Advanced Partial Differential Equations
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Lecture – 9
Mollification

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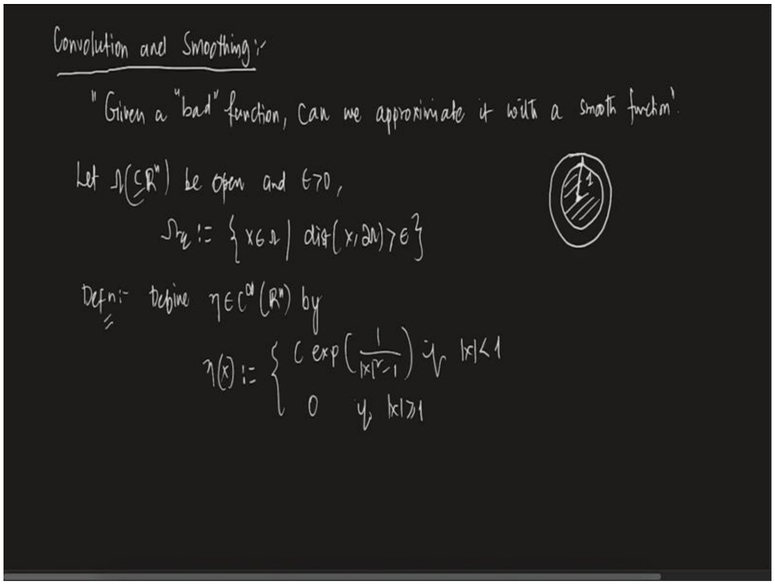

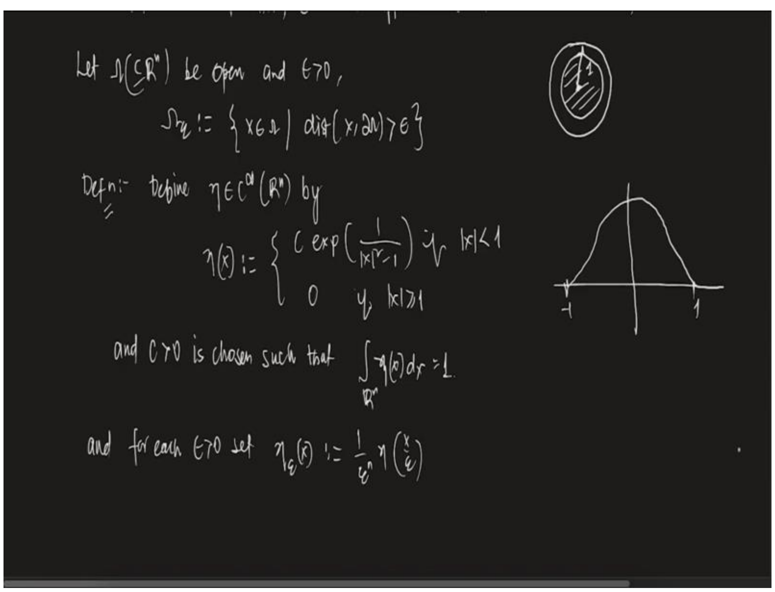

Welcome and in this small video, we are going to talk about something called convolution and smoothing, convolution and smoothing. What it means? See essentially, the point is this. Let say that I will start with so basically given any function, I want to approximate the functions smoothly. So, understand, so given a bad function; let us them call it a bad function, bad function can we approximate it, can we approximate it with a smooth function; is this clear. We already know that if you have smooth function; smooth functions are nice functions, so the question is this. Bad functions they may not work all the time.

So, let say but you cannot throw it out also. So, once you are given a bad function how do we deal with it? So, essentially what we do it? We take a smooth function work out our properties. And then approximate the bad function with the help of smooth functions and see things what (per). So, some notations actually we are going to do is this. So, let omega subset of Rn be open and epsilon greater than 0; we define epsilon, this is what we are defining. It is a set of all those x in omega such that the distance between x and the boundary; this is greater than epsilon. So, essentially think of a ball of radius 1 radius 1 radius 1.

So, essentially if I am taking an epsilon choose an epsilon, if I am writing the omega epsilon; it will be the inside ball, so that is your omega epsilon. Now, I will put some definitions down so definition. We define define a Eta the smooth eta, its infinity Rn by Eta of x; this is a constant time exponential, I will explain why this constant I am taking, 1 by mod x square minus 1. This if mod x less than 1; and 0, if mod x greater than equal 1. So, essentially it is a very smooth function which will die down, so something like this something like this. The minus 1, plus 1 in 1-dimennsion let us draw this thing; it will be a very nice function.

So, forget my drawing capability, it will be a very smooth round function. So, if you look like something like this; and in r2, we can think of this as a bowl kind of a thing inverted bowl. So, and this C, what is C? C greater than 0 is chosen is chosen such that such that integral over Rn Eta of x dx; that is 1. So, we will choose this C, C depends on R; will choose our C in such a way that the integral of Eta over Rn is 1. So, this Eta is a very special Eta, this Eta is a radial function as Eta. What is a radial function? It is a function of R essentially Eta; so, we look at this function on any in any ball of radius r that is going to be a constant.

So, and we also and and for each epsilon positive set, Eta epsilon of x to be 1 by epsilon power n Eta of x by epsilon. So, what we are going to do is we are going to use this radial function. See, first of all what you are setting out is a smooth radial function, C infinity function, which looks like the exponential of 1 by mod x square minus 1. You can understand that as mod x tends towards 1, this is going towards 0. So, essentially this goes towards minus infinity; so this goes towards 0, the exponential of this it goes towards 0. And this is always smooth, you can just prove it; it is not a problem.

And now what we are going to do is for every epsilon, we are writing an Eta epsilon; which is just a scale down version of Eta. We are just scaling it by epsilon, Eta of x by epsilon. Once we do that this Eta.

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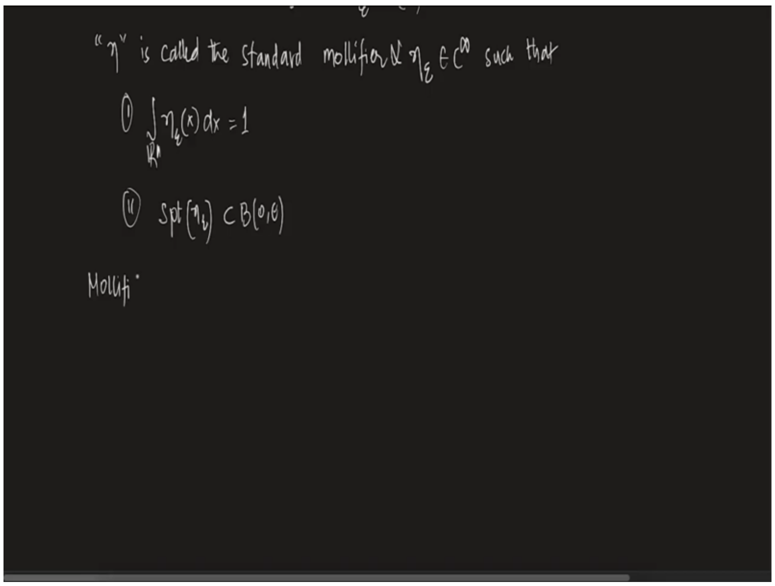

This Eta is called the standard mollifier standard mollifier clear; this is called the standard mollifier. This Eta epsilon and Eta epsilon is in C infinity, such that of course it is C infinity. Eta C infinity, so Eta epsilon is the C infinity, I mean there is nothing special going on here. So, Eta is called the standard mollifier Eta epsilon is in C infinity; and these are the properties of Eta epsilon. Number 1, integral over Rn Eta epsilon of x dx this is 1; and number 2, of course it is 1.

You can just take the integral, this Eta from for Eta n; the jacobian will be Eta power n which will get cancelled. And it will be integral of Eta which is 1; so that is why this is 1. And the support of Eta support of Eta epsilon; so, where Eta epsilon the closure of the set, where the Eta epsilon is non-zero; that is containing B 0, epsilon. Of course, because outside B 0, epsilon what is going to happen? Eta will be 0; so, Eta epsilon will be 0, outside epsilon radial ball.

So, the support of Eta epsilon is containing B, 0 epsilon. What it essentially says? Is beta epsilon is non-zero in this B 0, epsilon ball; and outside that it will be 0. Now, we are going to define something called the mollifier mollification. So, essentially what we said that we want to approximate a bad function with a good function.

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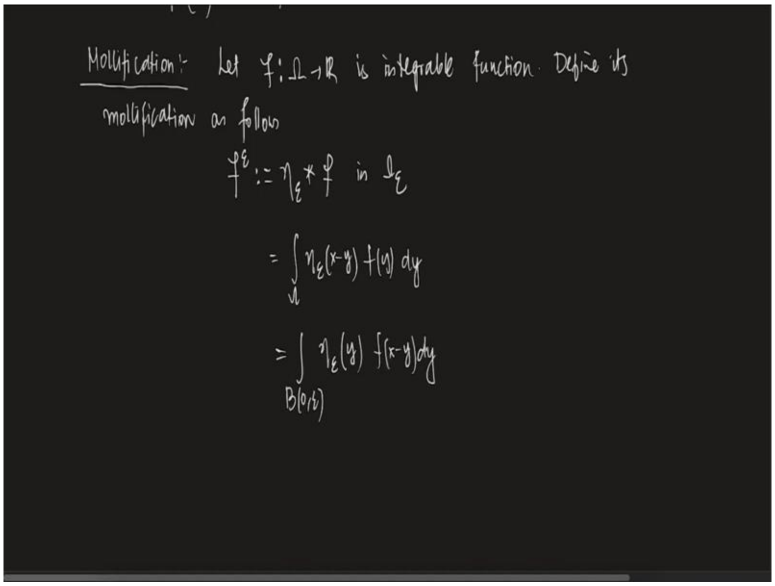

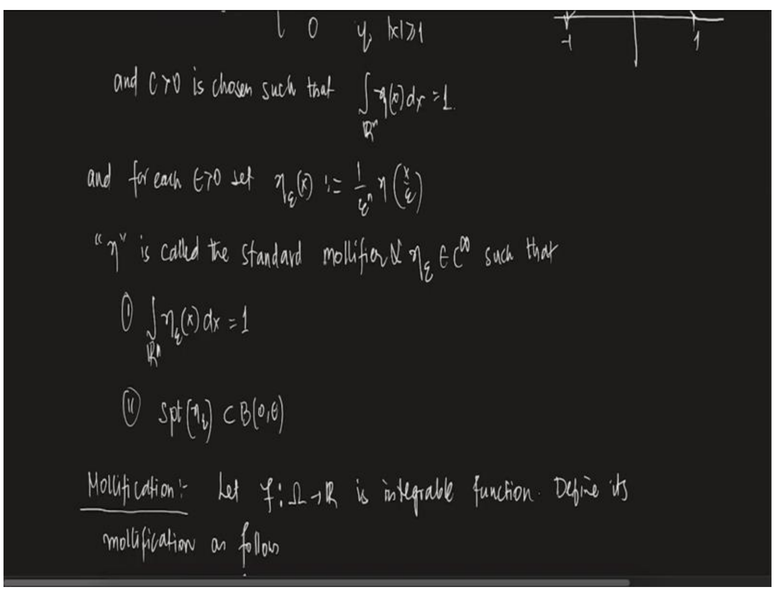

So, let f from omega to R is integrable is integrable; so, for now you can just think of this as a continuous function integrable function. Now, define its mollification as follows. F epsilon this is a mollification of f, so it gives me a continuous function; what I am going to do is I am going to make it a smooth function. How? I am going to define f epsilon, which is given by Eta epsilon;

this family of smooth function, convolution with f; so, this is defining omega epsilon. And how do if we want to write it properly how do you write it? It is given by integral over omega Eta epsilon of x minus y, f of y dy.

And again, you can use change of variable, and you can show that this is equals to B 0, epsilon Eta epsilon of y, f of x minus y dy. Please change its part that is your mollification. So, again what I am doing Eta epsilon is a family of mollifiers, which I made; family of radial this thing, radial mollifiers it is a family of mollifiers. Eta is a standard mollifier and Eta epsilon is a family of mollifiers which I constructed out of Eta. Now, what I am doing is I am constructing a new function f epsilon, which is Eta epsilon convolution f. So, I am using convolution and making a new function which Eta epsilon convolution f; which is define by this. So, that is the definition of convolution. And by change of variable, you can also write it like this. What we are claiming it, so this is called every x in omega epsilon; omega epsilon is outside the epsilon neighborhood.

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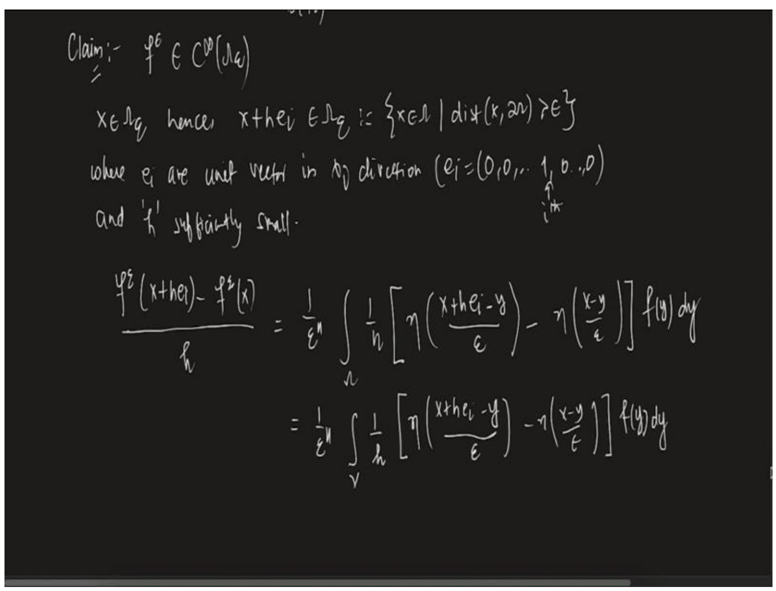

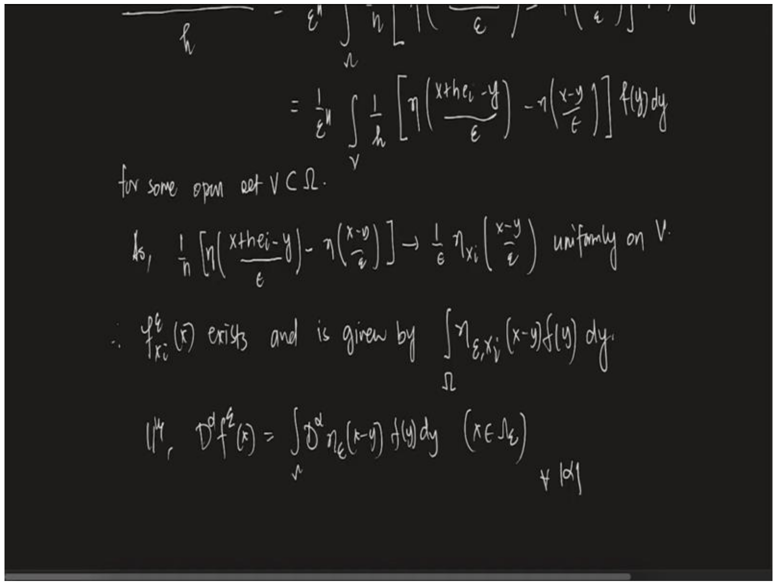

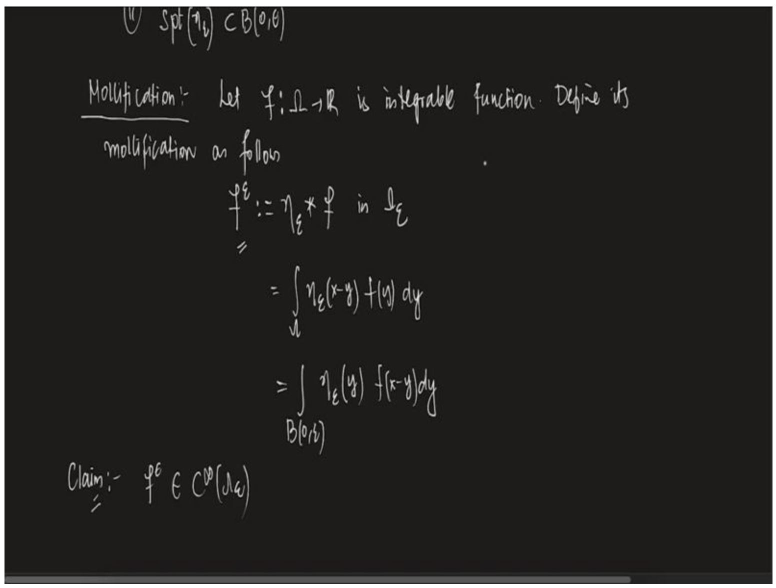

So, the our claim is f epsilon, this is in C infinity omega epsilon is it clear; so f epsilon which you are defining like this, we are saying that this is infinity define. F is not infinity differentiable; f is just an integrable function; I do not care I think of this as the continuous function. What it is saying is once we do this mollification, once you convolute it to the family of mollifiers; then that function will turn out to be a infinity differentiable function.

Now, how to prove this thing? What I am going to do is I am going to give you a hint; and I think you guys can manage it from there, the hint is this. So, let say x is in omega epsilon and therefore, hence we can take x plus h ei. This is in omega epsilon I can take this: what is omega epsilon? It is open set. How is omega epsilon defined? If you remember it is set of x in omega, such that distance between x and the boundary is greater than epsilon. So, if it is inside omega and it is an open set; so, if it is an open set, there is a small wall which contains, which is centered at x and contained in omega epsilon.

So, I have x plus h ei, I can take h sufficiently small such that this happens. What is ei square? Ei are unit vector in xi direction; so ei is a 0, 0, 1, 0, 0, 0 this is the ith component. And what is h? h sufficiently small. Why sufficiently small? Because I want my h to be small, such that x plus h ei is containing omega epsilon, that is all. So, once this is happening, I will show that the derivative of f first first derivative, first of all exist. So, let say f epsilon of x plus h ei minus f epsilon of x by h; let us calculate this thing and see what happens.

This is 1 by epsilon power n integral over omega, 1 by h Eta of x plus h ei minus y by epsilon, minus Eta of x minus y by epsilon; I am just writing down definition and nothing else f of y dy. Now, this is equals to 1 by epsilon power n integral over v; I am changing the integral over omega to integral over v.

Think about it from h to v I am changing; 1 by h Eta of x plus h ei minus y by epsilon, minus Eta of x minus y by epsilon, f of y dy. So, once I do this thing this is for some open set v containing omega. So, as 1 by h Eta of x plus h ei minus y by epsilon, minus Eta of x minus y by epsilon is converges to 1 by epsilon Eta of xi x minus y by epsilon this is true.

See Eta is a C infinity function; so, it is of course once continuously differentiable. If that happens, I am just using the definition of derivative of Eta, nothing else; it is same thing, you use the definition of derivative Eta. So, please remember this is 1 by n, this is 1 by epsilon. Now, once that happens, so this this is uniform region converges uniformly on V; it converges uniformly on V.

So, therefore the partial derivative f epsilon of xi at the point x exists and is equals to and is given by integral over omega Eta epsilon, xi of x minus y f of y dy; now, you do the similar arguments. Similarly, you can check that D alpha of f epsilon of x; this convolution as integral over n D alpha of Eta epsilon of x minus y, f of y dy; for x in omega epsilon. And this is for all (())(15:13) mod alpha; and so, the thing is u in f epsilon is infinity differentiable.