**Advanced Partial Differential Equations**
**Professor. Dr. Kaushik Bal**
**Department of Mathematics and Statistics**
**Indian Institute of Technology, Kanpur**
**Lecture No. 07**
**Uniqueness and Dirichlet Property**

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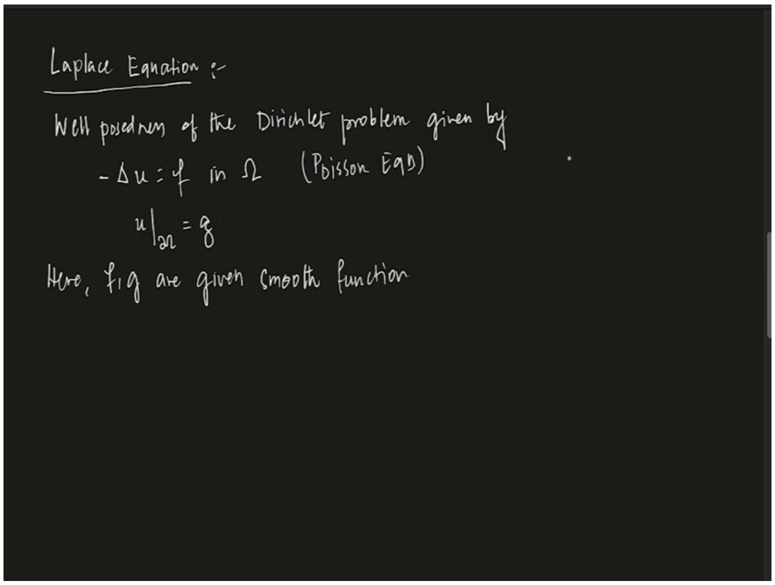


In this lecture, we are going to talk about fundamental solutions. Now before we move on, we are going to talk about Laplace Equation in this video, Laplace Equation. So, this is our continuation video on Laplace Equation and essentially let us just clear it up what we exactly want to achieve here. So essentially, we are going to talk about this is what we need to understand the well posedness of the Dirichlet problem given by, I will explain why I call it a Dirichlet problem given by minus Laplacian of u equals to f in omega as usual omega is an open subset of Rn and u restricted to the boundary of omega is g. This is the problem.

So, as you can understand this is a Poisson problem, this is a Poisson equation, because Laplacian of u this is called Poisson equation, if you remember, this is after a mathematician by the name of Poisson and French mathematician. And if f equals to that is Laplace Equation, if you have an arbitrary F, that is a Poisson Equation. And u when you restrict u to the boundary of omega that is going to be some function g.

Now here, f and g are given smooth functions, given smooth functions, I mean, here, I am not exactly specifying what sort of functions are there, but let us just assume there are smooth functions. So, why it is called Dirichlet let us understand this thing, see let us say you are given a Poisson equation minus Laplacian of u equals to f, with this you can specify different type of boundary conditions here. So, first of all, let us look at the equation and the boundary conditions.

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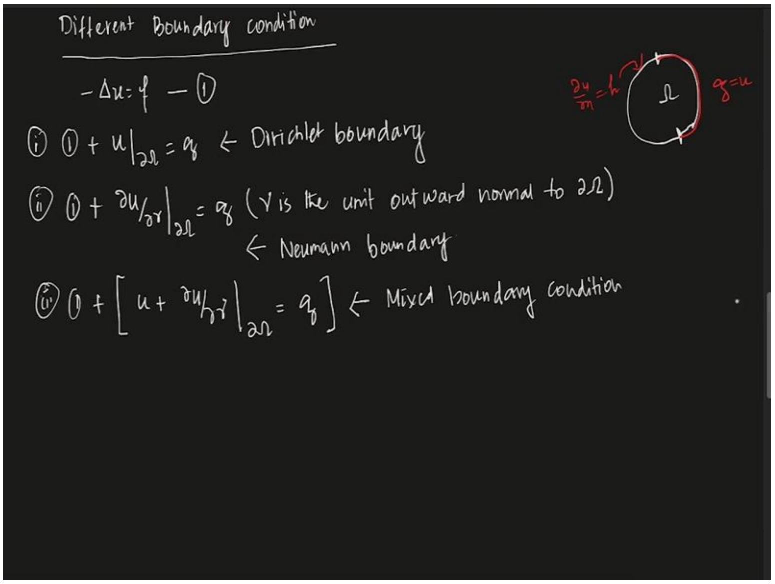


So, different boundary conditions, different boundary conditions. Let us assume that you are given a Poisson equation which looks like this Laplacian of u equals to f. Let us call that one. So, 1 plus this boundary condition that u restrict to the boundary is the g, g sum function. So, essentially you are given the information on u in the domain. So, Laplacian of u you know that it is equals to f in omega and the boundary, on the boundary the unknown function u will look like g. If this sort of boundary condition is given we call it a Dirichlet boundary condition. Dirichlet boundary, clear, Dirichlet boundary.

Now, let us say that 1 is given plus this sort of boundary condition is the there, when del u del gamma at the boundary, that is equals to g let us say, where gamma is the unit outward normal, outward normal to del omega. If that is the case, then this sort of boundary condition is called the Neumann condition, how do I put it, so that this is called a Neumann condition, Neumann boundary.

There are other boundary conditions also let me give you another example. So, let us say that is 01, that is 02, and 3. There are other boundary conditions you can just specify, but for now, we will just give another condition so 1 plus. So, obviously, this equation is there, equation plus a boundary condition. So, boundary conditions look like this something like let us say, u plus del u del gamma. So, this whole thing restricted to del omega, let us say that is equals to g. So, you have this equation and you have this boundary condition u plus del u del gamma restricted to the boundary is g. This sort of boundary condition is called a mixed boundary condition, mixed boundary condition.

Now there are other conditions also. So, let us say in some part of the boundary, let us say, this is your omega. That is your omega. Let us say, this is your omega and that is your boundary right. Now, it may happen that in this part of the boundary, let us put it in red color, it may have been that you are given a boundary condition like in this part of the boundary, it is g and in this part of the boundary, so u is g here on the boundary.

And del u, del eta let us say that is some h, some function h on this part of the boundary on this part. It may happen that you are given this equation and this sort of boundary condition. And that may also happen. So, there are different options, but for now you just restrict our self to the Dirichlet boundary condition. So, let us see what happen.

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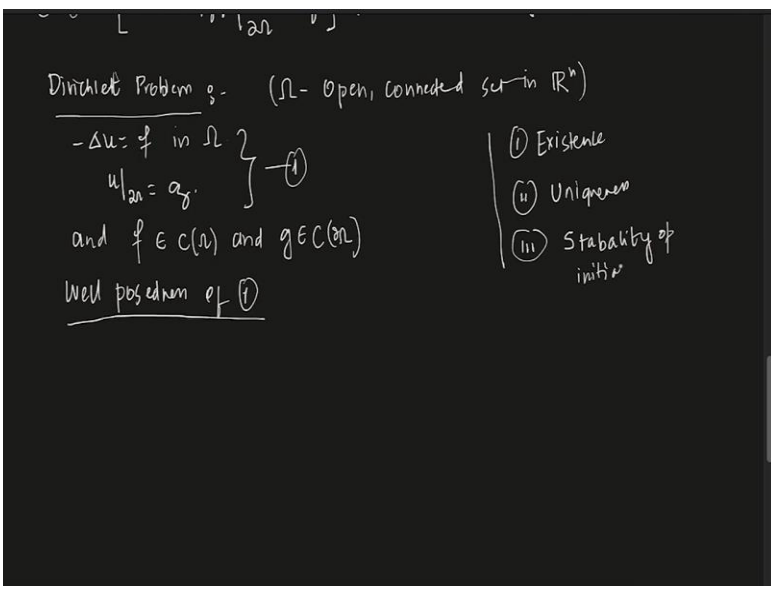


So, Dirichlet problem, we say, it is a Dirichlet problem clear. How does the Dirichlet problem look like, minus Laplacian of u equals to f in omega. And u restricted to the boundary del omega, this is going to be g. And we will assume if in c of omega and g is in c omega del omega. The continuous on the boundary and f is continuous on omega. So, this is the only condition, which I am just taking.

Now, the thing is this, what do we need to do here? If you remember properly, when working with this sort of problem, we said that our main motivation to study this sort of problem is the following. We need to understand that of course, here, whenever we are doing omega is assume to be a open connected set in our Rn. So, that is always assume. Now f is continuous and g is continuous on the boundary, we want to talk about the well posedness of the problem, well posedness of let us say that is your 1 of 1, well posedness of 1.

Now, first of all, let us start with this thing. See, we will not directly attach this problem. We will take it part by part. So, first of all, what we are going to do is we are going to assume, we are going to assume that, so well posedness what does well posedness, if you remember first thing is existence, existence. So, first of all, you have to find that if there is a solution or not. Number 2: uniqueness that is that if there is a solution if it is unique.

And number 3: is the stability, stability is I mean, how does the solution behave. So, the solution must be small when the initial data is more. So basically, let me write it as stability of initial condition, let us put it this way, initial data. Now, let us look at what are the problems associated to this.

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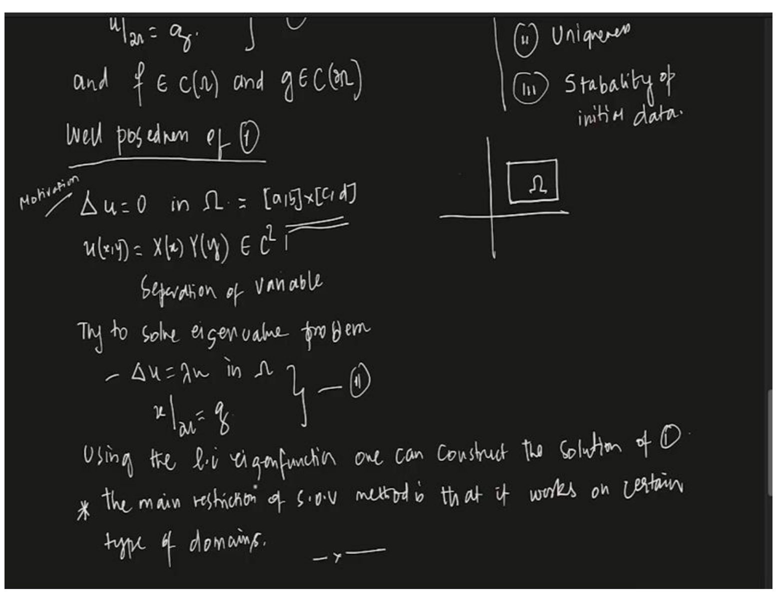

So, let us see most of you guys, I am quite sure, you know let us say, if I am asking you to solve this problem, Laplacian of u equals to 0 in omega, clear. And omega is given to be something like this, let us say in two dimension. So, this is just a motivation, motivation. In two dimension, let us say it will look like this, a, b cross c, d. So, in this thing, I want to solve this equation. So, in this rectangle, let us say, I do not know maybe something like this. So, this is a, b cross c, this rectangle R let us say, this is omega.

Now if I am asking you to solve this problem, what do you do? You can use a separation of variable and what is just roughly speaking, it will look like this. So, you are basically looking, trying to find a solution u of x, y, which will look like some function of x times some function of y. Now, of course, you are thinking of this as a C2 function. Now, since this is a solution, you are thinking of this as a solution of this so, you can calculate uXx, uYy and after that, put it in that equation, given the boundary condition, you can use Fourier series to find out what the solution is, you can do that.

Now, let us say from here if I am asking you to, so this can be done, so this is separation of variable. So, as you know in introductory video itself, I have told that we are not going to study about separation of variable in this course because that is not, to be very frank is not very interesting thing to do. I mean, it is just a method to solve it, very significant method, but nonetheless, just a one method to solve this.

Now, the thing is, let us say you are, you want I am asking you to solve this problem, f and omega is in a, b cross c, how do you solve this thing. So, if you remember, in separation of variable methods, what you do is first of all, you try to find the eigenvalues, try to solve the eigenvalue problem. Problem, Laplacian of u equals to lambda u, minus Laplacian of u in omega, that is the Omega a, b cross c, d, and you restrict it to the boundary detail omega is g.

You just solve, you just try to solve this eigenvalue problem. And then I mean, you can use the data from here to actually I mean, you can use the function from here, the solutions. And you can construct a solution for f. You can construct a solution for 1. So, using the linearly independent Eigen functions. So, once you solve this equation, let us say this is your 2, once you solve 2 you get some Eigen functions. So, when you use these Eigen functions using the circular. So, using the linearly independent Eigen functions, one can construct the solution of 1, you remember. So, that is the separation of variable.

Now, what is the restriction with this method? See, if this method works in any given domain, then we do not have anything to do, there is nothing to do here, you can just use separation of variable to solve this problem and you can look at the explicit solution, you use the explicit solution to, I mean, that is the point. Once you get an explicit solution, you do not have to anything existence, uniqueness, stability, all of these can be easily checked. But the problem is this separation of variable, please remember this thing, why we were doing this thing. The main restriction of separation of variable, this method is that it works on certain type of domains.

So, most, if you remember, just work out one problem within separation of variable you will understand it. See, if it is not in this form a, b cross c, d not in a rectangular form, then the separation of variable on what you can think of I mean of course, in there other coordinate system you can think of circle also on a disk and all you can actually solve it, but you are essentially whenever you are doing this change of variable to radial coordinate or spherical cylindrical coordinates, you are essentially doing something like this.

So essentially, for a domain, which looks like an rectangular kind of domain or like a spherical domain, that sort of domain separation of variable works. But most of the problems which we encounter that is not, I mean, not a given domain like this. I mean, it can be any arbitrary domain. What do you do there? That is the question. So, let us look at what do we do there? See, what we do is this.

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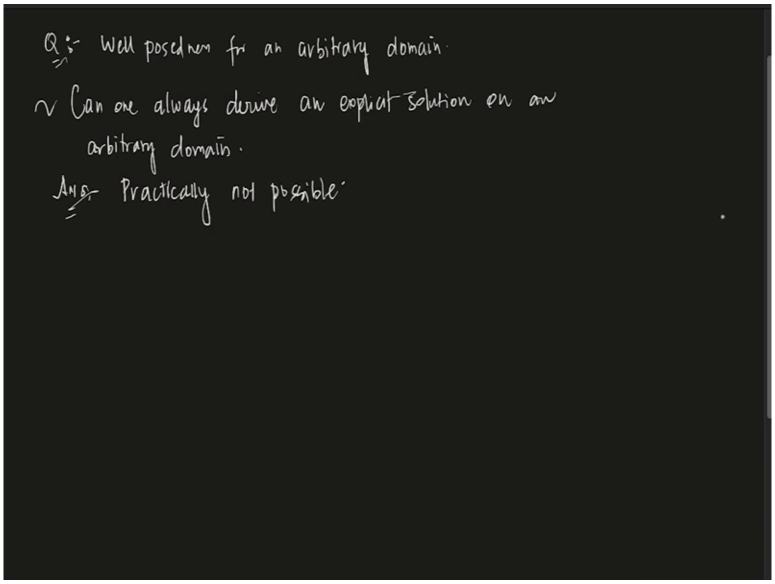


Now, question is well posedness for an arbitrary domain. I hope I made it clear what, why you need to study this. See, again, let me call the point is this, using separation of variables, you can of course, get an explicit solution. But the problem is that domain has to be very specific and to be very precise, the domain has to be a rectangular domain or a circular domain that sort of thing, a cylindrical domain. But most of the times in applications when you talk about domains, it may not look like that. The question is, how do you talk about well posedness for this?

So, you have to find an explicit solution. Can you always find an explicit solution? The question is, can one always derive an explicit solution on an arbitrary domain? So, this is the refreshed version of this. Well posedness, the refreshed version. But can you find an explicit solution for an arbitrary domain? So, the answer is this, theoretically it is possible, but practically you cannot find. So, practically not possible. Theoretically, it is possible, we will talk about that we will use Green's function to do that.

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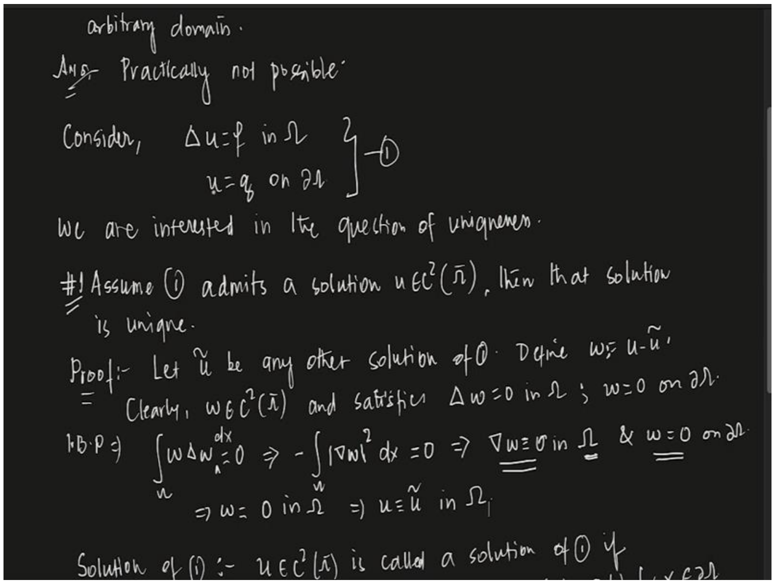


So, let us just explore more. So, let us just say that, so 3 are three components right of well posedness, existence, uniqueness and the stability with respect to the initial data. Let us talk about uniqueness first here, let us just get that out of the way. So, consider 1, consider this problem, let us start with Laplacian u equals f. In omega u equals to g on the boundary. We are interested in the question of uniqueness first. Now, please understand that we have not talked about existence, because that is a difficult part. First of all, let us say that we will assume that there is a solution, we want to see that if there is one solution, can we find other solutions from there for this problem? That is the question.

See, if we can find, if we can solve this problem, if you take f equals to 0, then the Laplace equation is also done. So, this is a much general problem. So, the question is can, so assume therefore, let us just put it this way assume, 1: admits a solution u in C2 of omega bar, clear. Let us say that one admits a solution of u in C2 of omega, then that solution is unique. So, this is a theorem, I do not know how to, let us say theorem 1.

So, what it is essentially saying is this, see first of all we need to study well posedness, we do not know whether that problem has a solution or not. But we are talking about uniqueness first, in the well posedness part, 3 part are there we just want to talk about uniqueness first, we are saying that if there is one solution, at least one solution, if you can just show that there is at least one solution, you can so that that solution is unique.

So, we are assuming that there is at least one solution will show that you cannot find any other solution that is only one solution, there is only one solution to work. So, how do we find it? Let us just take, look at the proof of this thing. Now, I hope you understand what I meant by one solution. So, there is a, maybe before I go on doing this thing, maybe I can talk about what do I mean by solution of this thing it will be better that way.

Let us say solution of 1, I just messed up the order a little bit solution of 1, what do I mean by a solution of 1 and then we can talk about this. So, solution of 1; as you can understand that u which is in C2 omega bar is called a solution of 1, if Laplacian of u at the point x is f of x for all x in omega and u restricted to boundary. So, u of x equals to g of x for x on the boundary for omega. That is the simple thing, it is only that u has to be in C2 to omega bar for now this is what we mean by a solution.

So, we are saying that if you can find a solution for this problem, then the solution is unique. How do I prove this thing? So, let u tilda be any other solution of 1. So, define w to be u minus u tilda. See, initially we have one solution u, I am also assuming that u tilda is another solution of 1. And you define w equals to u minus u tilda. So, clearly, w is in C2 of omega bar, see u is a solution and definitely this is in C2 of omega bar, u tilda is also a solution and what do I mean by a solution, a solution to this 1 is a C2 of omega bar function which satisfies this.

So, if u tilda is another solution u tilda has to be in C2 omega bar. So, u and u tilda are both in C2 omega bar you can see that the u minus u tilda the difference of those two, which is defined by w that is also in C2 of omega bar. And clearly, w is in C2 of omega bar and satisfies Laplacian of w equals to 0 in omega and w equals to 0 on the boundary, do you agree with me? See, if Laplacian of w Laplacian is a linear operator. So, Laplacian of w is Laplacian of u minus Laplacian of u tilda, Laplacian of u is also f, Laplacian of u tilda is also f. So, Laplacian of w is going to be 0. And moreover u is g on del omega and u tilda is also g on del omega. So, w is 0 on the boundary.

So, see, essentially if we define it like this w is going to solve the Dirichlet problem, w equals to 0, but not the Poisson problem. So, it is essentially a Laplace Equation, w solves the Laplace Equation. Now, what happens when we does that, then you take a integration by parts, again integration by parts use integration by parts, what does it do you. See, if you multiply this with a w, multiply both sides with w. So, let us say w times Laplacian of w integral over omega that is going to be 0, yes, because Laplacian of, I mean, let us just assume that w is not 0.

See, if w is, so let us assume that this is a problem given to us, if you can show that w equals to 0 is the only possible solution of this and then we are done. Now, because if w is 0, u is equal to u tilda and you are done. So, there is only one solution. Now, let us assume that w is not 0. So we can just multiply it by w so w Laplacian w that is also 0. Even if it is 0 it does not matter, but anyways. So, integral omega w Laplacian w is equals 0.

Now if you do an integration by parts here, what will it look like omega gradient of w square dx, this is with respect to dx I forgot to write dx dx. This is equals to 0, because there is a boundary term, but in the boundary it is 0, w is 0. So, the boundary term is not there. So, this is equals to 0, if you remember. So, this is the integration by part if you do not know this please look at the first lecture. There this integration by parts is done.

So, this will give you, see the gradient w square this is a positive function. You are integrating a positive function minus you can just throw it outside is not a problem. So, you are integrating a positive function over omega and you are saying that is equals to 0. So, the positive function gradient of w has to be 0. So, what does that means? It means the gradient of w is 0 in omega. The integral of a positive function is 0. So the function has to be 0. The function has to be 0.

Now, so if the function, so the function it has to be 0. So, here the function is gradient of w. So, gradient of w is 0 and, it is given that w is equals to 0 on the boundary. What is w? w is a C2 omega bar function, so it is continuous till the boundary, till the boundary of omega. On the boundary it is 0 and the derivative of w is 0 on the integer of omega, what does that say? It says that w has to be 0 in omega. So, that will imply, that implies w is 0 in omega.

See, if w is non 0 somewhere in omega is a constant that constant has to be there up till the boundary because gradient of w is 0 in omega, if the gradient is 0, then the function has to be constant. If it is a constant in omega and the constant is not 0, so on the boundary it cannot take 0. See, if w is a constant which is non-zero in the interior of the domain that cannot be 0 on the boundary because w is continuous function. So, w has to be 0 in omega and that will imply u is equivalent to u tilda in omega. And hence, there is a unique solution. So, you do not have to worry about this.

See, we also prove that the Laplace Equation has a unique solution. So, essentially what it is saying is this, if you have a causal equation which looks like this Laplacian equals to f in omega and u equals to g on the boundary of course, f and g are nice functions, if you are given something like this, then these functions admit a, I mean, you see, I do not know if there is a solution or not, if you just give me one solution, then I can guarantee you that that solution is a unique solution. So, uniqueness is out of the way, I mean we do not care about uniqueness anymore. Now, the question is if we can find such a solution or not and this is the place where it gets a little complicated.

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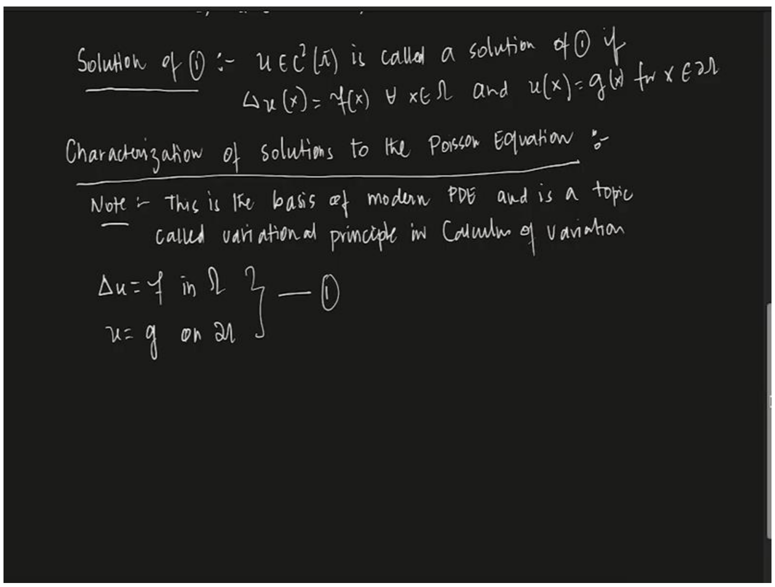


Let me give you a characterization of solutions and then from there we will do a list construction to find solutions. So, this is a characterization of harmonic functions, characterization, not harmonic functions, I mean, solutions of the Poisson Equation, so, characterization of solutions to the Poisson Equation. We want a characterization of this thing. So, this is actually, this characterization, this is not very, very important for this course, but I just want you to know how this works, but this is the basis.

So, essentially let me give you a small note this is the basis of modern PDE. If you want to do modern PDE, this is what you need to do, I mean this is all we do actually, this is the basis for modern PDE and is a branch and it is a topic I think, you can say. Topic called variational principle in calculus of variation, calculus of variation.

What I meant by this is the below thing, whatever I am doing right from the calculation of solutions, this for this course, it may not be very, very important, but I mean, of course, we will use this thing, but the thing is it is not very important, but this is most important if you are looking for a modern approach to PDE. So whatever we are doing is like 100 to 200 years old, in this course. But if you want to do the modern part, this is the modern part. So, just to give you the taste of that modern part I am just doing this part.

So, essentially, you are given this Poisson Equation. So, let me write it down. Let us say Laplacian of u equals to 0 equals to f in omega, u equals to g on the boundary, our usual problem, we are calling it 1. So, this is your 1.

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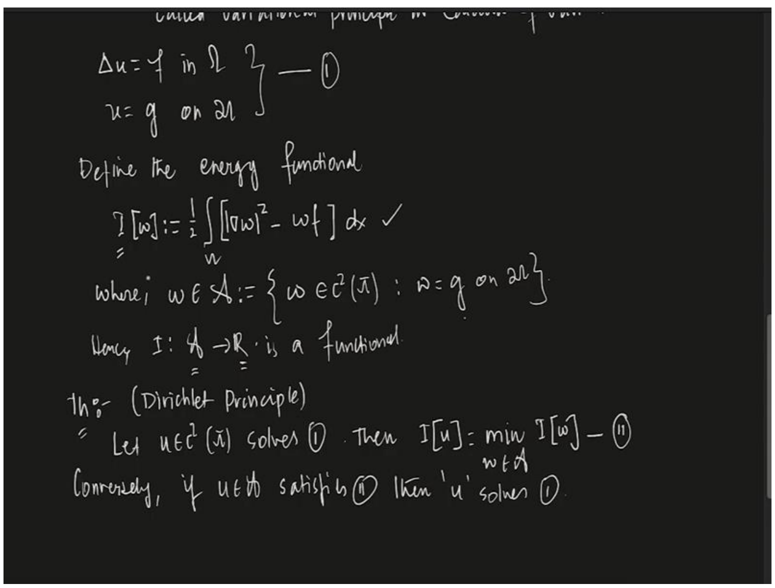


Now I want to talk about the solution of this problem. And while talking about the solution of this problem, I will define the energy functional. So, what I mean by this is I will define a new object for now you just think of this as a new object called I of w, and that will be given by integral over omega half gradient w square minus wf, dx. Maybe I can put it in a small dx. So, that is I of w and it is not like, I mean it is coming from here sum, it is not like that there is a proper scientific way of how this is coming and that is what we are going to, I mean, explain now.

Essentially let me just put down where w square w belongs to this set A, and how do we define this set A? It is defined as all the functions in C2 omega bar, such that, so you define a like this, you define set A and you take w like this, such that w equals to g on the boundary. So, you are looking at a C2 omega function such that w equals to g on the boundary and you look at, I mean you define a I which is from this set to R.

So, you define I hence, I is defined from this set to R, is a linear functional, functional is just a function, we call it a functional because it is a function from a function space to R it is taking a function and putting it into, giving you back a real number that is why it is called a functional. It is not a linear function. It is a function we call it an energy function. So, essentially this actually, so, I mean encapsulates the total energy of the system, but let us not go there. Let us just talk about what is so special about this and what is the relation between 1 and this synergy functional.

So, for that, we are going to talk about a theorem we are going to prove a theorem. So it is called Dirichlet theorem, Dirichlet principle what it says Dirichlet principle? It says that assume let u is in C2 omega bar. And it solves one here I of u which is the minimum, w is in A I of w. So, what it is saying is, if you have a solution it is saying that see. Let us say I am looking for a solution of this problem.

Let us assume that you guys already know what the solution is if there is a solution of this thing that solution actually minimizes this functional is it clear, this functional is some object if there is a solution that solution will be in this A of course, it is and see what it is saying is if you put that u in I, so I if you put u in that I, in that function then that will give you I of u and what is I of u that will be the minimum of all these Is'.

So, it minimizes the any solution of 1, we have seen that any solution of 1 there is only 1 solution to this problem. So, what it is saying is that solution actually minimizes the energy functional. This minimizes this particular stuff I w. So, you can take any w from this set A does not matter that w, the value of w is always greater than equal the value of I at the point u where u is the solution of this problem.

There is a converse part also, conversely and this is very important. Conversely if u is in A and satisfies let us call this thing as 2, this 2 the minimum of this is I of u satisfies 2, then u solves 1. So, very important here and very, very exciting thing what it is saying is this, let us say that you have a function if you have somehow a solution. So, first of all, it is saying that you give me a solution. Once you give me a solution to 1, once you give me a solution to this problem 1 you guys already know that if there is a solution it has to be unique solution.

So, once you give that unique solution to me, I can say that that solution minimizes this functional I w, which is defined by this that solution minimizes. Conversely, if you can find a minimizer to this solution, so, basically it is saying that if you can find a u in A in this set, which satisfies this, it means that see it means that there is a function u where the minimum is attained, where I is attending is minimum that is u.

So, basically it says that if there is a function u where the minimum is attained, then you can say that that function solves 1. So, both sides are true this is called a Dirichlet principle. So, essentially, see if you want to solve the problem, this problem the Poisson equation in arbitrary domain, what you can do is we will just look at the energy functional and try to minimize the energy functional. If you can, let us say somehow managed to find a minimizer to this synergy functional, then that minimizer will turn out to be a solution to this problem.

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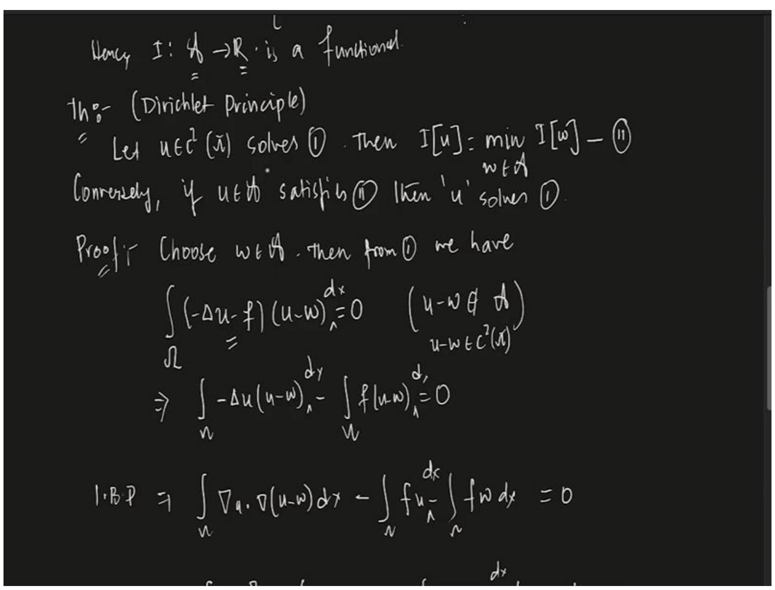


So, let us proof this theorem. So, choose, I will prove this part first. So, we choose w in A, then from 1 we have, see if I take everything together Laplacian of u minus f, and u minus w I am taking from A and u solve this 1. So, I am taking u minus w. So, this is on omega. So, you see Laplacian of u equals to f in omega. So, this particular thing is 0 irrespective of where w, u and w are in C2 of omega, in A, u and w is in A. So, u minus w is also in A, u minus w is not in A, u minus w is not in A, please remember this thing.

So, basically but it is in C2 of omega, but u minus w is in C2 omega par, why it is not in A because on the boundary this is going to be 0. But on A the boundary u minus w has to be g if it has to be in A. So, it is not in A, but in C2. So, basically the integral has to be 0 pie because this is 0, Laplacian of u minus Laplacian of u is equal to f in omega, if you can see, so let us write it like minus Laplacian this is just a convention nothing else. Minus Laplacian u equals to f, u equals to C.

So, see what I meant by this is, so it means minus Laplacian of u minus A is 0 in omega. So, if you multiply it by u minus w and integrate it (())(38:23) that is going to be 0. Now, we use integration by parts integration, if we use integration by parts, what happens? It is minus Laplacian of u times u minus w minus integral over omega f u minus w equals 0. So, what does that I mean there is a dx in all of these there is a dx, I am just forgetting to write it down there is a d of x.

Now, if you use a integration by parts here, so this is not an integration by part, it is just whatever I write here, I am moving integration by parts. So, that will give you gradient of u dot ingredient of u minus w dx minus equals to integral over omega fu minus integral over omega fw, again I forgot to write dx. So, I have this. Now let us just look at, now what happens see if you are doing integration by parts let us just write it like this. This is minus, this is equals to 0.

So, if you are doing integration by parts there is a boundary term, but u minus w is 0 on the boundary that is why we just multiply it by u minus w, u minus w that is 0 on the boundary. So, there is a boundary term if you remember in integration by that boundary term is not there. So, now what we do is, what it implies is this, see this implies that gradient of u square, gradient of u dot gradient of u square minus integral omega uf dx, this is equal to I will take that part here. So, integral gradient of u dot gradient of w, I hope this fine, minus integral omega uw dx. Again, I am always forgetting dx, dx is there.

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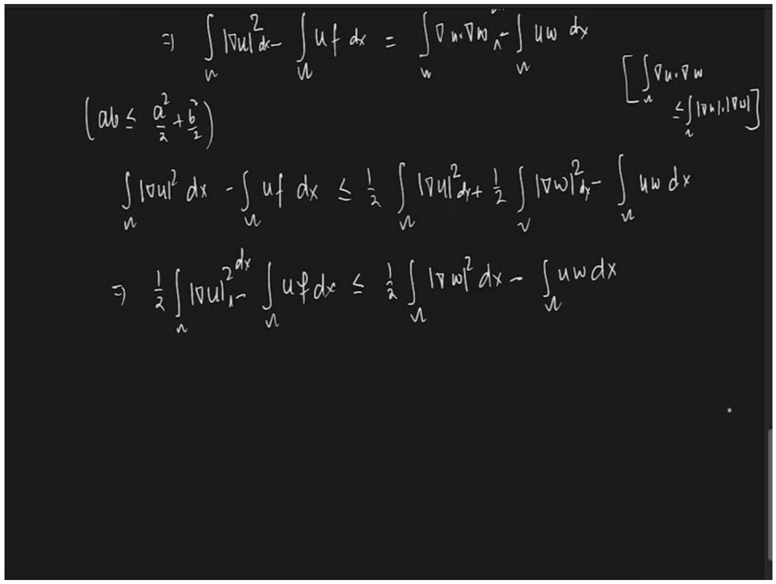


Now, if this is there see, what does that say this says that let us do this thing a little bit. You see, this is here I am using this thing, you guys already know ab is less than a square by 2 plus b square by 2. That is just an easy algebraic inequality. So, from here, what do you get? We get integral omega gradient u square dx minus integral omega uf dx. This is less than equals to half integral omega gradient u square, plus half omega gradient w square minus integral omega uw dx.

How is it coming, see integral over omega gradient u dot gradient, so integral over omega gradient u dot gradient w this is dominated by integral over omega gradient mod gradient u dot mod gradient w and then you use this inequality to get this. So, once you get this if you take this part here, it will give you half integral omega gradient u square minus integral omega uf dx.

Please forgive me I am always forgetting dx. So, this is less than equals to I mean, if you want you do not write dx, that is not a problem I am always forgetting dx. So, this is half integral omega gradient of w square dx minus integral omega uw dx.

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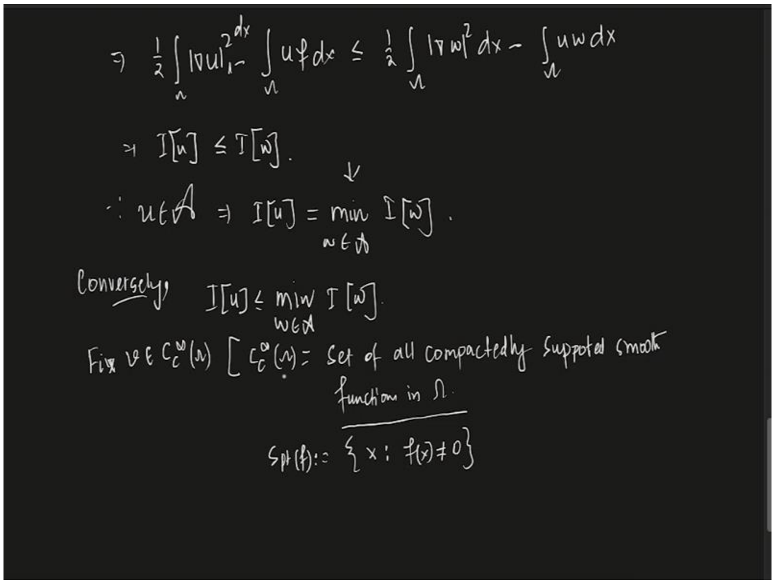


So, if you remember what is it, this is I of u and that is less than equal I of w. And since the u is in A. So, therefore, minimum I of u see w is any arbitrary if you remember, see w is an arbitrary point in A and u is in C2 omega such that u on the boundary is g. So, u is definitely in A. So, you see u is in A that implies I of u is equals to the minimum of I w is in A. So, it is saying that this is the infimum, but u since u is in A that infimum is actually and that is why this is the minimum. So, you have seen that if there is a solution that solution is the minimum of this energy function.

Let us look at the converse part. Conversely, let us say I of u is less than equal I of w and this w is in A, it is given to you. So, I should write it as a minimum of I mean, I want to write this, conversely I am assuming this I will show that you. So, maybe I can write it as conversely, let us say this is given to you minimum of w in A I of w. Now, I want to show that you solve the equation, how do I solve it?

So, you fix the v in cc infinity omega, so basically cc infinity omega I am not quite sure if I defined it. It is the set of all compactly supported smooth functions. Smooth functions in omega, compactly supported means what I meant by this is, so basically support of a function is a set of all those x such that f of x is not 0. The closure of this thing, if you take the closure of this thing that set this is called the support of f, this is the set support of f is the set where f of x is non 0, the closure of that set. So, you are basically if that this kind of things contain in omega, if you are looking at a function such that this happens, then we say that the function has a compact support in omega.

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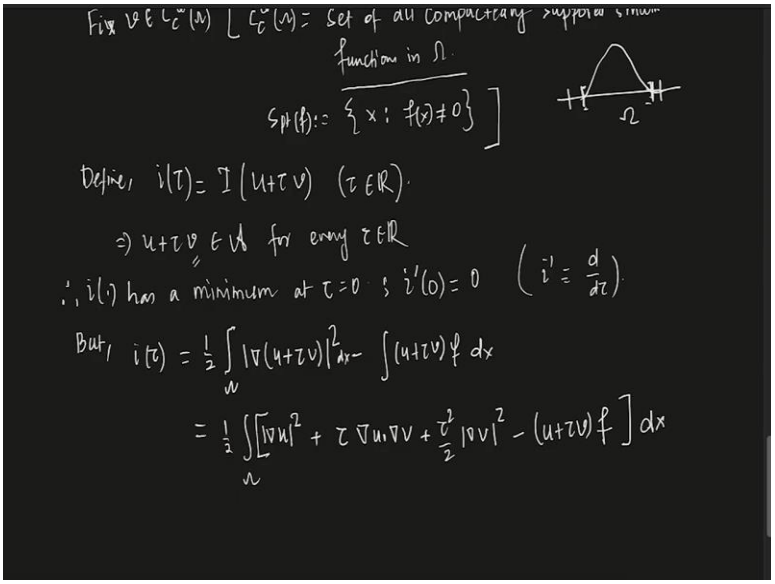


So, essentially, you can think of this, I mean, let us say a function which is 0 on the boundary and some non 0 part in the interior that kind of function has a compact support. For example, let us say that is your boundary, let us say this is your omega, one dimension, let us just look at one dimension and this sort of function are you looking. So, these are the compact support, clear, because it is non 0 in a compact set in this set, if you take the closure that is a compact set and it is 0 everywhere. So basically, that compact set is contained in omega and this is a, I mean, example of a function which has a compact support.

So, you start with a v which has a compact support in omega and it is smooth c infinity. So, infinitely valued and define, i of tau to be capital I of u plus tau v, u is this minimizer, it is given to us, v is a cc infinity function I am just taking the convex combination of just u plus tau v and why this tau, tau is in R. Now, you see, I want to look at what I does, see u plus tau v. Do you think it is in A? Of course it is in A, for every tau in R, of course it is why because you see this is the point why we are worrying we do want a compact support function because v is 0 on the boundary, for a complex of a function.

So, on the boundary u is basically g okay and since v cc infinity it is also C2. So, essentially this whole thing is in A, u plus tau. So, i this function therefore, i has a minimum at tau equals to 0.

Because I of u at the point u I attempts a minimum. So, the small i this will attend a minimum at tau equals to 0 capital I is u. So, capital I of u that is where the minimum is attempt. So, that is why small i attempts is minimum at 0. So, if it is attempts is minimum is 0, I can talk about i prime of 0 that is equals to 0, prime is with respect to tau. So, i prime is, d by d tau.

So, of course the derivative exist and it is I mean you can of course, show it if the differentiability is not a problem. If you are not convinced please check that the derivative exists here. So, i prime 0 is 0. Now, if i prime 0 is 0, see, but what is happening if that is 0. I of tau this is equals to half integral gradient of u plus tau v square minus integral u plus tau v times f dx. This is what happens and that is equals to half integral gradient u square, plus tau gradient u dot gradient v, again, I am forgetting dx all the time. So, let me do it like this, omega plus tau square by 2 gradient v square, minus u plus tau v f dx.

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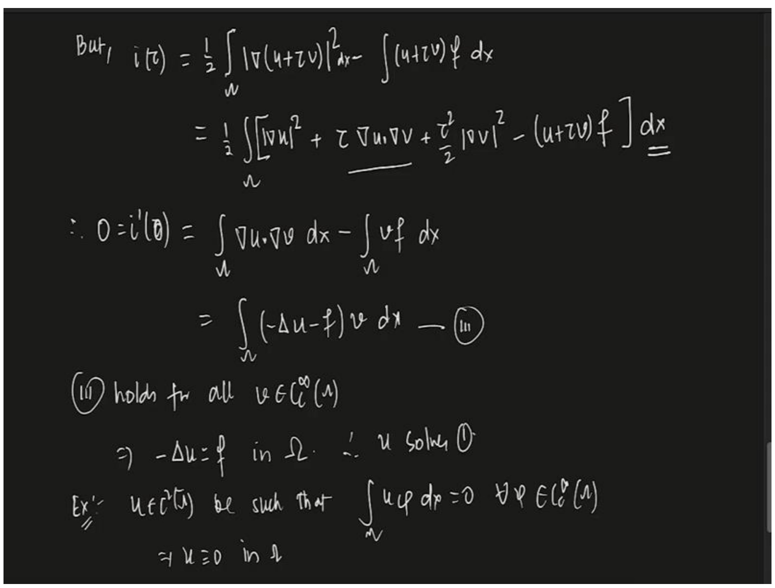


So, this is the case. So therefore, 0 equals to i prime of tau, i prime of 0 essentially. See I attempts minimize at 0 so i prime of 0 is 0. So, that will be giving, you just take the derivative with respect to tau, if you take that this is going to be 0, this is remaining and this is remaining. So, it will look like integral over omega gradient u dot gradient v, dx minus integral over omega vf dx. This is equals to integral over omega minus Laplacian of u minus f dx. Please understand i of tau will look like this.

So, if you take the derivative that derivative with respect to tau that will go inside the integral because this integral is with respect to x, it has nothing to do with tau so I can take the derivative inside if I am taking the derivative, this term is not there, this term is only, I mean, the derivative of this term with respect to tau is only gradient u dot gradient v, which is this and here it is tau, tau times this one and tau at the point 0 this term will not be there. So, the only term which would be there is minus vf, which is the derivative of this term is vf, so minus vf, right, so this this.

Now, you see this holds this identity. So, let us say that is 3. So, 3 holds for all v in cc infinity omega, right. If this holds for all v in cc infinity omega that will imply Laplacian of u equals to f in omega. Because see, if this holds for, so this is the small exercise if you want, exercise. So, let us say you is in C2 of omega but be such that u times v this is dx is 0 over omega for all v in cc infinity omega that will imply u has to be 0 in omega that is a small example.

So, it is basically saying that if you have a C2 function such that integral over omega uv dx is 0 for all cc infinity functions, then it means that u is 0 in omega. So, please do this thing, this I will put it in assignment. So, if this is true, then that will imply that Laplacian equals to f and hence, you see u is in alpha. So, u is g on the boundary it is always given and we have also showed that Laplacian equals to f in the omega and hence u solves, therefore, u solves 1.

So, what we proved is if you want to solve 1 this equation, if you want to solve the 1, then if you have to find a minimizer of this functional and vice versa. With this we are going to end this lecture.