Measure Theoretic Probability 1
Professor. Suprio Bhar
Department of Mathematics & Statistics
Indian Institute of Technology, Kanpur
Lecture No. 10
Properties of Measures – III

Welcome to this lecture. This is the final lecture of week 2. Before we proceed, we first recall what we have already covered in this week. We have looked at a way of measuring sizes of sets or way of identifying which events are more likely to occur in a random experiment. This we are doing via measures or in particular for random experiments via probability measures. We see the associated value, the size of the sets or size of the events and make the decision whether some events are more likely to occur. So, in this way, we have constructed the measures as countably additive non-negative set functions on σ-fields.

Now, we have also looked at many examples of σ-fields. In these examples, we have looked at the measures that we can define on top of them. And we have also looked at many of these algebraic properties and continuity properties of these set functions, of these measures. In this lecture, we are going to look at a slightly different way of constructing measures. So, so far we have seen examples, explicit examples of measures, but now, in this lecture, we are going to consider a way of constructing these measures.

So, the main difficulty arises in constructing examples is the verification of countable additivity. As you can guess, what you need to check is the, is that given any arbitrary sequence with pairwise disjoint sets, we need to see that the sizes add up and give you the size of the union. Now, this is not really trivial to verify in general spaces. And that is why we need some additional results which will help us verify this countable additivity property. This is what we are going to discuss in this lecture. So, let us move on to the slides.

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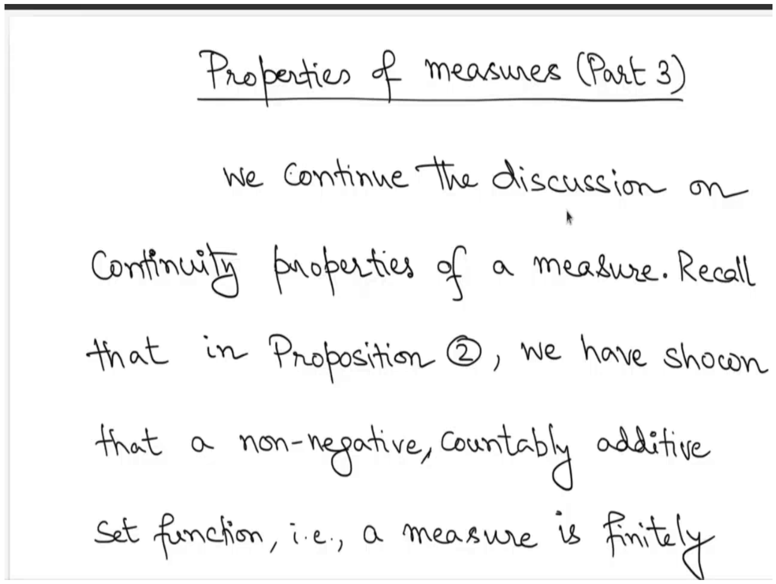

So, we are going to continue this discussion on continuity properties of measures and we are going to see that this is the property that will help us construct measures via these additivity properties. So, recall first that we have shown that are non-negative countably additive set function that is a measure is finitely additive. What do I mean by that? So, I take arbitrary finite number of sets $A_1, A_2, \ldots, A_n$ which should be pairwise disjoint, then the union, the size of the union should be the sum of the individual sizes.

Now, this was finite additivity and we are saying that this follows from countable additivity. In practice, finite additivity is easier to verify. So, this is our starting point of the discussion.

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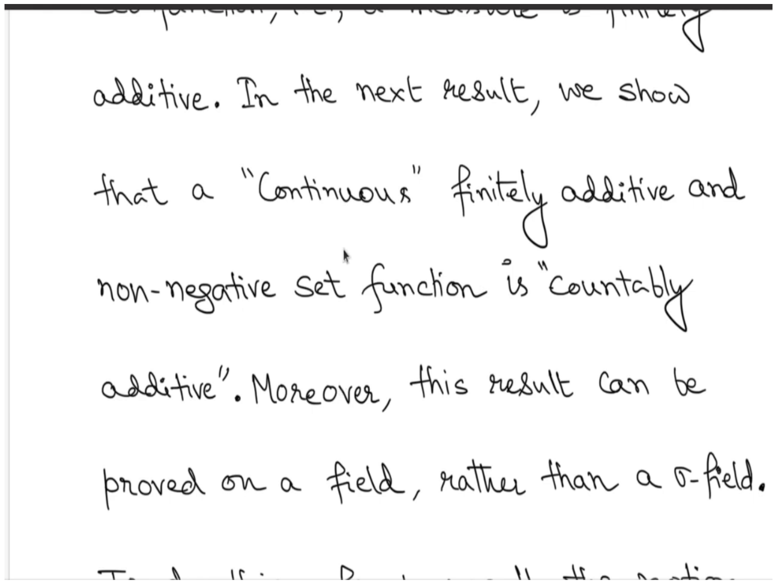

Now, in the next result, what we are going to discuss is that, we are going to talk about a continuity notion for finitely additive set functions. Now, we will take such set functions and we are going to show that that is countably additive. In all of these discussions, we are going to concentrate on non-negative set functions because that will help us in constructions of measures. So, we are going to start off with finitely additive set functions, verify certain continuity properties and we are going to claim that these are countably additive. So, this is the procedure.

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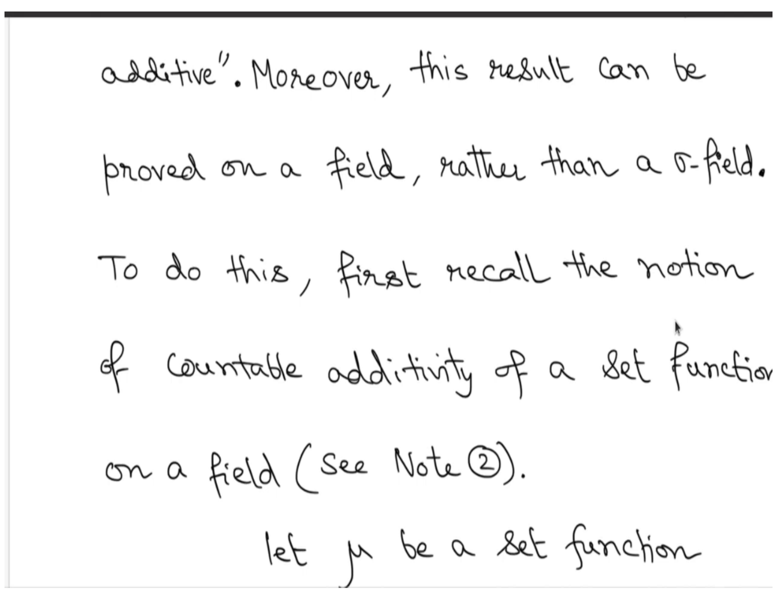

Now, this result can be actually be proved on a field. So, you can look at non-negative finitely additive set functions on fields. So, we have to clarify what do we mean by finding relativity on fields and so on. But what we are saying is that you do not need to start off with the σ-field, you can start off with a field and do these verifications. So, to do this, what do we need to recall first is the notion of countable additivity of an arbitrary set function on some collection, which in particular could be a field. So, this was discussed earlier in note 2.

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on a field (See Note 2).

Let μ be a set function defined on a field $\mathcal{F}$. For every sequence $\{A_n\}_n$ of pairwise disjoint sets in $\mathcal{F}$ with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, if we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

then we say μ is countably additive on $\mathcal{F}$.

We say that a set function μ on a field $\mathcal{F}$ is finitely additive, if for all positive integers n and pairwise disjoint sets $A_1, A_2, \ldots, A_n$ in $\mathcal{F}$, we have

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

So, let μ be our set function defined on a field. Now, what we need to verify for countable additivity is that you are going to take arbitrary sequences Ans which has, which are pairwise disjoint. And then you are going to look at the measure of the complete union, the countable union and you are going to claim that that is equal to the sum of the individual sizes. But since you are dealing with a field, since we are working with our field, what we need to assume is that this sequence has this additional property that this countable union is also in the field.

Now, this is the property, this is the restriction that we impose on the sequence rather than the field, because fields do not support countable unions. Arbitrarily, if you arbitrarily choose these sets, then the countable union need not be in the field. So, for this verification, what you do, you are going to choose those sequences with pairwise disjoint sets and the union being in the field for only those sequences you are going to make this verification. If you have this, then you are going to say that this μ, this set function is countably additive on the field $\mathcal{F}$.

Now, earlier, we made this definition for general collection. We are just recalling it from, for our setup, which is a field. We have just recall that same definition that was made earlier. Now, in addition to countable additivity, we are also have to look at the corresponding notion involving finite number of operations, which is finite additivity. So, we are going to say that the set function μ on our field $\mathcal{F}$ is finitely additive if for all positive integers and pairwise disjoint sets $A_1, A_2, \ldots, A_n$, so you are going to choose some positive integer n and then you are going to choose that many pairwise disjoint sets $A_1, A_2, \ldots, A_n$ from the field.

If you get such things, then by the structure of the field, they are finite union is already inside the field. So, you do not have to make this additional assumption of this union, whether it is there or not, it is already there by the structure of the field. So, you can talk about $\mu\left(\bigcup_{n=1}^{\infty} A_n\right)$. μ is defined on the field. So, you can talk about $\mu\left(\bigcup_{n=1}^{\infty} A_n\right)$. So, that is your left hand side. But you want to claim that that size can be obtained as adding up the individual sizes of the sets.

So, if you have this property for all possible n and all possible sets in the field, then you are going to say that this set function that you are going to consider is finitely additive. So, that is the only distinction between finitely additive and countably additive set functions that in countably additive set functions on fields, you have to ensure that you are working with sequences with union in the field. So, that is an additional assumption that you impose. But for finite additivity, the finite unions are already in the field. So, you do not have to impose those additional restrictions. It is already there for you.

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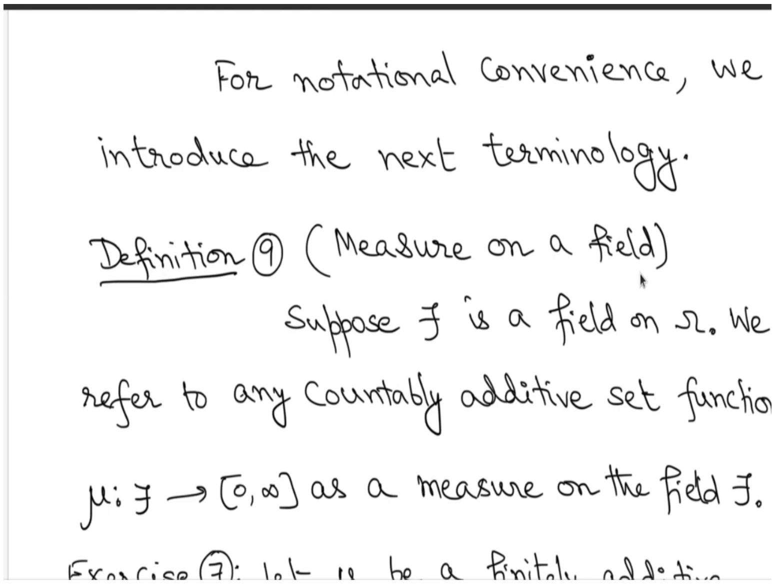

So, with these two notions at hand, we now introduce another terminology, which will help us in explaining the results better. What is this terminology? We are going to talk about set function being a measure on a field. So, so far, we have defined a measure on σ-fields. So on top of σ -fields, we talked about this non-negative countability additive set functions on σ-fields.

Now, since we already discussed a notion of countable activity on a field, it is reasonable to define this notion of a measure on a field. So, what we are going to do, we are going to just follow the same idea that we are going to take this collection, which is a field, $\mathcal{F}$, then we are going to refer to any countably additive set function or non-negative countably additive set function to be a measure on the field.

So, here again, just to highlight, we are talking about non-negative values, but ∞ is also included. So, sizes of the sets could be ∞. Again, you are talking about non-negative countably additive set functions on a field and those we are going to call as measures on the field $\mathcal{F}$.

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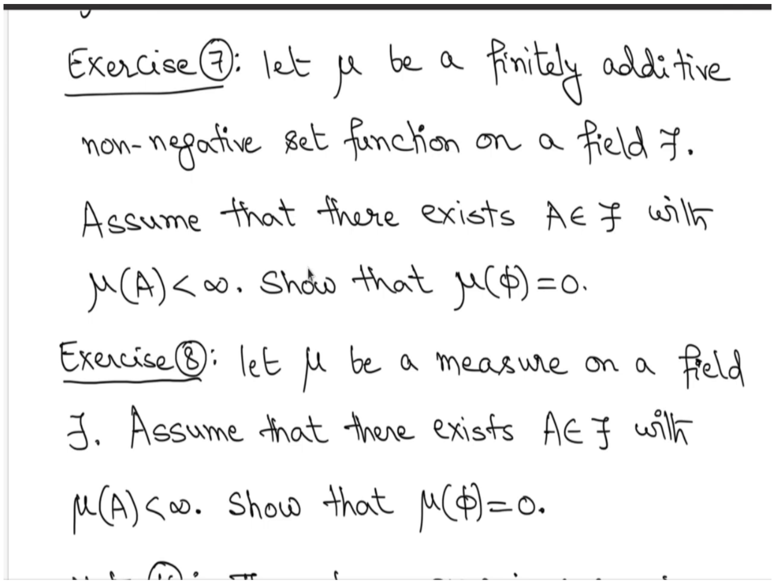

Now, as done for measures on σ-fields, we are going to discuss this property that measure of the empty set is 0 or not. Now, we obtained this property for measures on σ-fields by assuming that there exists at least one set for which the measure is finite. If you make the same assumption for these set functions on a field, then you again end up with the same property. So, remember, in your definition of a measure on a field, this example is still valid that where do you associate infinite mass to all the sets that is still a valid example.

If you want to remove that, if you want to get rid of that what you need to do, you need to ensure that there exists at least one set which has finite mass. So, these discussions was done for measures on σ-fields. We are making the similar discussion now for the fields case, measures on fields.

So, what we are going to do? We are going to look at exercise 7 and exercise 8. Exercise 7 is for finitely additive set functions and exercise 8 is for countably additive non-negative set functions, which we just called as a measure. So, this is the only basic difference between exercise 7 and exercise 8. But the same result follows that you assume that there exists a set A in the field with the measure of the set being finite, then using the exact same structure of the argument, using the exact same argument, you can show that the measure of the empty set must be 0.

So, again here, you can take a finitely additive non-negative set function or you can take a countably additive non-negative set function. So, then in the second case, which is countably additive non-negative set function, we called it a measure on the field. So, again, so for these two cases, in exercise 7 and in exercise 8, if you make that assumption for the existence of such a special set, you can immediately show by the same argument that measure of the empty set is 0.

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Note 16: The above exercises require our familiar assumption, discussed earlier in Notes 6 and 8. For the same reason, we shall continue to make this assumption for any set function on fields and σ-fields. In all

same reason, we shall continue to make this assumption for any set function on fields and σ-fields. In all
subsequent discussions, we continue to make this assumption implicitly and apply the Exercises 7, 8 as required.

Exercise 9: let μ be a non-negative

But then we are going to make this as part of our hypothesis implicitly, because we want to avoid that example where every set gets attached to infinite mass. So, we do not want to look at such a thing, because it is not physically relevant. So, in these type of exercises we are requiring this familiar assumption for the existence of a set with finite mass. So, now, this was discussed earlier in note 6 and 8 for measures on σ-fields.

Now, for similar reasons, we are going to ensure the same, we are going to make the same assumption for measures on fields also and also for finitely additive non-negative set functions on fields. So, in both of these cases, we are going to assume that there exists a set with finite mass, and therefore, exercise 7 and 8 remains valid. And then we are going to end up with the fact that measure of the empty set is 0.

But then just to clarify once more, in all subsequent discussions, this will be part of our hypothesis implicitly, but, to be clear, this does not follow from the definition of a finitely additive set function on a field or it does not follow from the definition of a measure on a field. We are making this assumption in addition to whatever we have already defined and we are going to make this part of the hypothesis implicitly.

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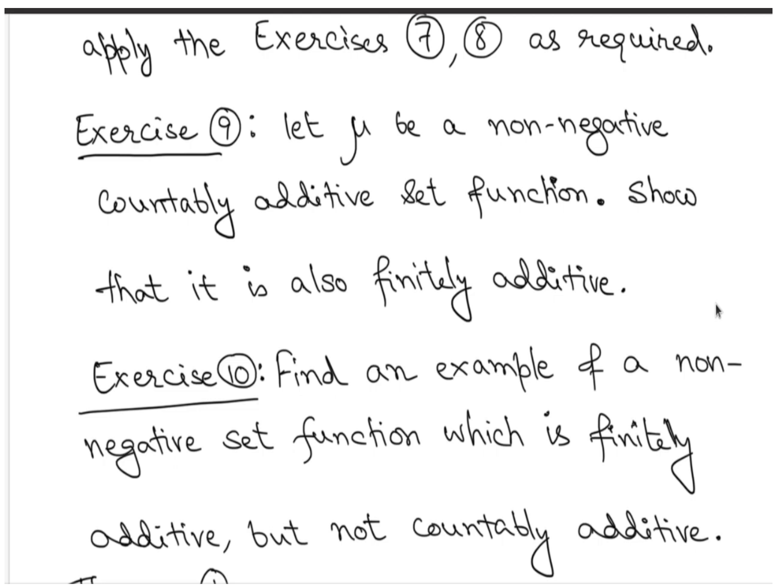

Now, as discussed for measures on σ-fields, you can show that a countably additive set function which is a measure, non-negative of course, which is a measure, you can show that it is finitely additive. Here you are going to use the fact that measure of the empty set is 0. So, again, as discussed earlier, for measures on σ-fields, it will again follow the same structure of the argument and you can show that a measure on a field is finitely additive.

But then, once you have that, you can now ask are all finitely additive set functions non-negative on a field are all these also countably additive? So, exercise 9 says that all countably additive non-negative set functions, all measures on fields are finitely additive. You are asking the converse question, given a non-negative finitely additive set function, is it true that it is also countably additive.

So, exercise 10 is going to tell you that it is not true. And you are supposed to figure out an example of such a thing. You are going to figure it out, you should figure out example of a non-negative set function, which is finitely additive, but not countably additive. Try to find this. It is an important example, which will help you clarify the difference between finitely additive and countably additive set functions.

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Theorem 1

let $\mathcal{F}$ be a field on Ω. Suppose that $\mu : \mathcal{F} \to [0, \infty]$ is a finitely additive set function.

(i) If μ is continuous from below at each $A \in \mathcal{F}$, i.e. for any sequence $\{A_n\}_n$ in $\mathcal{F}$ with $A_n \uparrow A$, we have

in $\mathcal{F}$ with $A_n \uparrow A$, we have

$$\lim_{n \to \infty} \mu(A_n) = \mu(A),$$

then μ is countably additive on $\mathcal{F}$, i.e, μ is a measure on the field $\mathcal{F}$.

(ii) If μ is continuous from above at the empty set ϕ, i.e. for any sequence the empty set ϕ, i.e. for any sequence $\{A_n\}_n$ in $\mathcal{F}$ with $A_n \downarrow \phi$, we have

$$\lim_{n\to\infty} \mu(A_n) = \mu(\phi) = 0,$$

then μ is countably additive on $\mathcal{F}$, i.e. μ is a measure on the field $\mathcal{F}$.

Proof of part(i): let $A \in \mathcal{F}$ and let $\{A_n\}_n$ be a sequence of pairwise disjoint sets in $\mathcal{F}$

But now, we come to the main result of this lecture. As before, we are saying that we are going to work with fields and you are going to work with finitely additive set functions non-negative on the field. Now, we are going to claim that with some additional continuity assumptions, we are going to, we can claim that it is also countably additive. So, this is some additional condition together with the finitely additiveness which will help us prove countable additiveness.

So, just on its own finitely additivity, finite additivity does not imply countable additivity, but with some appropriate sufficient condition, in terms of some continuity assumptions, will help us show that it is also countably additive. So, this is the discussion in theorem 1 which is the main result of this lecture.

So, let us see this. For measures, we already had this property. Now, we are starting off with finite additiveness. So, we do not have countable additivity which was used to prove this continuity assumption. So, now, from finite additivity, we are adding this as part of the hypothesis now, some certain continuity assumptions, which is continuity from below for this finitely additive set function. So, we are saying that $\mu(A_n)$ should increase to $\mu(A)$. If you have that, then you call that your set function μ is continuous from below at each A.

Now, if this is your part of your hypothesis, then we claim if this is your part of, this is the part of hypothesis that it is continuous from below at each set, then you are going to claim that μ is countably additive on the field and therefore, this finitely additive set function which is continuous from below at each set will become a measure on our field. So, together with finite additivity and continuity from below you can claim countable additivity and thereby you get a measure on a field.

There is a similar related assumption, but it is in terms of continuity from above. So, now again you are taking this finitely additive set function and assume that it is continuous from above at the empty set. So, what do I mean by that? So, you are going to choose sequences in the field again, which decrease and decrease to the empty set that is very important. So, decreasing means that their countable intersection becomes the empty set, but these sets be decreasing. For those sets, what you need is or the $\mu(A_n)$ should be 0, because or $\mu(\phi) = 0$ for all finitely additive set functions as discussed in the above exercises.

So, if you have this property, then you call that your finitely additive set function is continuous from above at the empty set. So, now with this as part of your hypothesis, you are going to claim that it is also implies countable additivity. So, together with finite additivity, continuous from above at the empty set, this property will imply countable additivity, and thereby, you will get a measure on a field, on the field $\mathcal{F}$.

So, these are two sufficient conditions that will help us prove that our finitely additive set function is also countably additive. Of course, once this becomes a measure, then you can immediately try to go by the earlier arguments and try to see that these continuity properties are really required. So, please check this that whether these conditions are also necessary or not. This is just helping you understand the topic better.

But now, what we are going to do is to prove this theorem that a finitely additive set function together with any one of these continuity assumptions will imply countable additivity. Let us start with part 1, which is continuity from below at each set.

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Proof of part (i): let $A \in \mathcal{F}$ and let $\{A_n\}_n$ be a sequence of pairwise disjoint sets in $\mathcal{F}$ such that $\bigcup_{n=1}^{\infty} A_n = A$. Take $B_n = \bigcup_{i=1}^{n} A_i, \forall n$. Then $B_n \in \mathcal{F}$ and $B_n \uparrow A$. Using Continuity from below at A and finite additivity of μ, we have

Then $B_n \in \mathcal{F}$ and $B_n \uparrow A$. Using Continuity from below at A and finite additivity of μ, we have

$$\mu(A) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Hence μ is countably additive.

Proof of part (ii): Take A, A_n's, B_n's as

So, how does this help? So take an arbitrary set in $\mathcal{F}$, call it A, and choose a sequence of pairwise disjoint sets in your $\mathcal{F}$ such that the union is A. Here, what you can do is that you can also look at these $B_n = \bigcup_{i=1}^{n} A_i$, some finite unions, and then we will get these examples of B_ns which increased to A. Now, you are going to use the continuity from below at this set A, which is part of the hypothesis, and use the finite additivity. So, what you are going to do?

So, we are going to say that as B_ns increase to the set A, then $\mu(A) = lim\, \mu(B_n)$. So that is why the continuity property that is assumed, but then look at $\mu(B_n)$, that is nothing but these B_ns are now pairwise disjoint union of these sets Ai, so, therefore, what you get is that limit of this finite summation becomes this. So, this is using finite additivity. But then, once you take this limit of these non-negative terms, sum of these non-negative terms, you end up with this summation, as expected.

So, therefore, you get countable additiveness with any arbitrary pairwise disjoint sets. So, this is what helps us in proving this countable additiveness. To prove the countable additivity, you are going to start with these pairwise disjoint sets. And then you look at this union, then these B_n's will increase to this set A and B_n's, since B_n 's are increasing then using continuity property from below at the set A, you are going to repeatedly apply this hypothesis end up with the property of countable additivity.

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Hence μ is countably additive.

Proof of part (ii): Take A, A_n's, B_n's as in part (a). By finite additivity of μ, we have $\mu(A) = \mu(B_n) + \mu(A \setminus B_n)$ and $\mu(B_n) = \sum_{i=1}^{n} \mu(A_i)$.

Now, $A \setminus B_n \downarrow \phi$. By the Continuity from above at ϕ, we have

$$\mu(B_n) = \sum_{i=1} \mu(A_i).$$

Now, $A\setminus B_n \downarrow \phi$. By the Continuity from above at ϕ, we have

$$\lim_{n\to\infty} \mu(A\setminus B_n) = \mu(\phi) = 0.$$

Hence, $\mu(A) = \lim_{n\to\infty}\left[\mu(B_n) + \mu(A\setminus B_n)\right]$

$$= \sum_{i=1}^{\infty} \mu(A_i).$$

This completes the proof.

And for part 2, we are going to take these same A_ns, B_ns as above, meaning, so this should be part I. So, this correction will be made in the notes. So, what do I mean by this? So, again, you are going to choose this sequence of pairwise disjoint sets and take this complete union, this countable union call it A and take these B_ns as this finite union which is of course a pairwise disjoint union of these sets A_i up to $i = n$.

So, take these same sets and you want to verify countable additivity once more. But now, first apply the finite additivity of the set function for the set A. So, since B_ns are subsets of A, then you get the remainder part as $A\setminus B_n$, so this is $A \cap B_n^c$. So, this is simply using the finite additivity. But then, look at the $\mu(B_n)$ term. Since B_n are finite disjoint union $\bigcup_{i=1}^{n} A_n$. so this will split by finite additivity. So, you have we applied finite additivity twice.

Now, look at $A\setminus B_n$. Since B_n increase to the set A, $A\setminus B_n$ meaning $A \cap B_n^c$ will decrease to the empty set. So, therefore, by this continuity from above at the empty set, here we use that hypothesis. We get that $\lim_{n\to\infty} \mu(A\setminus B_n) = \mu(\phi) = 0$. So, this is where we use the hypothesis that it is continuous from above at the empty set. Here we are applying it to the decreasing sequence $A\backslash B_n$.

Now, look at the original result that we obtained from finite additivity. So, this was, the $\mu(A)$ is the addition of these two terms. Now, you have proved that on the right hand side, the second term $\mu(A\backslash B_n)$ that term has a limit which is 0. You have proved that. But note that the left hand side is independent of n.

Therefore, what you are going to do is to let $n \to \infty$ in this relation and what will happen you are going to get that $\mu(A) = \lim_{n\to\infty} (\mu(B_n) + \mu(A\backslash B_n))$. Since you already know that $\lim_{n\to\infty} \mu(A\backslash B_n) = 0$ by the continuity from above at the empty set, then you are simply ending up with the limit of $\mu(B_n)$s.

But what are $\mu(B_n)$s? Again, by the finite additivity we have already know that summation i from 1 to ∞ this is the size of the individual sets. So, now, if you let $n \to \infty$ here, all you will get is the series and that will tell you that $\mu(A)$ is exactly called to this summation of the individual sizes. So, that proves that μ is again countably additive.

So, in both these properties, we are taking a sequence of pairwise disjoint sets, looking at the union and we are showing that it is the, this countable union of the individual sizes. And to do that, we are just making use of that structure of these sets B_n which increase to the union. So, that is what we are using.

So, for part 2, we do not need to look at arbitrary sets, but we just have to ensure continuity from above at the empty set. For the part 1 we need to have this continuity property for any set A, because you are going to show this for arbitrarily approximations from below. So, that is what is happening. So, there you have these two properties, one is continuity from below at every set or continuity from above at the empty set.

If you have that for any finite additive set function, then together with this structure on the fields, you have been able to construct or you have been able to show that this finitely additive set function is a countably additive set function thereby becoming a measure on the field.

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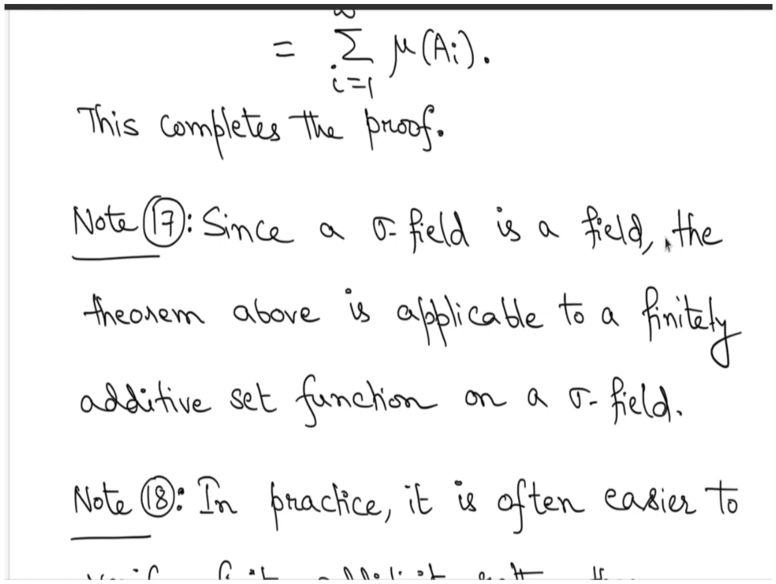

Now, what are these applications of this result? First, since a σ-field already a field, this above result which is actually proof for a field is applicable to any σ-fields. So, therefore, this theorem we can apply it to σ-fields and finitely additive set functions on σ-fields. So, remember this. We can apply this above theorem to finitely additive set functions on σ-fields. So, together with these appropriate continuity assumptions, we can construct, we can show these finitely additive set functions on a σ-field becomes a measure on a σ-field. So, that is the content here.

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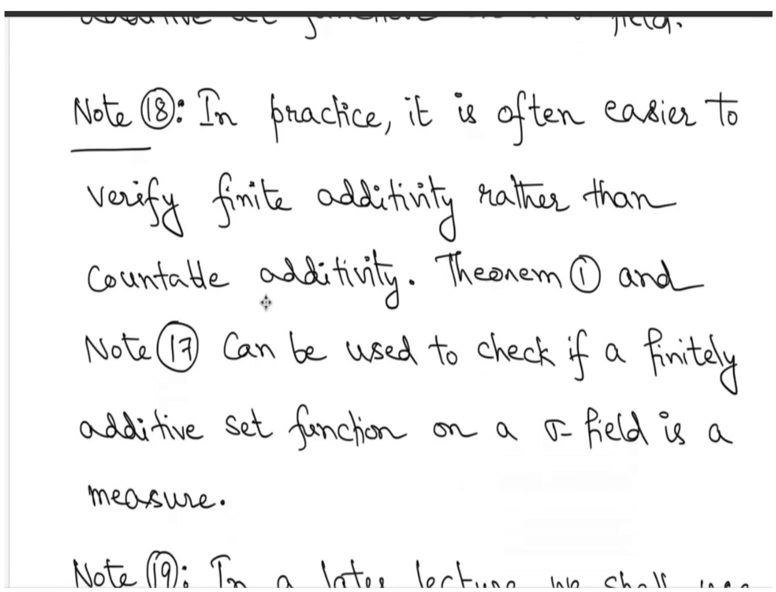

measure.

Note (19): In a later lecture, we shall use Theorem (1) and Note (18) above to construct measures.

We now discuss an important result for probability measures. This result allows us to construct events with probability 0 from sequences of events.

But then how do you really apply it or use this to construct examples of measures. So, in practice, what we are going to do is this. It is often easier to verify finite additivity, because it deals with finite number of operations. You just have to look at pairwise disjoint sets and then add up the sizes of the sets and see it matches with the size of the union. Finite additivity is usually easier to verify.

Now, what this theorem 1 and this note 17 tells you that you can take this finitely additive set functions on σ-fields, check the appropriate continuity assumption, if you have that, then as part of the conclusion of the theorem, you get that this set function is also countably additive. So, therefore, you get a measure. So, this is a way of construction of measures. So, this is what the content of note 19 is.

So, in later lectures, what we do, we are going to use this idea that we will take, first, we will take finitely additive set functions on σ-fields. We are going to check if that is continuous, in this sense, one of these appropriate sufficient conditions. And then we are going to claim that it is countably additive and thereby it will become a measure and it will allow you to construct measures. So, that is the content of this theorem.

We are now going to discuss some important results for probability measures. And this result allows us to construct events with probability 0 from sequences of events. So, you are going to take σ-fields, you are going to consider probability measures on top of that. So, this can come from some appropriate random experiment. So, you are going to look at these collections of results so it becomes a σ-field.

Now, we are going to look at certain specific type of sequences, certain specific type of sequences of events from the random experiment. And with some appropriate assumptions on the sequences, we are going to show that we can construct certain sets with probability 0.

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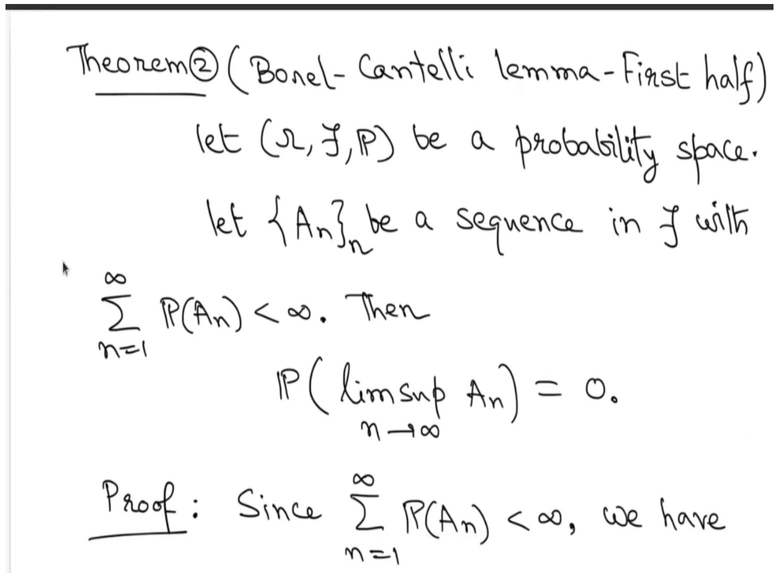

So, this is a very important construction, which allows you to construct sets or events with measure 0, probability 0. The sets need not be empty, but their probability may be 0. So, these are impossible events in terms of the probability that you will consider. So, these are non-empty sets, but their probability is still 0. So, this can happen. Now, what we are going to construct are these sets which has probability 0 from sets with small mass, not necessarily 0 mass, but small mass, so that is what is the content of this result, which is called the Borel-Cantelli lemma first half.

So, there is a second half of these results which deals with independence of events and which is not part of this course. So, we are only going to see this first half part in this course. Here, what we are saying is that we are going to take some specific type of sequences from any probability space. And with some appropriate sufficient conditions, we can construct some very specific sets with measure 0, probability 0. So, what is this? So, take $\{A_n\}$s to be a sequence in your σ-field with the condition that sum of the probabilities, sum of the probability of the events is finite.

So, here, you have to be careful. These Ans need not be pairwise disjoint. So, this summation condition that is there, this summation need not be the probability of the union of the sets An. So, this is some arbitrary sequence. But with this condition that sum of the probabilities makes sense, finite, so this sum is finite. So, that is what you require as a sufficient condition here. Then you claim that the $lim\ sup\ A_n$, so that is a genuine set that is in your σ-field, we have discussed this in week one. So, the probability of that should be 0.

So, that is what the Borel-Cantelli lemma first half tells you that with sequences where the probabilities are summable, you get that the probability of the limit superior of the sets is 0. So, how do you show this?

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<u>Proof</u>: Since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, we have

$$\lim_{k \to \infty} \sum_{n=k}^{\infty} \mathbb{P}(A_n) = 0.$$

Now, $\limsup_{n \to \infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_m$

$$\subseteq \bigcup_{m=k}^{\infty} A_m, \text{ for each } k$$

Hence $\mathbb{P}(\limsup\limits_{n\to\infty} A_n) \le \sum\limits_{m=k}^{\infty} \mathbb{P}(A_m)$,

for each k. ..... $(*)$

Since $\mathbb{P}(\limsup\limits_{n\to\infty} A_n)$ does not depend on k, letting k go to infinity on both sides of $(*)$, we have

$$\mathbb{P}(\limsup_{n\to\infty} A_n) \le 0.$$

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Since probability is, by definition, non-negative, the result follows.

So, you start with the series that is given to you. So, the sum of the probabilities this is finite. So, in particular, what does it tell you? It tells you that the tail probabilities, the tail terms, the remainder terms, after a certain stage onwards that contribution will be 0, if you let $k \to \infty$. So, if you are going to look at some terms from k onwards that summation should go to 0 as $k \to \infty$. So, that is another way of saying that this series will converge that the partial sum of the first k terms will convert, so the remainder should contribute 0 in the limit. So, that is another way of stating the convergence.

Now, look at the limit superior and recall the definition. So, what is this, this is $\bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_m$. So, first you will look at this union and then look at intersections of those. But then, if you look at this outer intersection, you can immediately claim that for each k your $lim\ sup\ A_n$ is a subset of this countable union from k to ∞. So, if you are going to look at this inner unions for each fix set k, then limit superior is a subset of that that is just by definition.

But now to claim that limits superior An has probability 0, we are going to look at the probability of this union. So, that is the attempt, that is the procedure that we are going to follow. So, let us look at that. But since you are having a probability measure, in particular, this is a measure. So, you have this monotonicity condition. So, $lim\ sup\ A_n$ is contained in $\bigcup_{n=k}^{\infty} A_n$. So, therefore, $\mathbb{P}(lim\ sup\ A_n) \le \mathbb{P}(\bigcup_{n=k}^{\infty} A_n)$. But these sets A_n need not be pairwise disjoint.

So for that we had proved this countable additivity property that it is actually given by a inequality. This was discussed earlier in a previous lecture. So, there we had discussed this algebraic property that for arbitrary sequence of sets, the union, the size of the union is dominated from above by this summation. So, this is what we had proved earlier in algebraic properties of measures. So, that is what we are applying here.

So, $\mathbb{P}(lim\ sup\ A_n)$probability of the limit superior of the Ans is dominated from above by the probability of the union and that is further dominated by $\sum_{m=k}^{\infty} \mathbb{P}(A_m)$. But then, this is true for each k. Since limit superior An is contained in these unions for each k, so you have this upper bound for each k. So, that is our first observation. But then you apply the fact for the convergence, you apply this fact, that limit as $k \to \infty$ of this summation is 0.

So, therefore, if you apply limit as $k \to \infty$ in *, this relation on both sides, your left hand side does not depend on k, but the right hand side depends on k and goes to 0. So, therefore, what you end up with is this relation that probability of limsup is less or equal to 0. But since probability is a measure and in fact this probability cannot be negative, so the only possible value is probability of limsup must be 0.

So, that is what proves that the probability of limsup must be 0. So, here we have used the convergence of the series, the finiteness of the series, and claim that the tail contributions are 0. And we have estimated the probability of the limit superior by the tail probabilities and since the tail probabilities, some of the tail probabilities go to 0, we get the property that probability of limsup is 0.

So, again, Borel-Cantelli lemma first half allows you to construct possibly non-empty sets, which is limit superior, very explicit set, from arbitrary sequence where the probabilities are summable. So, that is the sufficient condition that will allow you to construct these probabilities 0 sets which need not be empty and this has very important applications, which we shall see later on.

So, we stop here and we are going to continue these discussions in week three lectures. So, there we are going to start new topics involving certain functions. So, we stop here.