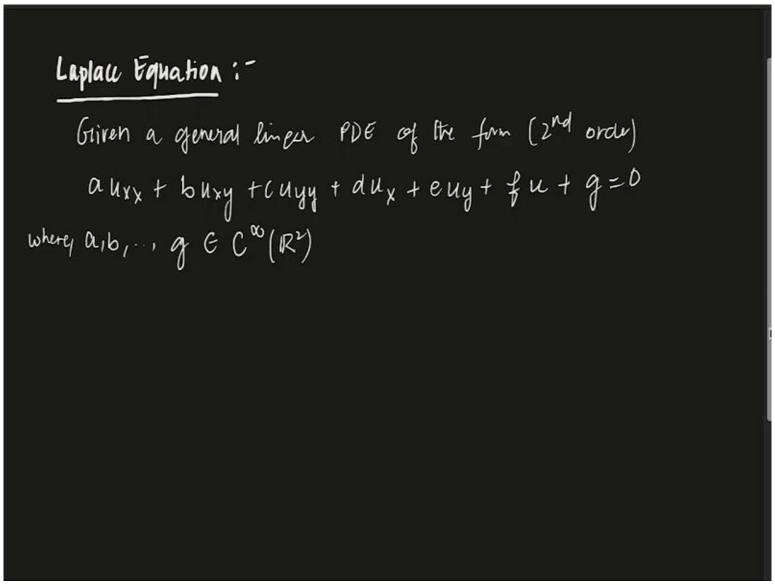
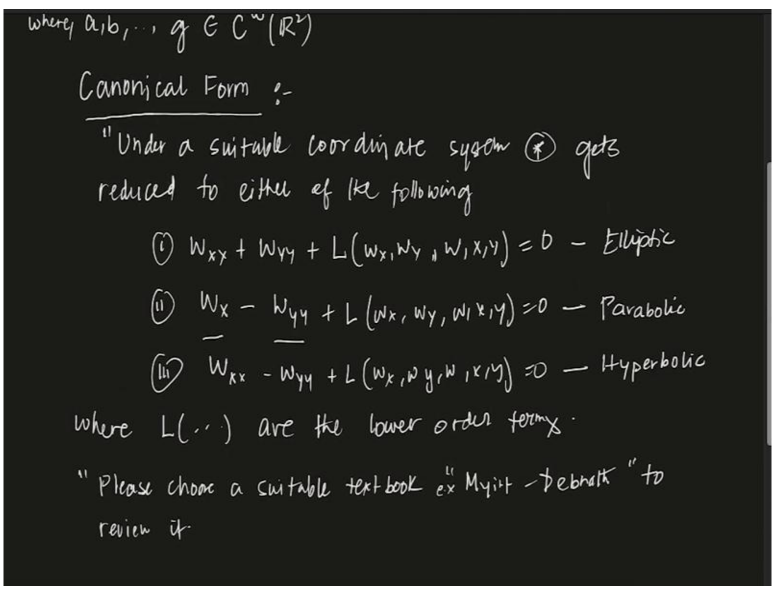
Advanced Partial Differential Equations
Professor Doctor Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Lecture 6
Laplace Equation

(Refer Slide Time: 00:14)





Today let us talk about Laplace Equation. Now, in the first few lectures we have talked about first order equation. So, ut plus ux equals to that sort of equation, transport equation, work dot equation that sort of thing. But now, we are transitioning into the second order equation. Why Laplace Equation? Let us understand this thing first why Laplace Equation. See, the point is this is the most basic second order equation of elliptic type. If you remember from earlier PDE courses, what happens is most of the, if given a general linear PDE of the form so second order how does it look like?

If you remember, it will look like this a uxx plus b, I am not writing all of this, but a depends on xy, so, I am talking two dimensions and b u of xy plus c u of yy plus d u of x plus e u of y plus fu plus let us say g equals to 0, that is a expression. So, all of these, here all of these coefficients a, b, c, these are all as in the, I am not writing all that. So, let us just do that all of these coefficients g these are all continuous, let us say these are smooth coefficients in R2. So, that depends on x and y where a, b and all of these is in say infinity R2.

Now the point is this, if you remember the canonical form, so the canonical form of this particular general equation, canonical form. I hope you understand so what I meant by this is see any second order linear PDE can be written in this form. Canonical form let us say that is your 1, let us say that is your star, it is 0 so let us call it star. So canonical form what it does is actually if you in a different in a more suitable coordinate system, so, what it does is under a suitable coordinate system, under the suitable coordinate system star gets.

So, under the suitable coordinate system star gets reduced to either of the following. So what it is, it is either Wxx plus Wyy plus some lower order terms, lower order terms containing Wx, Wy and Wx and y that is equal to 0 or Wx minus Wyy plus lower order terms containing Wx, Wy, Wxy equals to 0. Or you can have this form number 3, Wxx minus Wyy plus some lower order containing Wx, Wy, Wx and y.

What this canonical form does is, it does not change the nature of the equation. If you remember if you look at the discriminant of this equation and depending on the sign of the discriminant if it is negativity (()) (4:39), if it the positive, it is that the hyperbolic problem and if it is 0 it is a parabolic problem. Depending on that nature, if you do one suitable coordinate change, coordinate change then what happens is you can reduce this to something like this. This particular equation is an example elliptic equation. This equation is an example of a heat equation or a parabolic equation. And this equation is an example of a hyperbolic equation.

Let us understand what I meant by this. So, let me recall what I am doing, see, essentially we have a linear second order PDE which looks like this, all the coefficients are smooth this is what we are assuming. What canonical form does is? See, this is in some coordinate system x and y, you can change the coordinate system to uv. So, here you can change xy to uv but it does not matter, these are all variables. You can right xy you can right uv. So essentially, under the new coordinate system, you can actually change a star to one of the following expression.

So let us say if this star is a elliptic equation to begin with, there is a change of coordinate, which will actually carried from this to this, this is much easier to solve this expression. So essentially, we are only interested in the higher order terms, lower order terms, I mean, those we can handle, higher order terms. So it will look like Wxx plus Wyy, that is elliptic problem. Again, if the original problem is a parabolic problem, under the change of variable, it will look much simpler, it will look like Wxx minus Wyy plus lower order term.

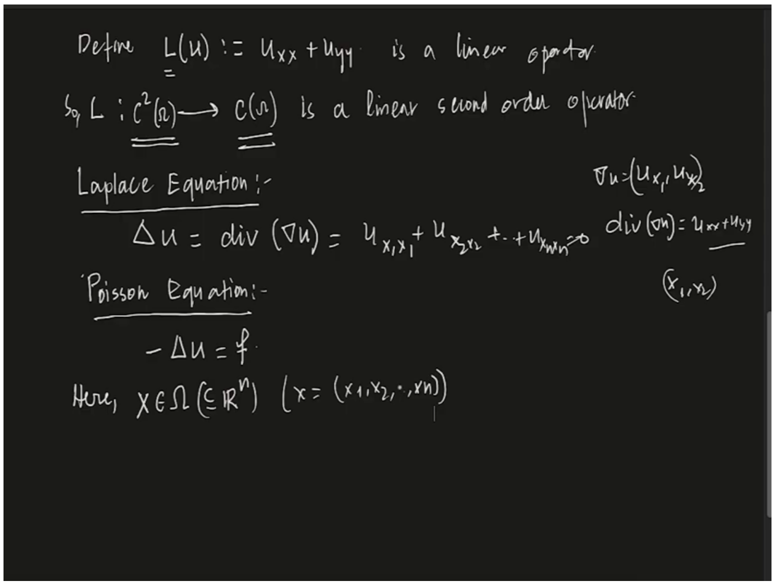
So, essentially, this is also a lower order term, but I am just writing it in this way just to make it look special, so, familiar just is like a heat equation, so that is parabolic form. Otherwise, you can just take this Wx over here also it is not a problem. And Wxx minus Wyy plus L of this is equals to 0 that is a hyperbolic form. So essentially this is like a whole equation form, two second order terms plus lower order terms like this. So, where let me put it like this, where L of are the lower order terms.

Now, so, let me make a small remark. I am assuming here that this is an advanced speaking course I am assuming that you guys know all of this, if you do not know please chose a suitable textbook. Whatever you want because most textbook contains this thing. Yes, please choose a suitable textbook for example, Myint and Debnath to review it.

See, this all thing, which I said if you guys are familiar with this thing, there is absolutely no issue is there. If you are not, then obviously you can just choose any textbook you want, I mean, this is just an example, Myint and Debnath, Partial Differential Equation this book, you can just look at that book and just review this whatever I said is not very difficult thing to do. Now, what is the point of all of this, the point of all of this is, see if we want to study most linear equations, you do not need to know, you understand you do not need to know how to solve a general equation like this. All you need to know is how to solve this three prototype equation.

If you know how to work with this three prototypes you are done, because ultimately any linear equation can be reduced to something like the second order. Any linear second order equation can be reduced to one of these three forms. So, if you can just study these three forms, these are the fundamental forms, if you can just study these forms. So basically, we call these an operator L of W is Wxx plus Wyy. How these operators behave? Then we are done. So then we can actually work out our problems.

(Refer Slide Time: 09:10)



So, let us start by first of all, with a example. So, we define L of u, so, this is what I mean by an operator. Define L of u to be Uxx plus Uyy. Before I move on doing anything, I just want you to take 15 seconds, take 15 seconds and think about if what is L? So L is supposed to as you can understand, it is kind of a function right is taking something and putting it into something, which is taking a function and giving back another function which looks like this. So can you tell me where is L defined from, just take 10 to 15 seconds and just think about it.

So let me just tell you where it is. See, L is an operator what I mean by operator here. So, L is a linear operator, what do I mean by a linear operator, what I meant by this is essentially see it is taking value from some vector space u. See if I am taking a u Uxx and Uyy has to be satisfied. So, u have to be accessed twice differentiable. So, it is C2 or whatever domain it is yeah, I do not care some domain let us say omega, I am starting out with an omega which is a subset of R2.

So, L is from C2 R2 I mean you are choosing an element of twice differentiable element and it is giving back Uxx plus Uyy, if u is twice differentiable Uxx is continuous and Uyy is continuous the sum of two continuous function is continuous. So, it is giving you back something like this.

So, L from is a linear of second order operator. This is what I mean by operator. So, see this is also a function, but in a special way. What is so special about it, that is not taking element from any ordinary set, but a vector space of functions, it is taking element from vector space of functions.

Now, let us look at where all of this comes from. So, let us look at some physical motivation. So, before I do this let us do put some name to this particular operator. So, Laplace Equation, Laplace Equation, if you are looking at equation which looks like this Laplacian of u that is given by divergence of gradient u. If you remember gradient u is let us just start with two dimensions do not worry about anything, it is exactly the same.

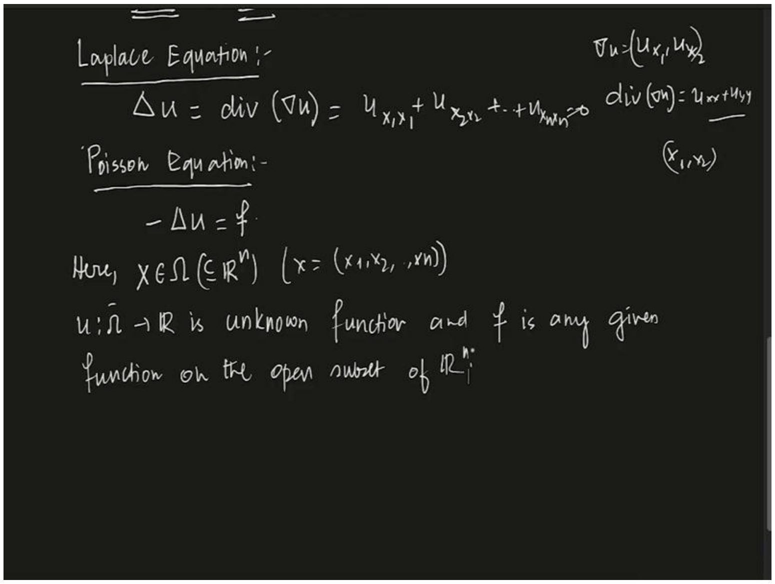
Let us just start with two dimensions. So, in two dimension gradient of u is Ux and Uy. So divergence of gradient of u that is Uxx plus Uyy. So this is the Uxx plus Uyy. That is your Laplacian. Now, there is something this equation, this is not an equation anymore, I mean this is just an operator, now this if it is 0 equals to 2 then it is called the Laplacian equation.

Now, we also talk about a similar equation which is called the Poisson Equation. So, the name is Poisson Equation, this is I mean most probably I am not 100 percent sure, but Poisson is just is like a fish in French. So, the equation looks like this minus Laplacian of u minus there is nothing special about minus you can write it, may not write it, this is just in front mentioned minus Laplacian of u equals to f that is called a Poisson equation.

Here we will assume x is in from omega which is subset of Rn. So, what I meant by x is an nxx, x1, x2, xn. I have used a something. So, let me change this thing. So, maybe, let me do it for, from now on I will do it for Rn itself. Actually, this is the problem, see I should write it like this. Let me change this part. So in two dimension it will look like this, x1x1, x2x2, so essentially your, I mean, x1 x2, so I am just writing a tuple in R2 as x1 x2.

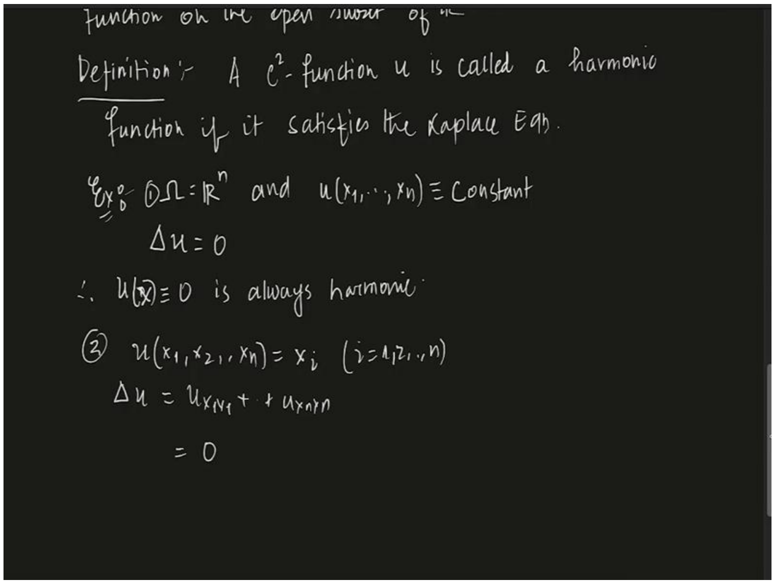
You can write it as xy also, but in that case, you cannot, I mean, I just want to I mean, reserve x for a element in Rn. So x is x1x2, so essentially, when I am saying it is a element of Rn, what I meant is, so let me put it here to the Rn. So this is uxnxn, n equal to 2 is just this I mean, here I just want to write x is in omega subset of Rn means x, so essentially x is x1x2xn. I mean, you do not have to worry about xn, the end component just two component, whatever, it does end component same sort of thing.

(Refer Slide Time: 14:59)



Now, Poisson equation is minus Laplacian of u equals to f, this f will be given to you, u from omega bar to R is the unknown function which you need to find unknown function. And f is any given function on the open subset of Rn, so, just think about it what I am saying, what I have said is you are looking at u which is from omega bar, omega bar omega is a open set. As I told you, if I am not specifying what exactly omega is most of, all the time in this course, just assume omega to be an open set in Rn. So, omega bar is a closed set the closure of omega to R that is an unknown function and f is any given function on a open subset of Rn. Now the question is you just have to find what u is that is the question.

(Refer Slide Time: 16:15)

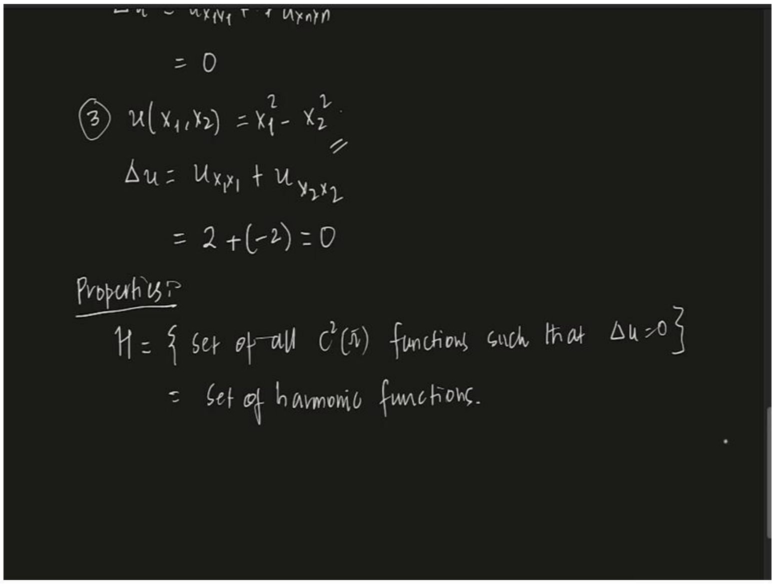


So, we start with a small definition here, definition. So, let us say C2 function u is called a harmonic function if it satisfies the Laplace Equation. So, if we are looking at a u such that Laplacian of u equals to 0 then that is, that sort of function is called a harmonic function. Now, example, take 5 seconds, just think about an example of a harmonic function. Let me tell you what is very easy example, let us say if your, let us say omega to be Rn and u of x1xn to be identically equals to a constant.

Now, if that is the case, what do you think Laplacian of u should be any derivative of this is 0. So, Laplacian equals to 0. So that is a trivial harmonic function. And, of course, one special thing about harmonic functions are 0 is always included. So 0, so ux equals to since constants always hold 0, therefore ux equals to 0 is always harmonic. And this is harmonic in any domain. I mean, it does not matter whenever you can take any omega you want omega is any open set, and u of x is always going to be a harmonic function. So that is one example.

Can you give think of another example? So let us assume that omega is an open set, and u of x1 maybe, I do not know, maybe x1, x2, xn let us just call it I am writing it for xn you can just think of two dimension also no issues. So let us say that is xi, whatever i is, i can be 1, 2, whatever, n. Now if that is the case, let us look at what Laplacian of u is, Laplacian of u in this case is ux1x1 plus u of xnxn. Now I do not have to calculate this thing, you guys can understand that this is going to be 0. So any coordinate function, this sort of function is called a coordinate function. Any coordinate function is harmonic.

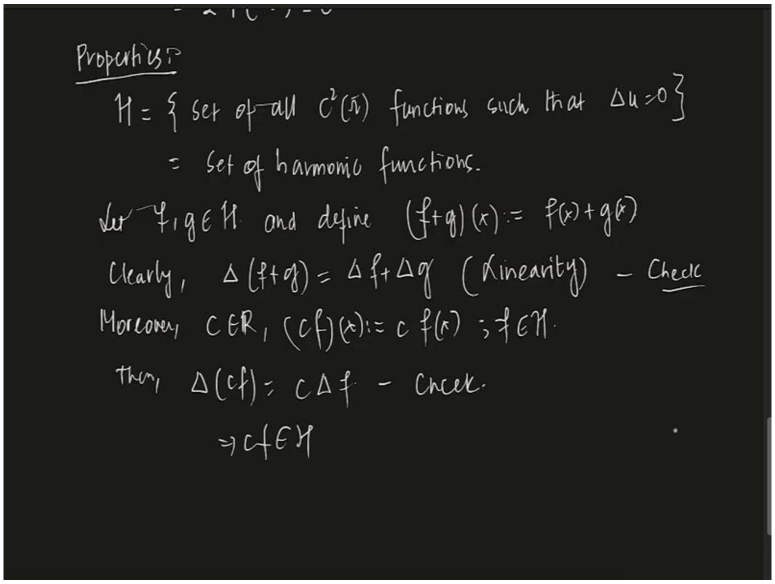
(Refer Slide Time: 19:19)



Let us take another non-trivial function here. So in two dimension, let us just think two dimension x1, x2, to be x1 square minus x2 square. Let us see if this is harmonic or not. It may be, it may not be, let us just look at it. Laplacian of u is Uxx, x1x1 minus Uy1y1 plus Uy1. Now, Ux1x1 as you can see, the first derivative is 2x1 second derivative is x2. So that is 2y1 sorry this is not y1 along, it is x2x2 plus 2 x2x2 is minus 2 that will give you 0. So Laplacian of u is 0. So, hence, this is an example therefore Ux1x2 given by x1 square minus x2 square is also harmonic.

So, I mean there are going to be other examples also, but these are more or less the basic examples of harmonic function. Before we move on let us look at what is so special about harmonic function. So, first of all properties, we define H to be the set of all C2 omega bar functions such that Laplacian of u equals to 0. So, this is the set of harmonic functions, set of harmonic functions. H is a set of all C2 omega bar functions such that Laplacian u equals 0. So, that is a set of harmonic function.

(Refer Slide Time: 21:20)

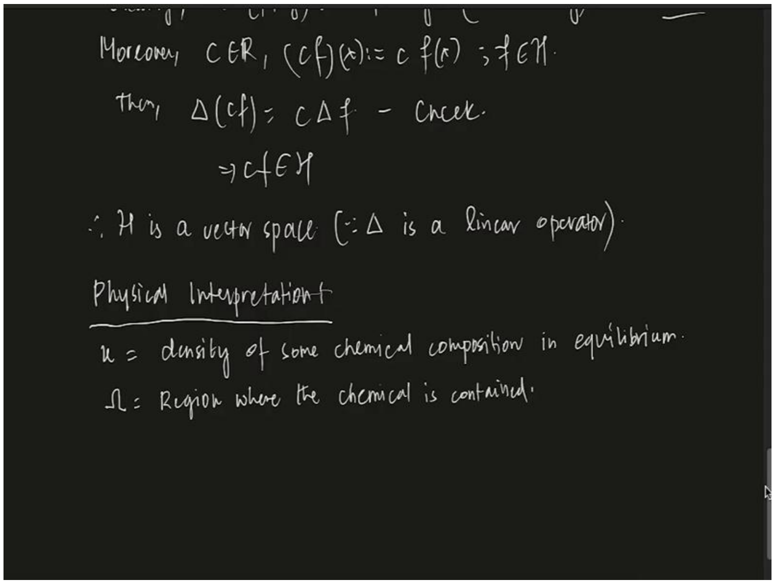


Now, I want to see what are some special properties of this set. Let us say that let f and g are in H, let us see what happen and define f plus g acting at x to be f of x plus g of x. If I define it like this clearly see if this is the case then Laplacian of f plus g, this is going to be Laplacian of f plus Laplacian of g, how is this true because of linearity. If you are not convinced about this thing, I am not proving all of this here, it is going to be time consuming I mean, it is not, it is just two lines, but please check this part here it is not very difficult.

Moreover, you see for a c in R, if you define c times f acting at x to be c times f of x and f is in H, you start with a f in H and c in R and you define cf like this, then Laplacian of cf let us see what happens, it is definitely c times Laplacian of f this also you can check, please check this part, check this.

Now, think about this. If you consider this property that given two functions f and g the sum is linear and constant times f is also in H. So, this definitely belongs to H, it means that cf belongs to H. What does that say, it says therefore H is a vector space, H is a vector space. So, this is very special about H. All of this happening because the operator is a linear operator. Since delta, so this is called a delta, this symbol is called delta is a linear operator. Now, what we are going to do is we are going to look at some physical interpretations how, where do we use this sort of operator.

(Refer Slide Time: 23:55)

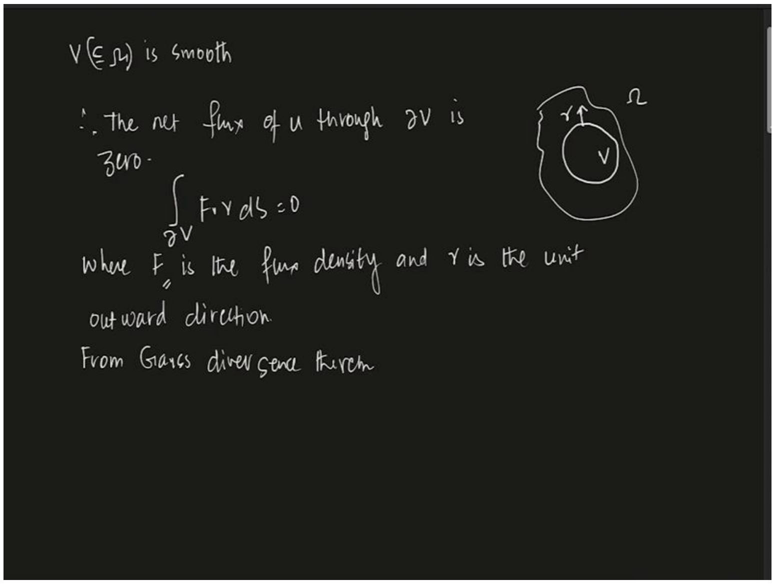


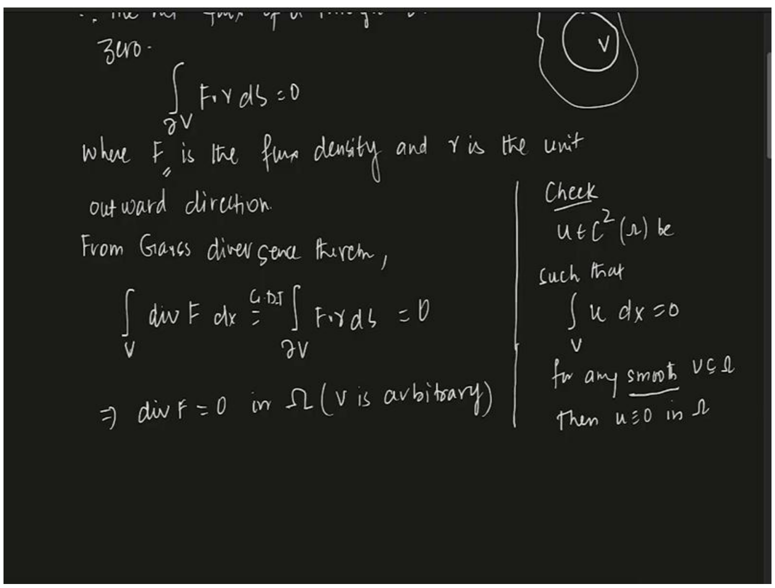
See this is the most widely used operator in all of partial differential equation and probably one of the most important objects in all of mathematics Laplacian. So, it is very important that we know understand and I mean appreciate what, how beautiful this operator is. So, let us look at some physical interpretation. So, first of all, let us assume that u is the density of some chemical concentration in equilibrium.

Let us assume u is some function, so this represents the density of some chemical composition. I mean some substance is given you are looking at a chemical composition of that thing in equilibrium, equilibrium means when it is stable. Now, let us say that the chemical is contained in some I mean region which we will call as omega.

So, the region where the chemical is contained. So, that is your omega, omega is our domain. So, let us see what happen. See, let us say that is your omega. Now, let us assume that v is some. So, let v be a smooth region contained in omega. So, that is your v let us say any smooth region which is contained in omega. If that is the case, there seems to be some problem with the software.

(Refer Slide Time: 26:28)





Let us just take another page. So, if that is the case then, so basically what I am saying is if v is a smooth so containing u smooth, is smooth. So, what am I doing that is your u and this is where the chemical, so the chemical is and u is the density of the chemical and that chemical is in equilibrium and you are just looking at another small sub-region, which is v, sorry this is not u only is in omega. v something in omega.

Now, since the I mean the chemical, the concentration that is in equilibrium, the chemical is in equilibrium. So, therefore, the net flux, so net flux means whatever is going in or coming out of u through the boundary of v, del v is 0, this is quite easy. And if that is proved what does that say del v if F dot gamma ds is 0. So, basically there is no flow of, the flux density is not changing in the unit direction. So, let us say gamma is this, gamma is the unit outward direction. So, where F is the flux density and gamma is the unit outward direction.

So, what I meant by this is, see the whole liquid or whatever the chemical is it is in equilibrium. So, if you look at a small region wherever on the omega, small smooth region the net flow of the chemical, the density, the change in the density basically the net flux of u, through this boundary of the v that is going to be 0. So, if F denote the flux density then I mean in that direction in any direction I mean whatever the gamma is here, in any given direction gamma, the flux density if you take the integral of that that is going to be 0.

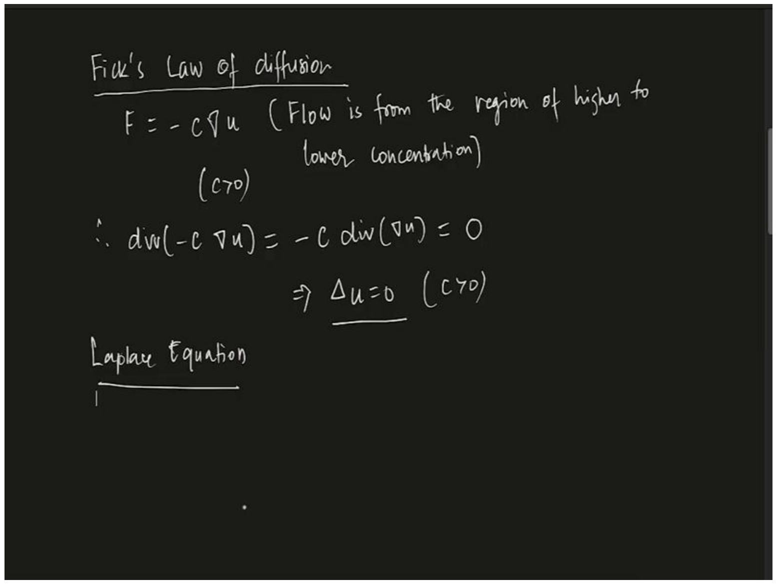
Now, if that is the case from Gauss theorem actually, from Gauss divergence if you remember when you looked at I mean integration by parts, I said that it is going to be one of the most important how do I put it most things important thing you can learn integration by parts. We are going to use that integration by parts over here.

So, let us say that integral del v F dot gamma ds is 0, then use Gauss divergence and say that you see integral over v divergence of F d of x that is equals to integral of del v F dot gamma ds and that is going to be 0, yes. So, how are you getting, this is Gauss divergence theorem. So, basically GD Gauss divergence theorem, Gauss divergence theorem says this and from here we get this flux is 0.

Now see this v is arbitrary, this v can be anything. So, you are saying that a object when you integrate that object over any sub-regions, move sub-region that is going to be 0, what does that say that the divergence of F is going to be 0 in v, v is arbitrary of course, in omega, arbitrary, in omega, I have to say it is in omega. Why it is in omega, see v is contained in omega, v is arbitrary. So, you are saying that you are taking the divergence, you are taking some object and integrating it in any smooth sub-region of omega and that integration is 0, so definitely the object in question that is going to be 0.

If you are not convinced here please check this part. So, check this, check that let us say u is in C let us say 2 of omega be such that integral over let us say u, and ds over v this is 0 for any smooth v containing omega, then u has to be 0 in omega. See one thing is this why am I talking about smooth v in omega. If you remember Gauss divergence theorem says that you have to have this region where you are integrating, that region has to be a smooth region at least C1. So that is why I am just assuming it to be smooth.

(Refer Slide Time: 32:04)



Now, you see there is something called, since we are talking about chemical concentration, there is something called a Fick's Law of diffusion this is from your physics, Fick's Law of diffusion that says that the flux density F is proportional to the gradient of u. So, basically what it is saying is minus some constant times gradient of u. So, why minus because the flow, the flux density the flow, is from the region of higher to lower concentration.

See, if there is a flow in the system here it is none, if there is a flaw in the system that is always going to be from a region of higher concentration to the lower concentration. So, that is why this negative sign is there, you just say that the flow is somehow in a negative way. And this C, of course, is a constant or proportionality, I mean, we are just saying that it is proportional to gradient of u, and gradient of u is just the change of the flux. So, basically it is the change in the density.

And this C is positive we are assuming because otherwise there is minus C we will get it. So, if f is minus C times gradient u, let us just put it here therefore divergence of minus C times gradient of u that will give you minus C times divergence of gradient of u that will be 0, because divergence of f is 0, f look like this. So, divergence of gradient u is 0. So, that will give us that

Laplacian of u is 0 because C is positive. Since C is positive, that will give you our minus Laplacian, I do not care.

Now, see, the very important thing is this is from Fick's Law, we got that Laplacian of u is equals to 0 now you just if you want to, so this is the Laplace Equation, that is the Laplace Equation. The whole idea of this is see here, if it is chemical concentration, you are talking about Fick's Law. Fick's Law will give us the Laplace Equation. Actually, if it is a Fourier law of heat conduction that is also similar to this, if the Fourier law of heat conduction, then that also will give you a Laplace Equation.

In that case, u will consider so let us say for heat conduction, u will be your temperature, the function u. And if it is a Ohm's Law, electricity conduction then u will be the electrostatic potential. So this is a very, very important thing to understand. So with this what we are going to do is in the next lecture set of lectures, we are going to talk about how to work with Laplace Equation, yeah. So with this we are going to end this particular video.