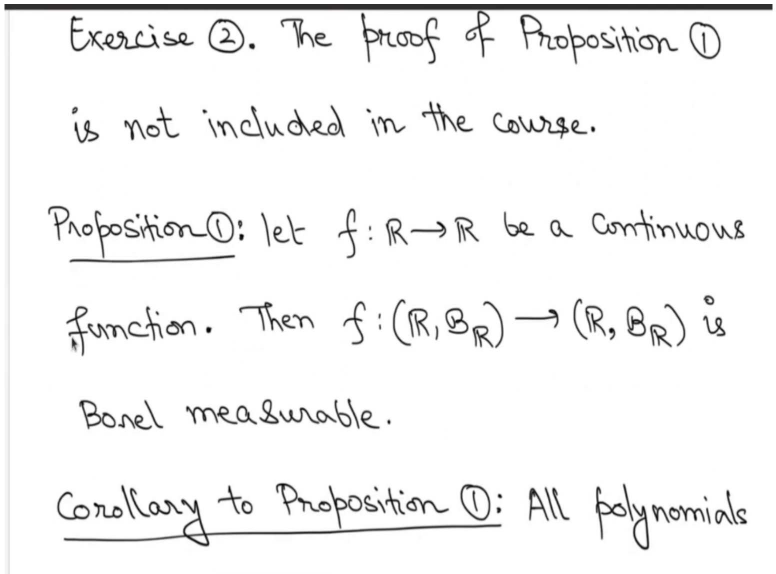

Using the principle of good sets what you can show is that these continuous functions turn out to be Borel measurable. So, you get a large class of examples. So, here what we are doing, we are looking at real valued continuous functions on the real line and we are choosing the Borel $\sigma-$ fields on both sides of the range and the domain and we are saying that if you take a continuous function then it is automatically Borel measurable.

So, that is the statement of the proposition. But since this requires the principle of good sets which we have skipped earlier we are not going to discuss the proof we are going to assume this as a fact. And we are going to use this.

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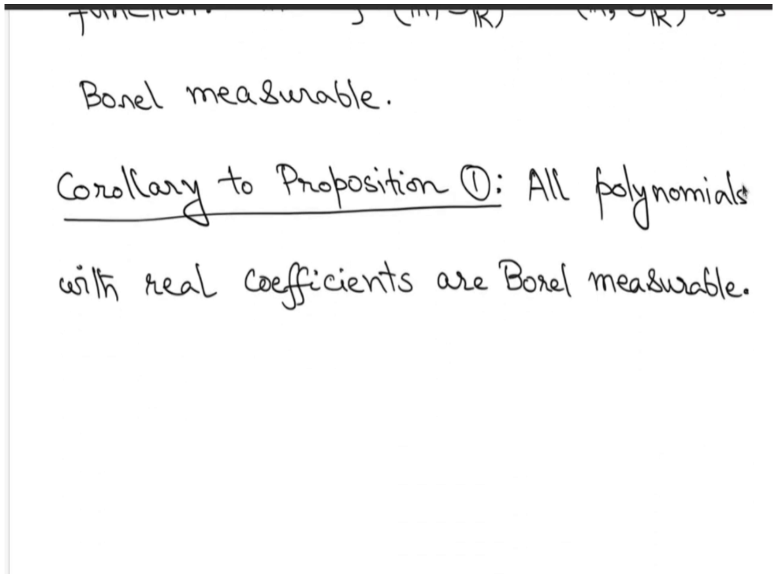

So, for concrete examples of concrete continuous functions and hence measurable functions you can choose to look at the polynomials with real coefficients. So, if you choose any polynomial with real coefficients, they will turn out to be Borel measurable. Because, these are continuous functions. Moreover, you can think of other nice continuous functions like trigonometric functions, like sin functions and cosine functions they are also continuous functions defined on the real line and therefore they will turn out to be Borel measurable.

So, therefore you have a large class of examples immediately after you get this proposition. And other than that, we have discussed these two examples, one was the constant functions, another was the indicator functions. So, we are going to continue these discussions for, in Borel measurable functions but we are going to look at functions taking values in higher dimensions. In Euclidean spaces higher dimensional Euclidean spaces. This we are going to discuss in the next lecture. We stop here.