Advanced Partial Differential Equations
Professor Doctor Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Lecture 8
Mean Value Property

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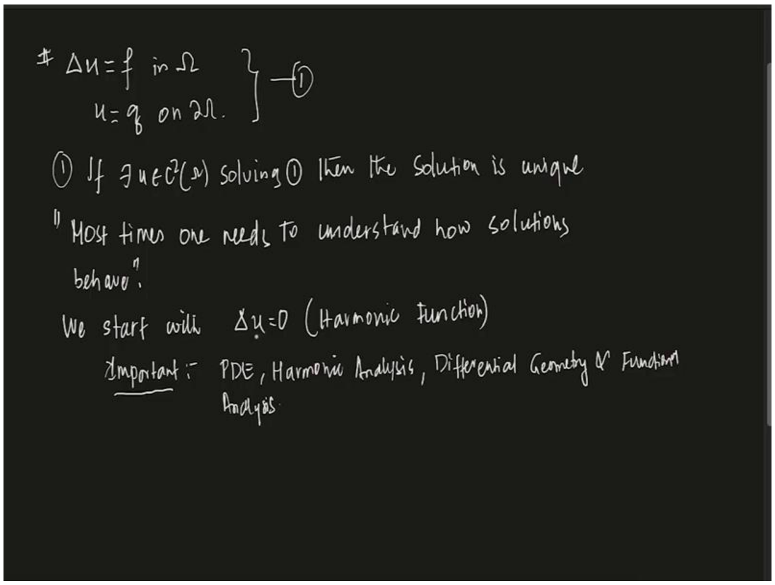

In this lecture we are going to talk about some properties of Laplace Equation. So, as you have seen let us say you are given a problem like this to solve, we were interested in solving this problem is equals to f in omega and u equals to g on the boundary. So, what we have seen, we have seen that if there exist u in C2 omega let us say that is your 1, solving 1 then the solution is unique.

So, what by this is if there is a solution and that solution is unique and we also saw that if you need to find a solution, you just look at a corresponding energy function of that particular energy function if you remember is half of gradient u dot gradient w minus half of, minus integral fw. If you,, on that admissible set A, which is equal to g on the boundary on that set if you just minimize that energy function then you get the solution of that problem. So, these are the two properties which you have seen.

Now, the point is see Laplace Equation most of the times, the idea is this, most of the times if you need to solve this equation. So, why do we need to solve this equation, essentially we want to see what are the functions which satisfies this equation. And most of the times it is actually very difficult to actually explicitly write down that function, the solutions.

What is the next best thing? The next best thing is to look for some information on the solution. So, let us assume that we cannot solve the problem. And most of the times we do not need to solve the problem, we just need to understand how the problem behaves. So, let me put it in this note, most times one needs to understand how solutions behave, you understand behave.

So, let us say that where the solution attempts a maxima or let us assume that what sort of solutions are there, are all solutions bounded or not that sort of question. So, we want to look at those properties. Now, to study this equation, first start with the Laplace Equation. So, basically, we start with this equation, we start with Laplacian of u equals to 0. So, basically harmonic function. So, we start with harmonic function, you remember any function which satisfies this equation is called a harmonic function.

So, essentially what I meant is, see, we want to talk about the properties of harmonic function. See this is why it is important, let me tell you why it is important, where it is important, what can you do with this information? So, let us say you want to study later on in your course, of course this is, the fundamental operator in PDE, if you want to study let us say harmonic analysis, harmonic analysis then also this is the fundamental object which you study in harmonic analysis, analysis, harmonic function. And this is also a fundamental object which you study in differential geometry. And you also get it in functional analysis.

So, you see there is a lot of, other than mathematical physics there is a huge application in physics for this operator, but along with that, if you just concentrate on mathematics, there are a lot of different places where this kind of operator pops on and you have to know what are the properties of this operator, of the solutions of that operator without explicit solving. So, these are called harmonic functions as I have told you.

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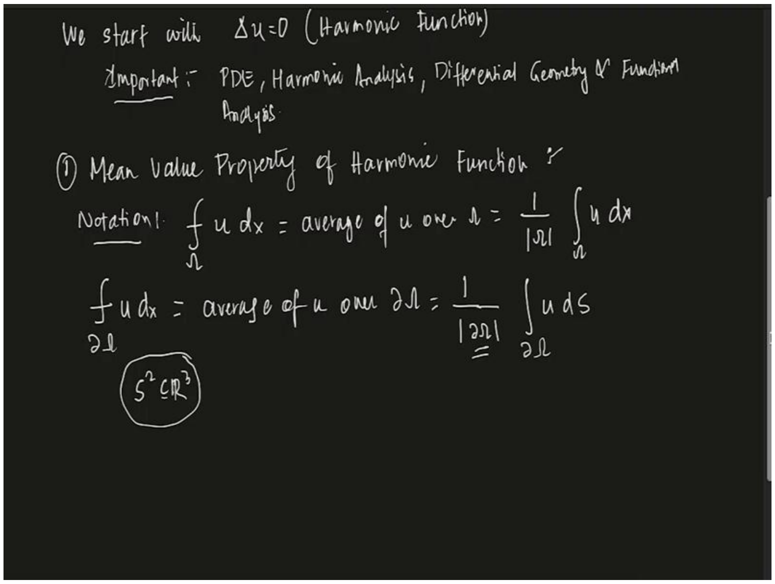

And the first property which I am going to prove is a very important property is called a mean value property. So it is called the mean value property of harmonic function. So please remember this thing, that this is not a property for any function, you cannot expect that any function will satisfy this property, but only harmonic functions. So what it says is this, so, of course, we are assuming omega is open and all that, I am not going on about it.

So, we will define something. So, clear simple notation, which I want to write here is if I write it like this over omega integral, this means the average of u over something, let us say d of x, the average of u this over omega, this means the average of u over omega. And what does it mean? It means that you take the integral of u over omega and divide it by the volume of omega.

If you know, measure theory basically you just divided with the measure of omega, so 1 by in terms of measure theory, you divided it by measure of omega integral over omega Udx. If you do not know, measure theory, do not worry about it, it basically says that you just divide it with the volume of omega. And similarly, if I write it like this del omega Udx, then it means that the average of u over the boundary of omega and that is defined by 1 by the measure of del omega.

This is the surface measure, integral omega Udx. So, let me explain again, if you know measure theory, you do not need to know it. But if you know that this is essentially the integral of Udx, so, basically, the sum total of whatever u is in the area of u and you divide it the area under u and divide it with the measure of the set over which you are integrating. If you know measure theory in that terms I am saying.

If you are not comfortable with measure theory, it essentially says that you take the total area under the curve u, and you divide it out with the area of, the area in R2, in R3 or Rn you call it a volume, volume of omega. So, let us say in a ball it is just basically whatever is inside the volume of the ball, if you take omega to the ball and we just divide it out that is your average. And similarly here on del omega, it is the same average of u over del omega. And in terms of measure theory, you say it is integral of u, sorry, this is d, sorry, I have to write it properly. This is mistake. It is ds. And this is del omega.

So basically you take or whatever u does, so basically, you take all the points in u sum it up, that is what integral is. And after that you just divide it out with a measure of the boundary, this is the surface measure, surface measure of the boundary of omega. So if you are not comfortable with measure, then what it says is you take the integral of u over the boundary and you divide it out with the surface area. So, let us say if you are thinking of a ball in R2 think of a sphere in R2, so let us say S2 subset of R3.

Think of a ball, just your, let us say cricket ball kind of thing. You take the surface of the ball, that is the surface and that area of the ball you guys already know. So basically, this is 4 pi R. And this is 4 by 3 pi R cube, pi R square sorry, yeah, pi R square. So essentially, what I am trying to say is, you are basically looking at the surface of the ball, you take the surface area of the ball and you divide it out, omega is an arbitrary thing, you just take the surface area and divide it to out.

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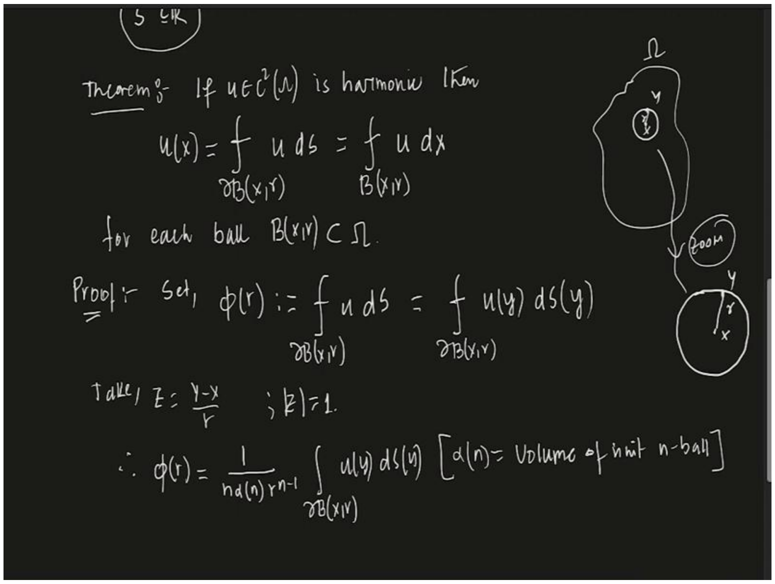

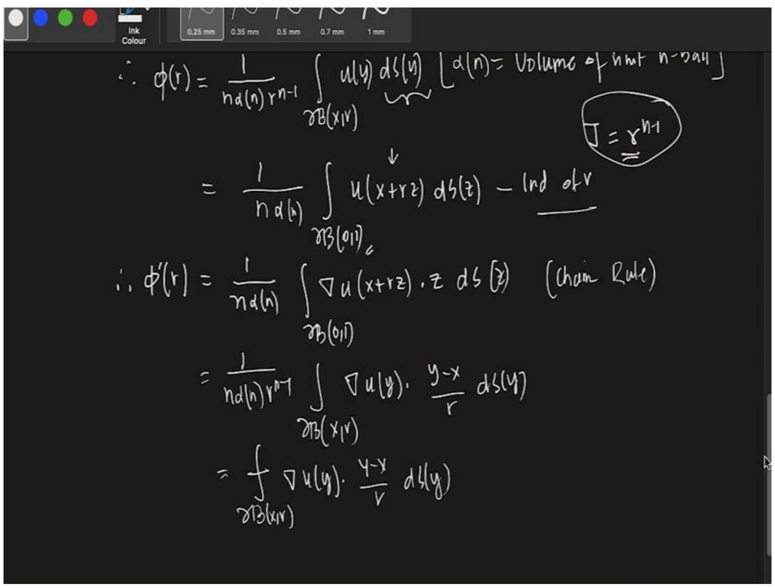

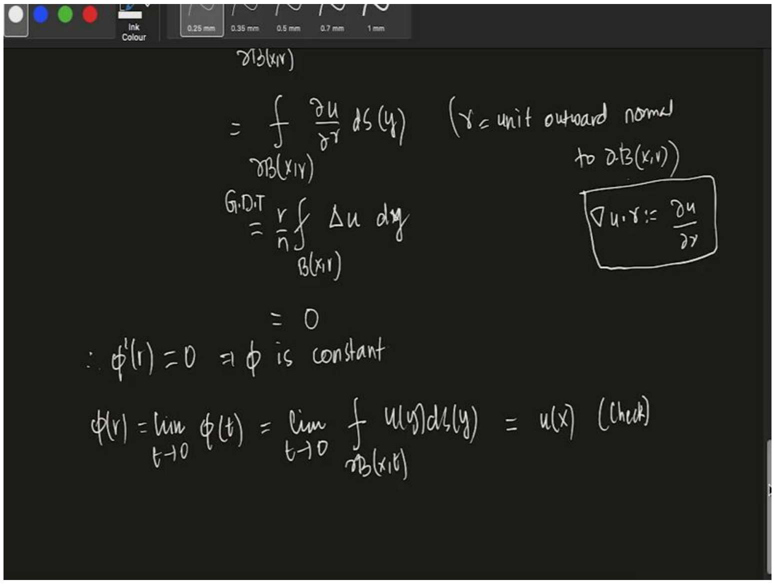

Now, let us talk about the theorem. So this is one of the most important theorems in all of analysis or at least PDE. It is called a mean value theorem for Laplace equation. And it says that if u is in C2 omega is harmonic, so if you are looking at a harmonic function, which is in C2 omega, then u of x can be written as integral the average of u x, r Uds, this is equals to integral Bx,r Udx. So what it is saying is if you take the average of u over the ball, the surface of the ball, so this holds for all for each ball B x, r which is containing omega.

So essentially it is saying that you take omega, whatever omega is let us say omega and you are given a harmonic function in omega, then you take any point x, you take any point x, you want to know what is the value of u at that point x, you take any ball around that point, if you take the average of u over the ball, over the whole ball, then that is going to be u x. And similarly, if you just take the average of u over just the surface of the ball, then also you are going to get ux. That is harmonic mean value property means, very, very important.

So, essentially, it is saying to think of u as a temperature distribution in a equilibrium. So, basically after some time the temperature stabilizes. So, temperature distribution what it is saying is, if you just look at, if you want to find out the temperature at one single point, you look at the neighborhood of that point. The average temperature at that point, if you divide it out by the total area of that whatever the region you are looking at, then that will give you that temperature at that point. So that is what it is saying, that is intuitively quite clear.

So, the proof, the proof is a little technical proof. So, let, I will explain every step of it, but what I need you to do is please, I am requesting you to go through this proof yourself, just write it down yourself properly, every single detail, I will explain every single line here, but you have to do it yourself. So, what you do is this. So, first of all, fix or maybe set a new function phi of r, phi of r, which is integral over the average over the ball, the boundary of the ball Uds. This, I am defining. See, this is a function of r. Why? Because if you change r, r is greater than 0, r is greater than 0 radius of the ball.

So, basically, this is a function of the radius, the radius changes this phi of r changes, that should be the case. So this is a function of r. So, I have defined this thing. Now, this let us write it properly, this is essentially del B x, r u of y ds y. Now, what I am doing is this y what is y, y is on the boundary, y is on the boundary here and that is x. So, if you take Z to be y minus x by r, then y is basically x plus Z r. And, what do you think Z will be, Z will be on the surface of the ball with center at 0 and radius 1. So therefore, if you see, what is the mod of Z, mod Z is 1. Because y minus x is basically r.

So this is the ball with radius r essentially, let me zoom it. Zoom version. That is y and this is the center x, this is the point where we need to find u, I want to find what is the value of u at that point. And I am taking any, I am just fixing a ball, fix r so the ball is fixed. I am taking any y on the surface of the ball, on the boundary. So y minus x, this y minus x this is vector and the radius is r, the radius is r. So if you take Z to be y minus x by r what is mod Z? mod Z is 1. So therefore, I can write so essentially, what, where is Z lying on? I am just changing the variable in such a way that Z is lying on the boundary of the ball, which is centered at 0 and radius 1.

So phi of r can be written like this, just change the variable here. See, when I divide this thing out, so basically I am taking the surface area of the ball, just the surface area of the ball. So that is given by in Rn that is given by 1 by n alpha n, alpha n is the total volume of the ball alpha n is the volume of in dimensional ball, r power n minus 1, if you write this thing properly, del B x, r, U of y ds y. And here alpha n is the volume of unit n ball. Unit n ball means, you take a ball with center at 0 and radius 1, alpha n is the volume of that ball.

See, now, if you write it properly, you see, I just want to change it to Z. So, that will give me n alpha n integral del B 01, u of I will change y to x plus rz. See y is x plus rz and ds y is getting changed to ds z. And for this change of variable from y to z there is a Jacobian. What is the

Jacobian? It is the r power n minus 1. Please check this part from this, y I am taking this y to be, y minus x to be rz.

So, if I did this change of variable, the Jacobian of this change is r2 over n minus 1. So that r2 over n minus 1 and this is getting cancelled and I have this. So, this is their and therefore, therefore, phi prime of r, can I talk about phi prime of r? Of course, you see, because phi is a differentiable function u is a continuous function, the integral of a continuous function is differentiable, so phi is differentiable.

Now you see, it is n alpha n. If I take the integral with respect to r, this derivative, phi prime of r is the derivative with respect to r, if I take the derivative that I can take inside y, because this integral is with respect to Z, it has nothing to do with that. So I can take it inside. And you see this ball, this is a fixed ball with center 0 and radius 1. If you do that, then that will give you the derivative of this with respect to r so which is gradient of x plus rz, dot z, ds of z.

Now, what I want you to do is take 2 seconds, take 5 seconds and think about why did I do this change of variables. So maybe you have got it, maybe you did not. What happened is, see I want to see what is phi prime of r. If I want to do phi prime of r over here, if I just take the derivative here, this here, the integral is with respect to r, so if I am taking the derivative, I do not know how to handle this something. So because this integral is with respect to r, and I am taking the derivative with respect to r that is a problem.

So, that is why what I did is, I did a change of variable and made this integral independent of r, you understood here R is fine, but here on the integral if there is r there, we are in a problem. So, we do not want that that is why I just made a change of variable made this integral independent of r, so independent of r. Once it is independent of r then I can take this derivative and this ddr that will go inside without affecting this integral it will go inside and it will, using chain rule I can just write it down so this is chain rule.

Again, I am urging you to please go through this calculation. So, if this is the case, now, you see, I can use my Green’s theorem, here. So once it is inside, now I can just replace it by y again, let us just change it to y. So, if I change it to y, again r power 1 by r power n minus 1 the Jacobian thing will come out and this will become 1 by n alpha n r power n minus 1. See, I just wanted to take the derivative that is why I changed it to z. Now I am just changing back to y. So, this is x, r, the integral, and this is gradient of u, y; z is y minus x by r and ds of y.

See from here to here, the Jacobian is r power n minus 1 if you are going the other way, the Jacobian is 1 by r power n minus 1 that is why this. So, this is essentially again del B x comma r gradient of u, y x minus sorry, it is y minus x by r, ds of y, this is there. So, if this is there then what can we get? You see from here del B x, r, gradient of u, y times y minus x by r ds y, this, what is y minus s by r, if this is the ball y minus x by r that actually points to the unit outward normal.

So, this is essentially del u by del gamma ds of y, that is clear why it is gamma, what is this gamma? Gamma is the unit outward normal to del B x, r. See if this is the ball, what is the unit outward normal in this direction, it is y minus x by r. So, and you know that what is del u del gamma, del u del gamma is gradient of u got gamma, gamma is this one. So, that is why it is del u del gamma. See gradient of u dot gamma that is defined by if you do not know this thing please read it, it is del u del gamma. This is what we define and your gamma in this case is y minus x by r, because it is the unit outward normal and that is why you write it like this.

Once you do that, you see if you, now can use Gauss you remember Gauss divergence theorem, you use Gauss divergence theorem and you can actually say this is r by n, because, you see if you multiply it by r by n it will be you can write it like B x, r. Please, I want you to check this part how this r by n is coming. Why r by n is coming, because you see I am defining, I am changing del B x, r to B x, r. So, from the boundary of the ball, I am changing it to the ball itself, inside of the ball. So that is why when I am changing it, the volume is r to the power n, alpha n r to the power n, that is and here there is r power n minus 1, that is why this r is there.

So please check this part from here to here, just break it up, it is not very difficult just break it up it will come. And this is Laplacian of u dx, if you remember this formula, we just did this thing first class. Again, I am urging you to please check that how this r by n is coming, just break it up this is equals to 1 by n alpha n r power n minus 1. And when you change this thing integral over B x, r you can divide it, so basically without this cross sign, you just write it as 1 by alpha n r power n that is why that n extra is there and r extra is there.

Now, Laplacian of u on B x, r, we already know that it is 0 because Laplacian of u is 0 on the whole domain, sorry this is y, so, on B x, r this is going to be 0. See Laplacian of u, I have assumed that u is a harmonic function in omega B of x, r is contained in omega. So, Laplacian of u is 0 on B x, r, so, this is 0. Now, so phi prime r is 0, therefore, phi prime r is 0. What does that say? It says that phi is constant.

And so, phi of r what is phi of r, phi of r is limit t tends to 0 phi of t and there is limit t tends to 0 del B x, t u of y ds y of course it is because phi is constant. So, phi at the point 0 and phi at the point r they are going to be same essentially. So this is true and it seems phi of t is defined by these I am just writing it like this limit t tends to 0 this.

Now, this thing if you calculate please check this part this is going to be u of x. So for a continuous function, you can check this integral if you take with the limit t tends to 0 this will be u of x. So this part, you have to check it yourself. Take a continuous function, so essentially, how do you check this thing? So a small hint, how do you check this thing, you take 1 by this is n alpha n t to the power n minus 1, u of y ds y minus u of x, you have to show this is small as t tends to 0.

So these you can write it as 1 by n alpha n t to the power n minus 1, del B, integral over del B u of y minus u of x, ds y. Please remember this thing see, what is happening is this, I can take this u, x inside. Why I can take this u, x inside, It is less than equals to. Why I can take, see this is equals to if you take equals to you can write it as, u y minus u, x ds y, if you do that, there is a, if you take this integral here also, then there is a surface area of this ball which will get included that will get cancelled out and that is why this 1 is there.

So, if you break this thing up, it will look like this and after that I am just taking a mod inside that is why it is less than equals to. If you are getting confused here, you can just later ask why it is happening. But it is not very difficult you please check this part and after that you see u is continuous. So, on this ball as t tends to, as the radius of the ball decreases, this actually comes together, comes very close to each other and that is why you can make it very small, hence this. So you have to check this part.

So, that is what we showed is this phi of r, this is equals to so integral over, so the average of u over the boundary is equals to u of x. Now, I just have to show the other way that u of x is also the average of u over the ball. Now, integral over B x, r u dy, this is equals to if you remember this is the radial how to integrate radial functions, del B x,s u ds. This is a surface integral, ds, this is small ds. So, essentially what am I doing?

You see, you want to integrate u over this ball. So any function let us say you want to integrate over some ball. So, essentially, what you can do is you can take a small boundary of a ball, some small boundary like this of some smaller ball. So I want to integrate it over for this whole ball. What I am doing is I am taking the same center, I am taking a circle with the same center and just looking at the boundary of this thing.

So, first of all, if you integrate this boundary and after that you run this radius of this smaller wall from 0 to whatever let us say here it is r, 0 to r then you get the whole area. So, that is what the idea is, first of all integrate over an arbitrary s radius ball s is less than r and after that you run s between 0 and r and you are done. Now, if you calculate this thing, what exactly is this thing, this is 0 to r, if you know from here what do you get is n alpha n, it is n alpha n s to the power n minus 1 if you calculate this thing, because u of x is constant.

See all of this is u of x and there is no x here included. So u of x is there and t of s. So, u of expenses you can take it outside and when you integrate this particular thing, you get alpha n r power n. And hence, therefore, if you multiply this alpha n r power n here 1 by alpha n r power n B x, r u dy that is going to be equals to u of x. So, u of x is the average of u over the ball or the average of u over the boundary of the ball. And this is the mean value property. Now, what we are going to do is, so, this is, please remember this, that this happens, because this is a harmonic function. So, given a harmonic function, it will satisfy the mean value property.

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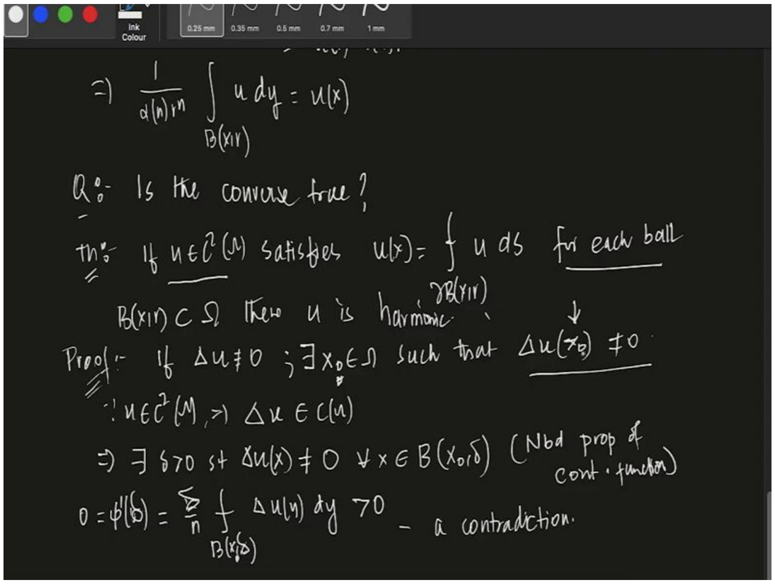

Now the question is this, let, is the converse true. So, question is, is the converse true So, there is this theorem which I am going to prove now, the theorem says that if u in C2 omega, satisfies u x equals to integral over the average of u ds, if this satisfy this for each ball B x, r which is containing omega then u is harmonic. Let me write it properly because I wrote, I cannot read what I wrote is harmonic.

So, essentially what it is saying is this theorem and the above theorem if we just add it together it says that if you have a harmonic function it will satisfy the mean value property and if a function is there, which satisfy the mean value property then that function is going to be harmonic. So, this is an exclusive properties of harmonic function. So, let us look at the proof, the proof is quite simpler so it is not very difficult.

So, if Laplacian of u is not equals to 0. So, essentially what is happening is this, see we have to show that if it satisfies the mean value property then u is harmonic. So, let us say u is not harmonic, what does that mean? It means that there exists x naught in omega such that Laplacian of u is at the point x naught and the point x naught is non-zero. And see u is in C2 omega, since u is in C2 of omega this will imply the Laplacian of u is in C omega because Laplacian of u is twice differentiable uxx plus uyy, u is in C2. So, uxx plus uyy that is continuous.

Now, so, Laplacian of u is a continuous function, at some point on the domain that is non-zero that implies there exists a neighborhood delta greater than 0 such that u of x is non-zero for all x in B x naught delta. So, essentially what it is saying is it is saying that u x so this is by the neighborhood property of continuous function, neighborhood property of continuous function. I am quite sure all of you guys know this.

So, see what is happening is I am assuming that there is a point that is non-zero, if this is not harmonic, if that happens since Laplacian of u is continuous function u is given to be C2. So, Laplacian of u is continuous since Laplacian of u is a continuous function and u at one point is non-zero then by neighborhood property of continuous functions we can say that there is a neighborhood around x naught where u is non-zero. If that happens, then you can define phi as above. So, define phi as above phi of r to be that whatever we define just earlier.

If we do that, then we have seen that phi prime of r is 0, just earlier whatever we did phi prime of r is 0, for the same phi and that is equals if you remember just look at the earlier calculation, I am not doing that party again r by n integral B x, r, Laplacian of u y dy, just look at the earlier calculation see you do realize that just a few minutes back we have proved this thing, just look at the other part, it is r by n integral B x, r the average of this Laplacian of u y dy.

Now, see this I can change this r to delta, this holds for any ball, so, I can change r to delta and this happens now on B x, x you change it to x naught, x you change it to x naught. So, you see the ball with center at x naught and radius delta what is Laplacian of u, Laplacian of u will be non, not equal to 0. So, for now, let us just assume that it is positive, if it is non-zero in this place, it is either positive or negative. So, that is the property of a continuous function. If it is non-zero it is either positive or negative.

So, let us say it is positive. If it is positive, then this particular thing has to be greater than 0. If it is negativity it is strictly less than 0 it cannot be 0 of course. Because in that ball it is either positive or negative in the whole ball. So, the whole integral is going to be positive, if that happens, you are basically showed that 0 is equals to this is equals to this which is greater than 0. So, basically you showed that 0 is greater than 0, that is a contradiction. So, I think it is clear.

So, let me explain again, what we were doing. See, essentially, we are saying that if you have a C2 function, and it satisfies u x equals to this mean value property, the average of u over the boundary of a ball, if that is u of x for each ball, then for each ball, it is very important not for a fix ball, but for every ball in omega, then u is harmonic that is what it is saying, how do I prove it? Let us say it is not harmonic, if it is not harmonic, then essentially there is some point in omega where Laplacian of u at the point x naught in non-zero.

Now, see, if u is in C2 of omega it is given. So, u is in C2 of omega means u of xx and u of yy are both continuous, so the sum is continuous, so Laplacian of u is continuous function. Now, we just assume that there is a point where u is non-zero. So basically, Laplacian of u is non-zero. So, basically Laplacian of u is a continuous function, which is non-zero at single point neighborhood property gives us a neighborhood of x naught when Laplacian of u is non-zero.

Now, you just do whatever we did earlier, we define a few of our like the average of u over the boundary, and we have shown that phi prime of r is 0, that we did, so, that is equals to what will happen, if we just look at the earlier calculation, which is given by delta by n integral over the ball the average over the ball of Laplacian u dy, just the calculation which you did please just look at that calculation.

Now, what is happening is this see in these ball, so, this holds for any ball which is in omega. So, I can just take this particular ball, x naught and delta. So, this particular thing u at the point x naught is given by this phi prime of r, this is phi prime of delta let us write it as delta, phi prime of delta that is going to be 0 if you change r to delta, and b of r naught, see in this thing, either Laplacian of u positive or negative in the whole ball.

So let us say it is positive, if it is positive, then we have shown 0 is greater than 0 if it is negative, then we have shown 0 is less than 0. In both cases, it is a contradiction. So, what is happening is, there cannot be a point x naught where Laplacian of u is non-zero and hence u is higher. With this we are going to end this lecture.