Advanced Partial Differential Equations

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Lecture 4

Method of Characteristic: Quasilinear Case

Welcome everyone. So, today we are going to continue our study of method of characteristics.

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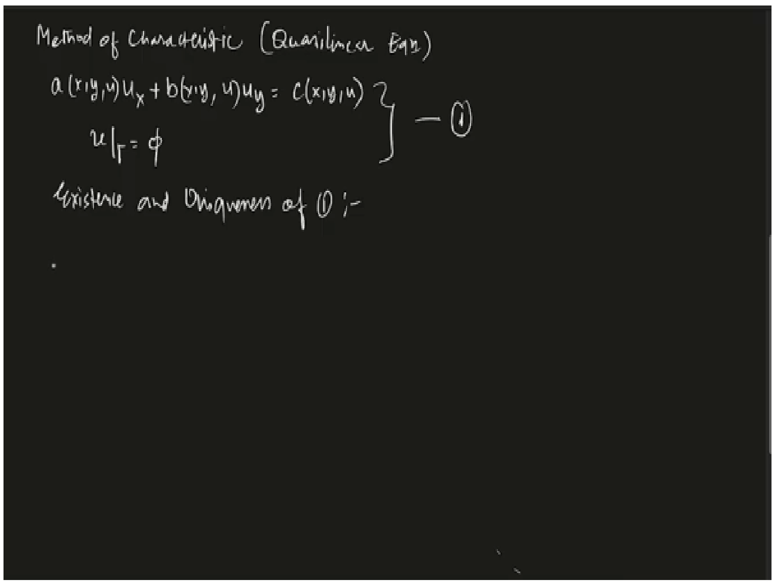

So, method of characteristics, but we are going to do it for a quasilinear case, quasilinear equation. So, let us look at what sort of equation are we talking about. So, basically we are talking about an equation which looks like this a of x, y, u ux plus b of x, y, u uy equals to c of x, y, u.

And u restricted to gamma is let us say phi. So, this is the equation which is given to us, and we call it a quasilinear equation because, see this is a nonlinear equation. I mean, it is not a linear equation, but the coefficient of a, b and c these depends on it, these may depend on u.

So, it is not a linear equation, but is not fully nonlinear equation, it is just a quasilinear equation because it depends u, here also u and here also u. I want to solve this equation, so I want to start with the well posedness of 1, so essentially in this case well posedness whatever I say I mean only the existence and uniqueness part. So that existence and uniqueness of 1 that is what we need to understand.

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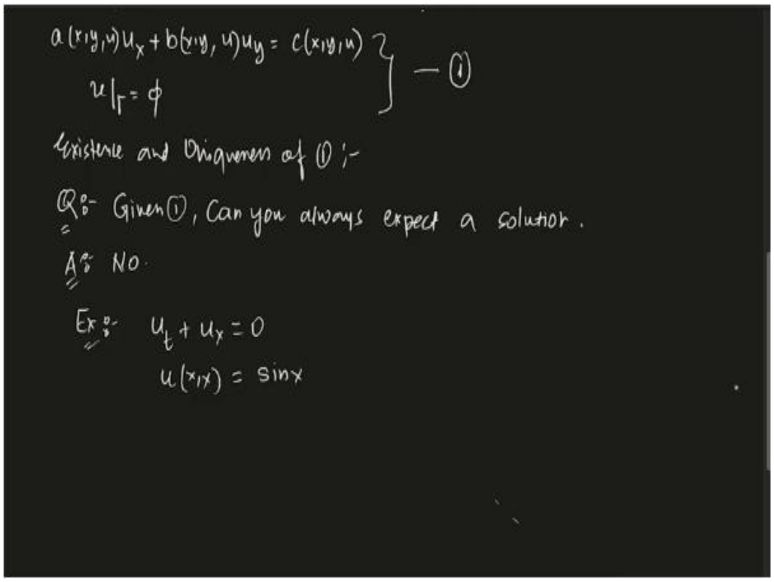

So, for first of all, let us understand that, given any equation like this, can we always expect that there is a solution for, so the question is this question, given 1, can you always expect a solution, expect a solution or a unique solution, I mean, first of all, let us just understand that if we can expect a solution and then we can talk about uniqueness later, but the answer is this do you do you think we can expect any solution out of it? No, the answer is no. Why? So the thing is, you see, so let us take an example and see, see ut plus ux equals to less than 0 and u at the point xx is sin x.

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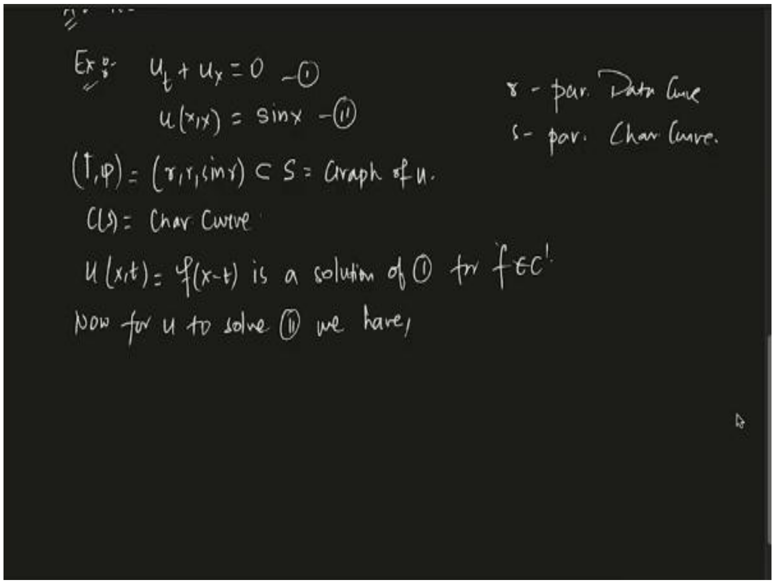

Let us take up this example. This is a very, very simple problem to solve. We already know that if, let us say gamma and phi, this is the initial data curve, if you want to write it in terms of a parameterization this is r, r, sin r, this will lie on S which is graph of u, graph of u, I want to solve this and C of S is the characteristic curve, S is parameterization the characteristic curve, r is parameterizing the data curve, so r parameterization data curve and S is parameterizing the characteristic curve.

Now, let us solve this problem. See, you guys already know I mean, I am not writing all these characteristic equations and all because you guys already can do that, we saw in the earlier classes, u of xt, if I just solve 1, let us say that is 1 and that is 2. If I just solve 1, what do I get? u of xt is some arbitrary function of x minus t, is a solution of 1, solution of 1 for f in C1, for any f in C1 this is the solution.

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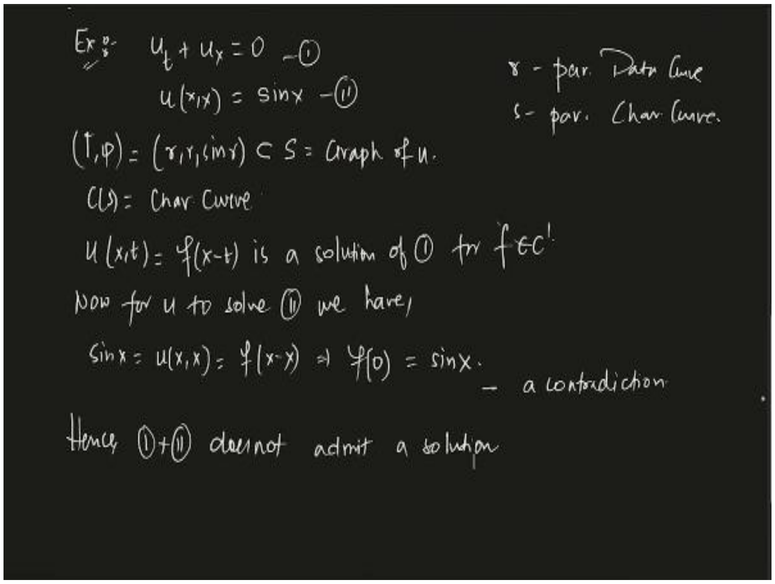

Now, for u to solve 2 we have, now let us see u solves 2, now, if u solves 2, u at the point xx is sin x, so sin of x should be u at the point x, t equals to x, and then you have f of x minus x, so that will give you a f of 0 is sin x. Now, do you think that can happen, see f is, if you want to write, f is an arbitrary function that is fine, but for 1 solution f is fixed, so for fixed f u of xt given by f of x minus t is a solution.

Now, f of 0 in that case will be a constant, f is a C1 function, f at the point 0 is a function, you just can take one value here and it is a constant and that is equals to a sin x you are writing and x is varying in whatever, I mean in let us say r, so that is a function this cannot happen, so a contradiction, you understand, you cannot find a solution hence, 1 plus 2 does not admit, does not admit a solution.

Let us understand what is the problem here, so just take one, take thirty seconds and think about what is the problem, see the problem is this, let us say if I am putting u at the point x0 is equal to sin x, there was absolutely no problem we can just write it as sin of x minus that is a solution, here the problem is this, what are the characteristics curve, the problem is this.

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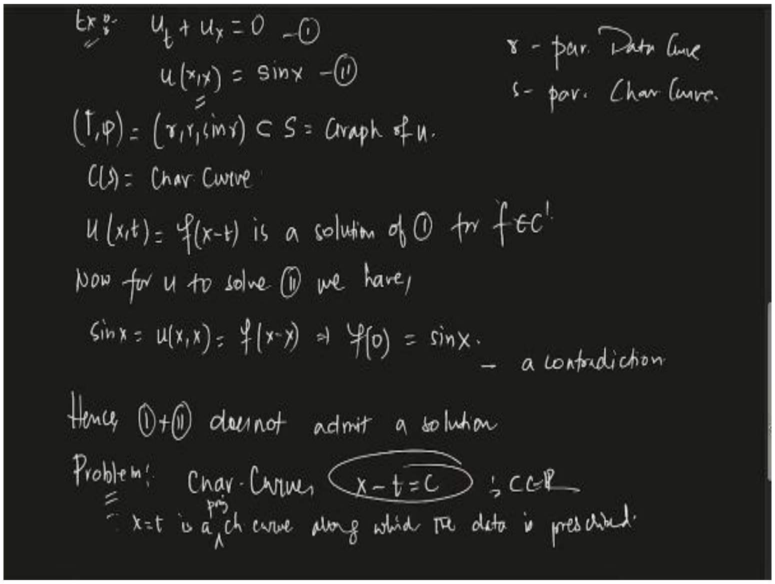

Let me explain what the problem is, look at the characteristics curve, what are the characteristics curves? Characteristics curves are given by x minus t equals to constant, those are the characteristic curves, c in in r. See that initial data here is given along one of the characteristic, so c equals 0, x equals t is a characteristic, x equals to t.

So, x equals to t is a characteristic curve, characteristic curve along which the data is prescribed, prescribed, you understand, see x equals to t, so basically same coordinate x equals t line, on that line that data is prescribed and that is why what is happening is f of f x minus f which is becoming f of 0, and that is why that constant cannot be the because function.

So, essentially what is happening is here x equals to t is a projected characteristic, sorry, this is the projector characteristic curve. So, the data is prescribed along one of the projected characteristics and that is why, what do you have that is why you have this contradiction. So hence, you cannot have the data prescribed along projected characteristics.

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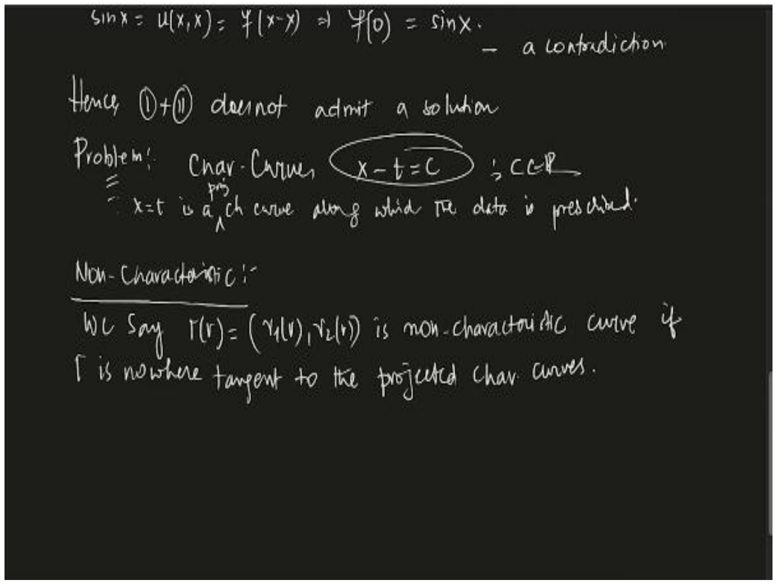

So, here comes a concept which is called a non characteristic data curve, so non characteristic data curve. So, you see the thing is this, what I meant by all of this is you cannot have data along any curve and expect the problem to have a solution, you understand, you need to have data on a, so the curve gamma, so you understand, so for the solution to exist the curve gamma must be a very special curve.

So, let me write it like this, we say gamma of r which is gamma 1 of r, gamma 2 of r, this is non characteristic curve if gamma is nowhere tangent to the projected characteristic curves.

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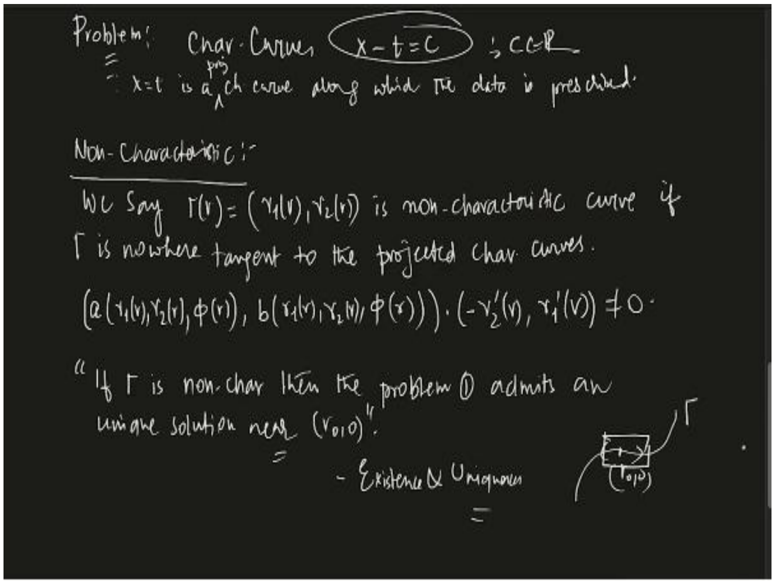

See, if gamma is one of the tangent, so here tangency is not coming because all of the projected characteristics are lines and the tangent to the lines are the lines itself. So, if gamma is nowhere tangent, then we are good to go. So, essentially what it is means in case of a so, let me write it in this thing in case of a quasilinear equation.

So, what are the projected characteristics? It is basically this gamma 1 of r, gamma 2 of r, if u of gamma 1 of r, gamma 2 of r which is phi of r in our case times b of gamma 1 of r gamma 2 of r phi of gamma, u of gamma 1 of r, gamma 2 of r which is this, this vector field times minus gamma 2 prime of r, gamma 1 prime of r this is nonzero, is it clear.

See, what is this, this is basically I mean this is, this particular thing is saying this is a normal thing. So, this is basically saying that this particular projected characteristic curve, projected characteristics are given by x prime y prime, x prime is a, y prime is b, so it is basically saying that the projected characteristic curve, sorry, the non characteristic, this curve, the data curve is a non characteristic curve, if the data curve is nowhere tangent it cannot be tangent at any point, tangent to the projected characteristic curve, so that is the relation so, this dot minus gamma 2 prime, gamma 1 prime this has to be nonzero.

And if this happens, then you can show that, so the existence and uniqueness theorem, let me just try to write down the existence uniqueness. Okay, we will do the existence uniqueness later for now we just give you a short idea. So, essentially, if gamma is non characteristic, then the problem 1 admits an unique solution near, so let us r naught is some point r naught 0.

So, S equals to 0, so you take any fix, so let us say that this is your data curve gamma, let us say that the point r naught 0. So, near the point r naught 0, you can actually find a unique solution for this problem provided this gamma is a non characteristic. So this is the existence theorem, existence and uniqueness and uniqueness theorem, we will do it properly later but for now, just keep it in mind. But please understand that then unique solution is near r naught 0. So basically in some neighborhood here, some neighborhood here.

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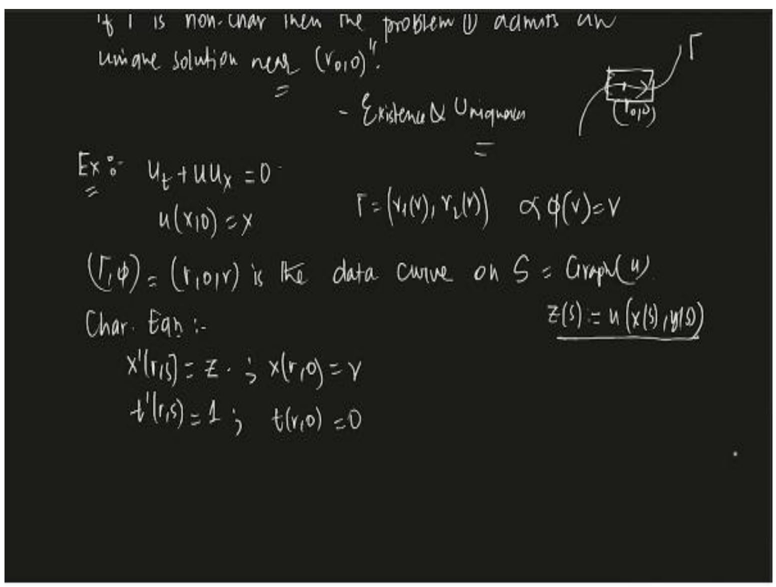

Now, so, this is the case, so let us say you are given a problem, first of all how do you solve a quasilinear equation? So, let us take up an example and see how do we solve it. Example, let us say ut plus uux equals to 0 and u at the point x0 is x, let us take this equation and let us see how to deal with this problem. So, as usual gamma phi, the parameterization will be given by r, 0, r, gamma phi is gamma 1 of r, gamma 2 of r and phi of r is given by r. So, is the data curve on S, S is the graph of u, what is u? u is the solution of this problem.

The graph of u, the surface is given by S and the data curve r, 0, r is lies on S. So, the characteristic equations, what are the characteristic equations? As I told you, you see this is exactly in that form and you can use the whatever we have been doing up till now. So, x prime rs, this will be with respect to x this is u and so this is Z, we will also define Z of S to be u of xs, y of s.

See along the curve whatever the height of u is, x and y is in r2, u at the point xy that is the height, so that is given by Z, so that is we are writing ZS like this, so x prime is u and u is Z, so this is Z and moreover x at the point r0, so at the point x equals to 0, what is the value which is r, y prime r s is given by, sorry, this is t prime rs, t prime rs is given by 1 and t at the point r0, this is 0.

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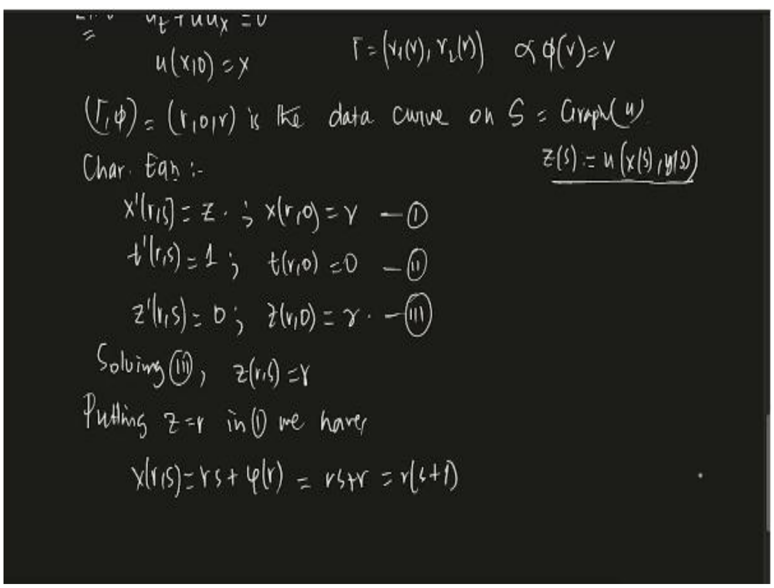

Because r, this is 0 and Z prime rs is 0 and what is Z rs r0 this is r, so that is your characteristic equations. Now I want to solve this equation, see, let us say this is 1, this is 2 and this is 3. So solving 3, what do we get? Z of rs is r, Z of rs is constant with respect to s, clear?

At the point S equals to 0 Z is r, so Z is always r for all S, so solving 3 we get Z of r is r, now putting it 1, putting Z equals to r in 1, we have x of rs is r, so x of rs if you just solve it, it is rs plus a function of r, and so if you put this initial condition at the point r0, this is going to be r, so this is rs plus r, so essentially this is rs plus 1.

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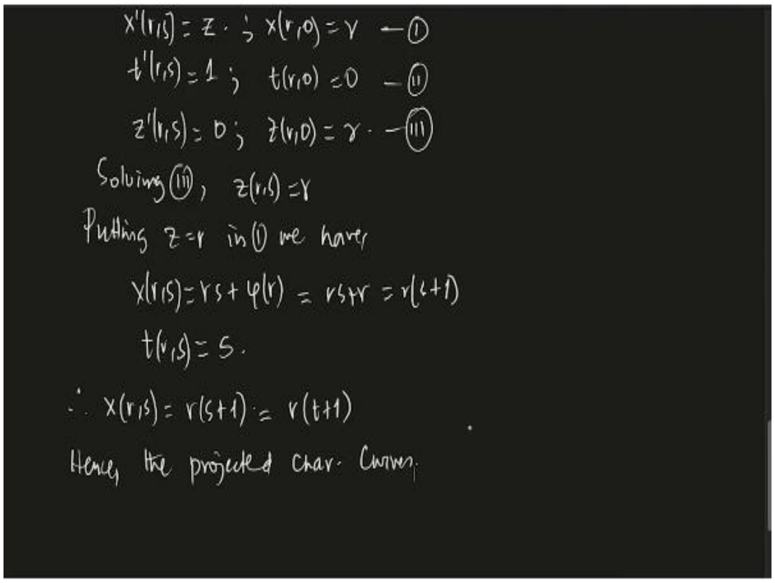

And similarly, what is t of rs? t of rs is s plus phi of r, s plus some function depending on r and since t of r0 is 0, so t or rs is essentially S. So please check this part if you are not convinced please check this part. Now, if we put this value here, therefore, what do you get? x of rs you get it to be t is s, so r times s plus 1 which is r times t plus 1, hence, the projected, let us write down the projected characteristic, what are the projected characteristics curves, what are the projected characteristic curves?

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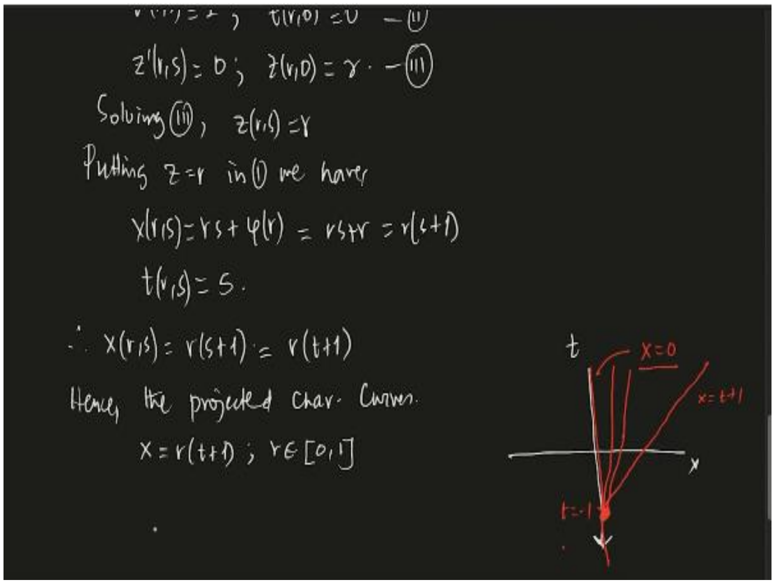

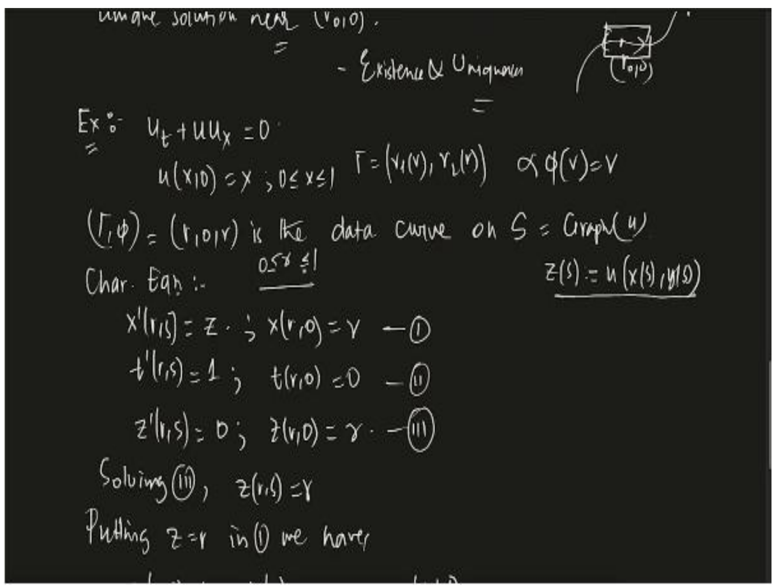

They are given by x equals to rt plus 1, r is in r. So you see projected characteristic curves here, they look a little different. It depends on r, for now let us take these initial condition to only depend only restricted to 0 and 1 let us put that, let us see what happens, see here so this r only varies between 0 and 1, let us put this restriction and see what happens. I mean, without that also it is not a problem. So in this case, what we will do is r will only vary between 0 and 1.

I want to see what are the projected characteristics? So let us say the projected characteristic will be in xt plane, xt plane, let us see, so first projected characteristic corresponding to r equals to 0 is x equals to r equals to 0 with x equals to 0. Okay, so x equals to 0, x equals to 0, this is one of the projected characteristics, I mean maybe I can use a red for this thing, this is a protected characteristic.

Now, the other protected characteristics is, see r equals to 1 let us say, so x equals to t plus 1, x equals to t plus 1, so something like this, t equals to minus 1 this point, this is x equals to t plus 1, this is another projected characteristic and if r varies between 0 and 1, all these projected characteristic, every projected characteristic will lie here, these are all straight lines, I am not very good at drawing, so you understand these are a straight line and this will all meet at this point t equals to minus 1, this will all meet at t equals to minus 1.

So, these are our projected characteristics, so it will spend the whole thing and it will meet up here. Now, the thing is, let us write down the what is the exact solution and then we will talk more about what is happening here.

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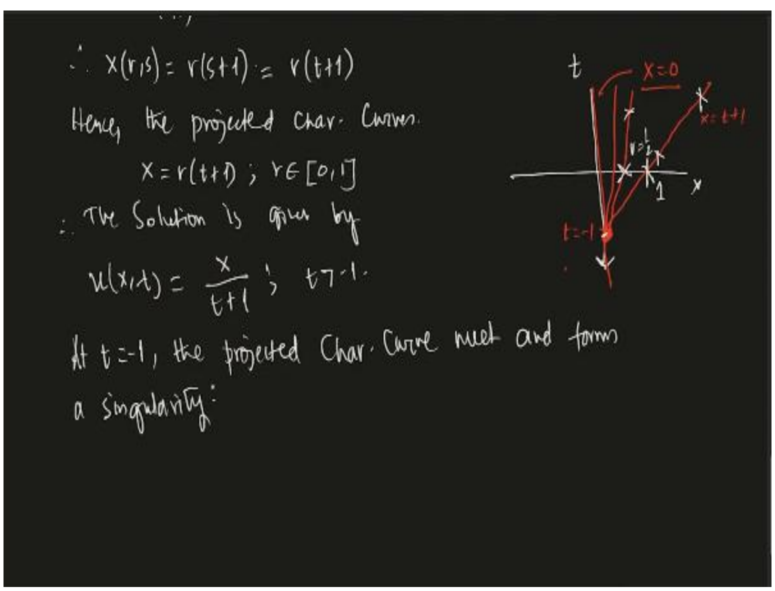

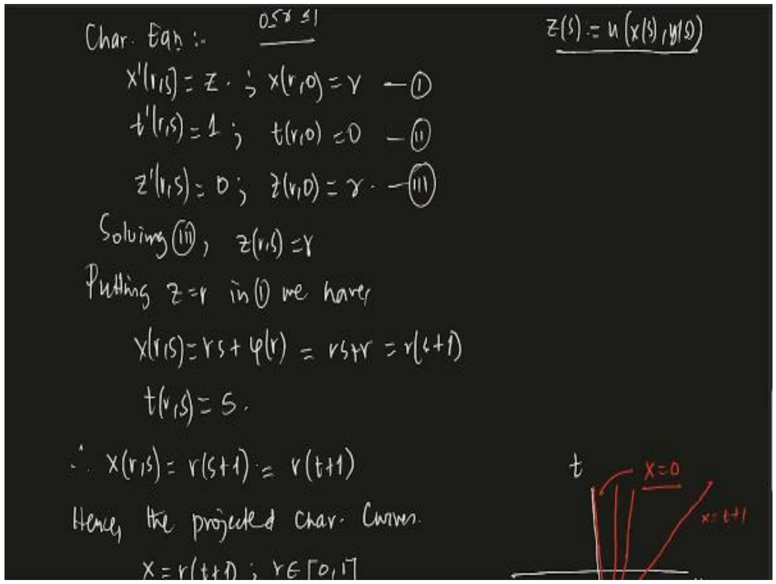

So, the solution and therefore the solution is given by, see what is the solution? Z is r, Z is r. So Z u of x and y, is u of xs ys which is Z of S which is given by r, and r is x by t plus 1, that is a solution. So, this t obviously cannot be minus 1, otherwise this blows up. So t is greater than minus 1 and this is evident from the projected characteristic. See, what is happening is this, along the characteristics, along the characteristic Z is r.

So, Z is constant along the characteristic, along the characteristic Z is constant. So, just think of the graph above for every point here what is the value of Z, it is constant. So, here, here, every point it is constant, at this point what is the value of Z, let us say this point is 1, for r equals to 1, so Z equals to 1 here, here also Z equals to 1, but again if you carry on along this line Z has a different value, but along these lines Z is always constant, so let us say this is r equals to half, r equals to half.

So, along this line Z is equals to half is going, going, going, but when it reaches here, what is happening it will when at t equals to minus 1, what do you think this will go? You see, along this line if it goes at this point, it will take half, it should take half, along this line if it goes it should take 1.

So, what is the value of Z at this point or u at this point? There is a problem, I mean, both the lines do intersect somewhere, but what is the value of u at the intersection, at that point it is undefined because along two different characteristics you are getting two different values.

So, and this is given by, you can see u is given by this. So, here you understand that things are getting complicated in terms of when you look at the quasilinear equation. So, u have xy, sorry, u have xt will be given by this. So, you understand that at t equals to 0, at t equals to minus 1 the projected characteristics curve meet and forms a singularity, singularity. Later in the course, we will talk about the singularity, this is a very important kind of singularity, we will talk about the singularity and we will see how to work around all this singularity. So, this will be done later.

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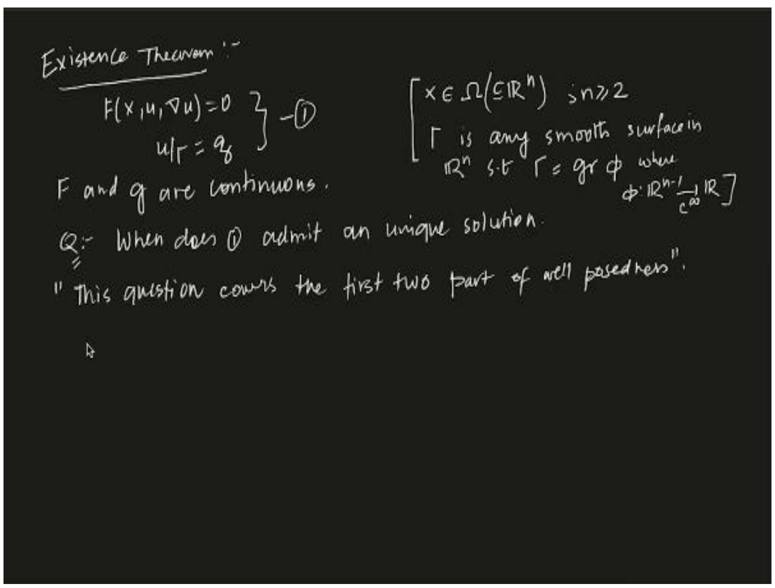

So, now, we want to talk about existence theorem. So, essentially, what is so special about non characteristics, we want to look at it and do that. So, we want to talk about existence. So, existence theorem, so what is the existence theorem? Essentially, let me tell you what is the exact idea, the exact idea is to talk about this sort of thing, F of x, u, gradient u equals to 0 and u restricted to gamma is let us say g, this sort of thing is given, F and g are continuous, are continuous wherever they are defined, I mean, I am not writing all that it is continuous.

Now, this is a very, this is a general format of a PDE. So, that is your 1 and we want to actually explain, we want to see that so, F and g are continuous, so the question is this, question is when does 1 admit a unique solution. And whenever we talk about solution, we mean of course, u is in C1 of omega wherever this u will be define C1 such that it satisfies this, that u restricted to this gamma, gamma is some curve, that is equals to g.

And of course, I mean, here you can think of this x, so here we can think of this x is in from omega, which is containing Rn, we are restricting ourselves to R2, but the same exact same thing works for Rn and there is nothing special about it and this gamma so this is the extra assumption x is in omega which is in Rn and n is greater than or equal 2 and u restricted to gamma is g.

So, what is gamma? Gamma, see if you are taking your domain to be open set in Rn, so gamma is any curve, any smooth curve, we are talking about smooth curve in Rn such that gamma is the graph of some function, graph of some phi. So, essentially what is gamma?

Gamma, sorry, this is not a smooth, this is a smooth surface, sorry, this is not a smooth curve, in two dimension curve, in three dimension it will be a smooth surface. It is a smooth surface in Rn such that u is equal to, so basically you can think of u as a graph of phi where phi is from Rn minus 1 to r.

So, the graph of a function from Rn minus 1 to R you think of that graph so, that will give you a surface in Rn and the information of u is given on that surface and you are saying that you are looking for function u such that when you restrict u on that surface gamma, then you get this function g, this is an Rn of course, when it is R2 then essentially you are looking for, I mean gamma is just a graph of some smooth function, this is a smooth function, it is smooth, c infinity let us say, c infinity function from R to R and so, the graph of phi which is gamma will be a curve in that case, a smooth curve which lies in R2 and then I mean the real thing happens, so x is in R2 in that case.

Now, the question is this, so when does 1 admits a unique solution? Now, if you remember why are we suddenly talking about it because this is the, this actually, so this question. So, please understand, this question, see for a given problem, you want to talk about the well posedness of the problem. And this question covers the first two part of well posedness.

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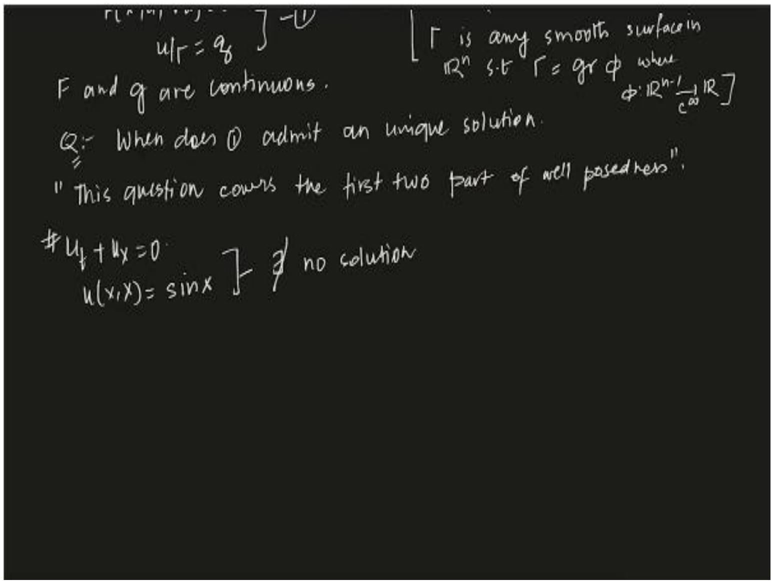

So, we want to prove this thing and we saw that I mean, for any problem. So just a small revision. So, we saw that for any problem let us say ut plus ux equals to 0 and u restricted to x and x is less than sin x, this problem, so there does not exist less than no solution, you can show that. So, it is not that, I mean, it is not a like a guaranteed thing that any problem, like if we have a solution, I mean, forget about uniqueness, there may not exist any solution, because the data is given under one of the characteristic lines.

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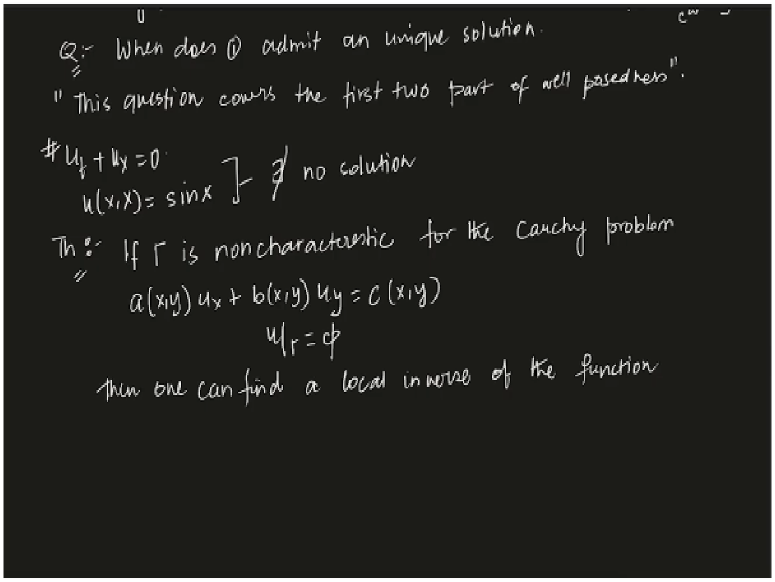

So, what can we do about it? We talk about non characteristics. So, let me give, so essentially if you want to find a unique solution of some problem, something like 1, then we have to somehow take into account that non characteristic condition. So, from there on itself, we are going to talk about the theorem which we will deal as uniqueness theorem.

So, the theorem which we want to say is a state is this, the theorem. So, this is the uniqueness theorem very important. So, it says that, if gamma, gamma is that curve, gamma is non characteristic. So, what is the non characteristic? So, basically gamma is a non characteristic curve essentially or surface whatever I mean for two dimensional I am just writing, you can do it for n dimension also. Non characteristic for the Cauchy problem, what is the Cauchy problem? For now, I am just writing to do as a quasilinear problem or semi linear let us say, you can do exact same thing for quasilinear and a nonlinear problem.

So, a of xy ux plus b of xy uy equals to c of x comma y and u restricted to gamma is phi, so see here I am just writing it for semi linear, you can also replace it with a quasilinear or you can do it for a nonlinear problem also. So, if gamma is a non characteristic for the Cauchy problem, remember the non characteristic conditions.

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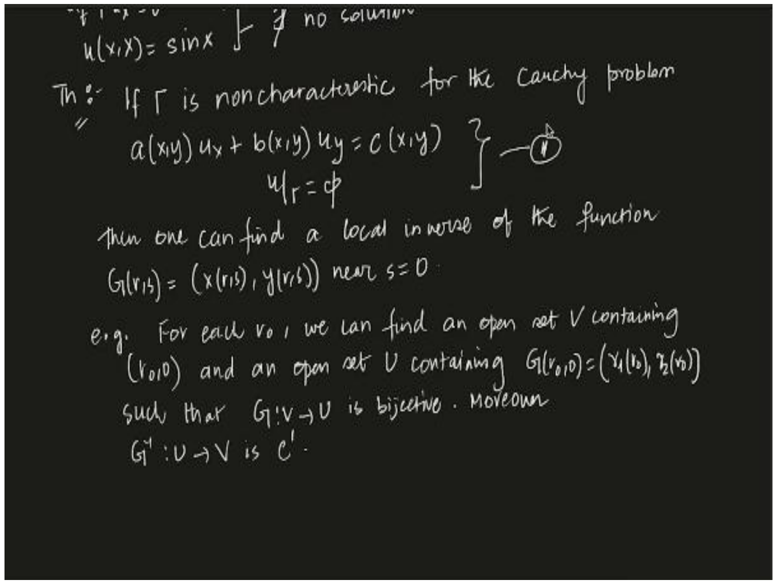

Then one can find a local inverse of the function G of rs which equals to x of rs, y of rs near s equals to 0. So, what it means is that is for each r naught we can find an open set V containing r naught 0 and an open set u containing G r naught 0, what is G r naught zero? It is x r naught 0 and y r naught 0, what is x r naught 0, y r naught 0? Which is gamma 1 of r naught 0 so it gamma 1 of r naught and gamma 2 of r naught such that G from v to u is bijective. Moreover G inverse from u to v is C1.

So, essentially what it is saying if the non characteristic condition is given you can actually confirm that a problem like this, let us say that is your 2, a problem like this admits a solution, but the solution, this is very important, please understand this thing.

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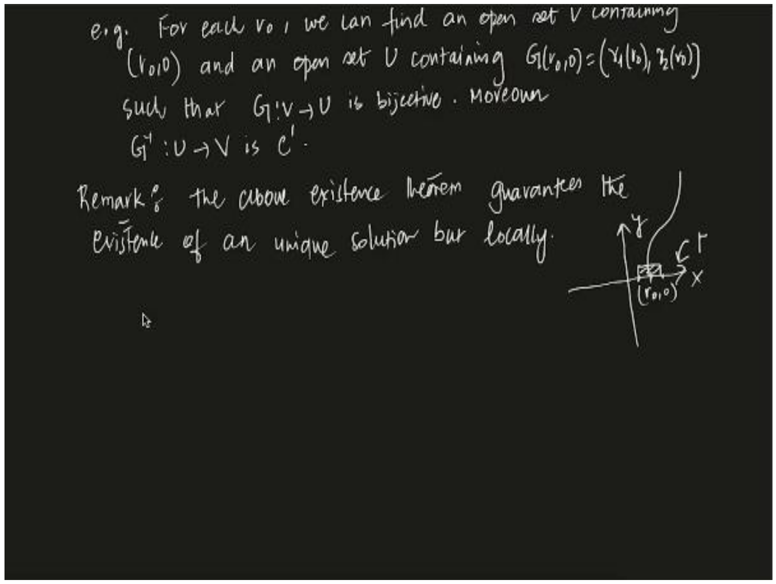

So, before we prove anything this is a small remark the above existence theorem guarantees the existence of an unique solution. See, it is saying that there is a neighborhood of r naught 0 where the solution is unique, unique solution but locally. See, it is not saying, so let us say you were coming, think about this thing, let us say this is your domain.

So, let us say the whole domain you are there and the initial data, initial curve is this, let us say that is your x axis, this is your y axis and let us say that is your initial curve gamma, this is just an example. The initial curve gamma, what this theorem is saying is for arbitrary problem, if the gamma is a non characteristic, if gamma is a non characteristic for that problem, then you can actually find a solution but not everywhere I mean, you cannot say that the unique solution exists in the whole domain, that is your domain x y plane.

It may not exist in the whole domain, but you can say that there is a small neighborhood, let us say something like this, in this part of the domain, small level, this is just an arbitrary, there is a small neighborhood like this, where you can actually, I mean corresponding to this you have some u, and that will give you some small surface. So, you can have a solution which is unique, which is unique and satisfies that problem.

So, it satisfies this problem, u aux plus b uy equals c and u restricted to gamma is u restricted to gamma is for only on this part, only this, not in the whole gamma, so just this part of the gamma, so let us say that is your r naught 0. So in a small neighborhood of r naught 0, you can actually say that there is a unique solution. See, near s equals to 0.

So, s equals to 0, see s equals to 0 we start from here and let us say s goes like this so it is saying in a small neighborhood you can get that, you cannot get it in everywhere in the domain. So, it's always local, so please remember any existence, you know most of the existence theorems are always local condition.

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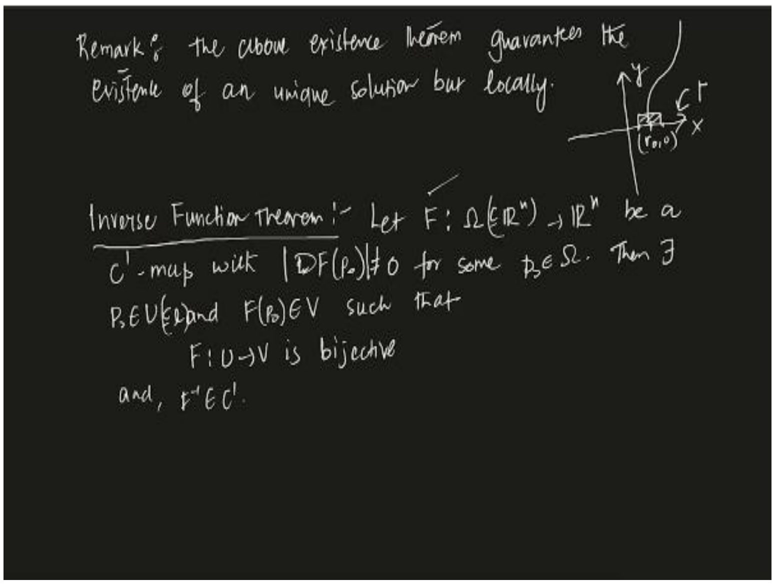

So, now, we want to prove this thing, now, how do you prove it? So, before we prove this thing as you can see from the statement of the problem, this has something to the inverse function theorem. So, let us talk about inverse function, so we need inverse function theorem to be used here. So, what does the inverse function theorem says, as you know that I mean inverse function theorem is one of the most cornerstone theorem in all of analysis, it is probably the most important theorem in all of the analysis.

So, what it says is this, so given, we basically are talking about a vector field. So, let us say F is from omega subset of Rn to Rn be a C1 map, so we were looking at that obviously, please remember inverse function theorem is always from the same dimension, dimension is same Rn to Rn, C1 map with DF at the point P naught not 0 for some P naught in omega.

So, inverse function theorem, what the gist of inverse function theorem is that you are basically given a function, C1 function you want to see whether there is a inverse function, inverse exists, whether F inverse exists or not, definitely this is not true for any given function, what inverse function theorem says is if the Jacobian of F, so if DF P naught, if the Jacobian of the derivative of F at the point P naught is nonzero for some P naught in omega, then there exists u containing P naught and v containing F of P naught such that F from u to v is bijective and F inverse is in C1.

So, what it says is, see if you are starting out with the F which C1, if you are starting with F which is C1 which is very important, then and if we get a Jacobian at some point P naught is nonzero, if the Jacobian is nonzero then what you can say is around that point P naught you can have an (())(37:56) u and look at the image F of P naught you have another (())(38) v such that F from u to v.

So, you have to restrict F obviously, u is containing omega, so this is a u which is containing omega, u is containing omega and F of P naught which is containing v such that when you restrict F to u, from there F to v that is bijective map so, basically F inverse exists and moreover F inverse is also C1, so if you take F to be C infinity F inverse is C infinity, if F is Ck, F inverse is Ck with this property obviously, this is important.

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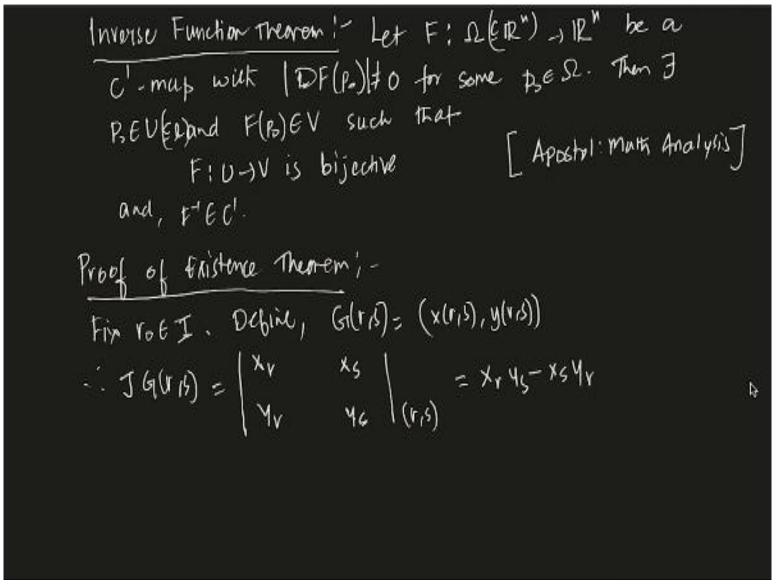

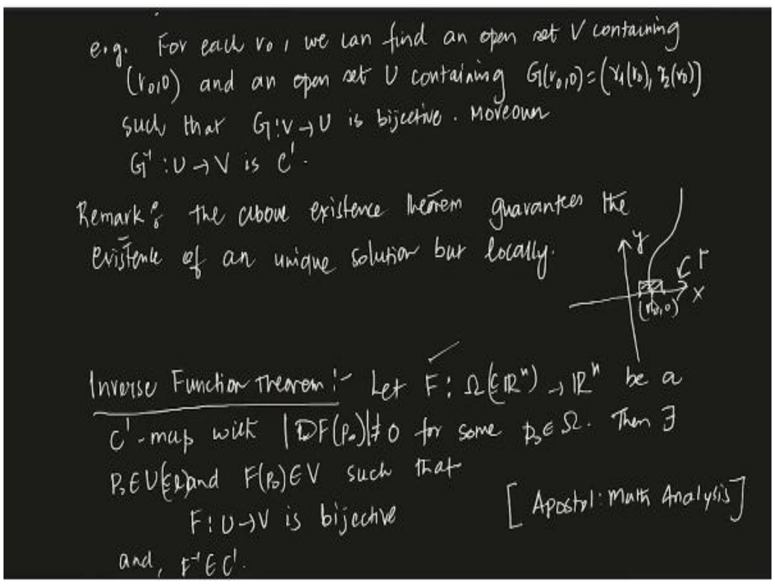

Now, let us prove the existence theorem, proof, so proof of existence theorem, before we go on the proof, I mean if you guys do not know this theorem please read it, it is very, very important theorem and the proof of this you can get it in a Apostel Mathematical Analysis book, please look at that book.

So, let us look at the existence theorem, so for existence what we do is you first of all fix a r naught in i, whatever in that interval wherever you are taking the parameterization so fix r naught so, what are we doing you see you fix it r naught here.

So, r naught is the parameter, r is the parameter, a parameter is in the initial curve and r naught is one of the points. So, you see, we are defining and define, it is already defined, so we are just writing it like this G of rs is to be x of rs and y of rs. Now, therefore, Jacobian of G at the point rs, what happens to this thing, this is the determinant of course, of X with respect to r.

So, this is xr or rs I am not writing that all and this is xs, yr, ys I hope you guys know all this and if you do not know please just read about it, I mean this is not very difficult thing to understand, the Jacobian is given by, you take the first component, take the derivative of x with respect to r, take the derivative of x with respect to s, this is of course, at the point rs I am not writing all that, this is at the point rs, so this is given by xr ys minus xs yr, this is there.

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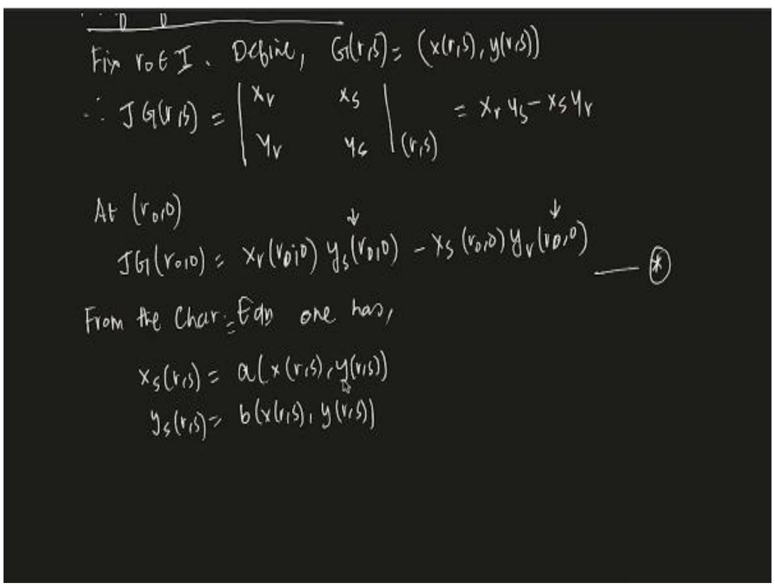

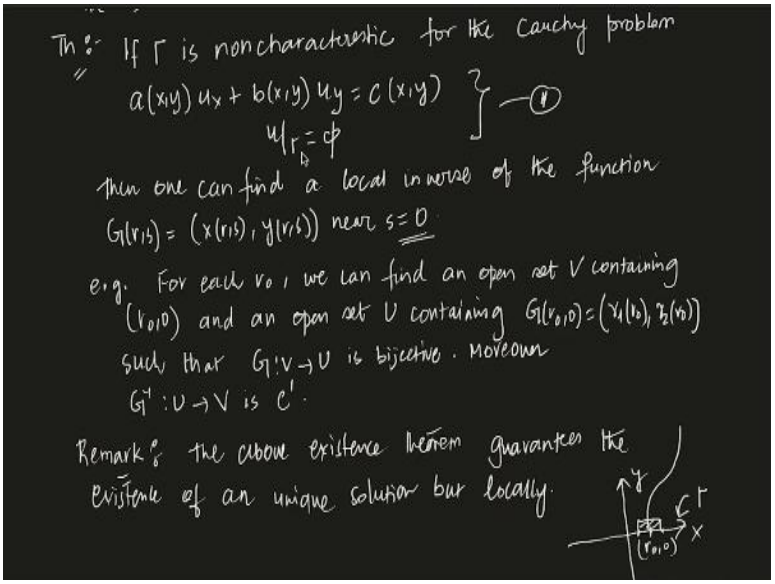

Now, see at the point r naught 0 let us see what happens, Jacobian of G at the point r naught 0 will given by xr at the point r naught 0 ys at the point r naught 0 minus xs at the point r naught 0 yr at the point r naught 0, now what is xr at the point r naught 0, if you remember, see first of all ys at the point r naught 0, do you remember what is ys at the point r naught 0?

See, xs and ys at the point rs is given so now what happens is there is for this problem I have skipped a little bit, you see for this problem, during this problem you can talk about, I mean just go as we have done up till now, so think of the characteristic curve which is parameterized by gamma, so the gamma is parameterized by the characteristics curve and you can write the characteristics equation, if you write the characteristics equation what do you think xs is, x with respect to s, that is a and y with respect to s is b, that we already know.

So, let us write down the characteristic equation, so this there so we have to calculate this things, let us say, this is I do not know maybe star, we have to calculate this thing, now from characteristics, from the characteristics equations one has xs at the point rs is a of x at the point rs, y at the point rs and ys at the point rs is equal to the v at the point x rs and y at the point rs, this is the characteristics equation if you remember.

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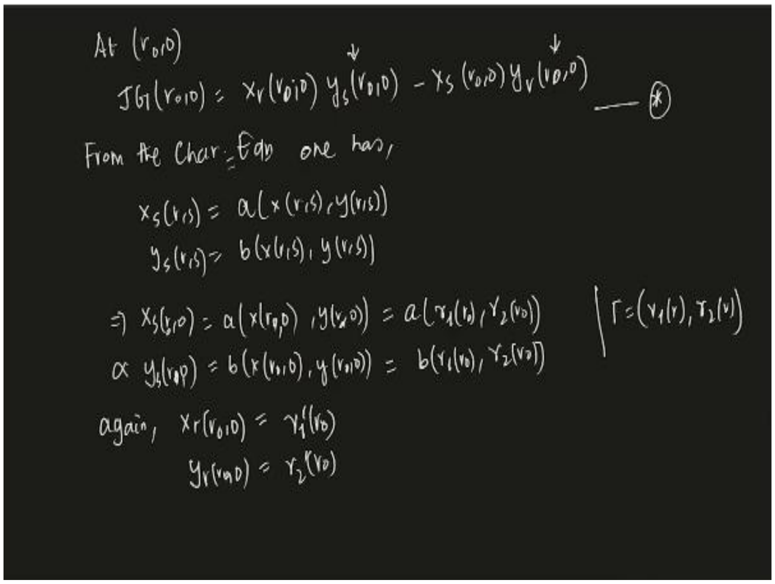

Now, this implies xs of r0, this is a of xr0 yr0 and ys of r0, sorry r naught 0, so I am calculating at the r naught 0, this is b of xr naught 0 y of r naught 0, now what is a of xr naught 0 and b of xr naught 0 we have to understand that, so you see this is a of r naught is, this is gamma 1 r and gamma 2, sorry, gamma 2 of r naught, so gamma is parameterized if you remember, it is gamma 1 of r, gamma 2 of r, so at the point r naught it definitely gamma 1 of r naught, gamma 2 of r naught, and this is given by b of again gamma 1 of r naught, gamma 2 of r naught.

Now, again you see what is xr at the point r, so xr at the point r naught 0, what do you think that is? So, x at the point r naught 0 is gamma 1 of r naught, so this is gamma 1 prime of r naught, and yr at the point r naught 0 is gamma 2 prime of r naught, so that is given.

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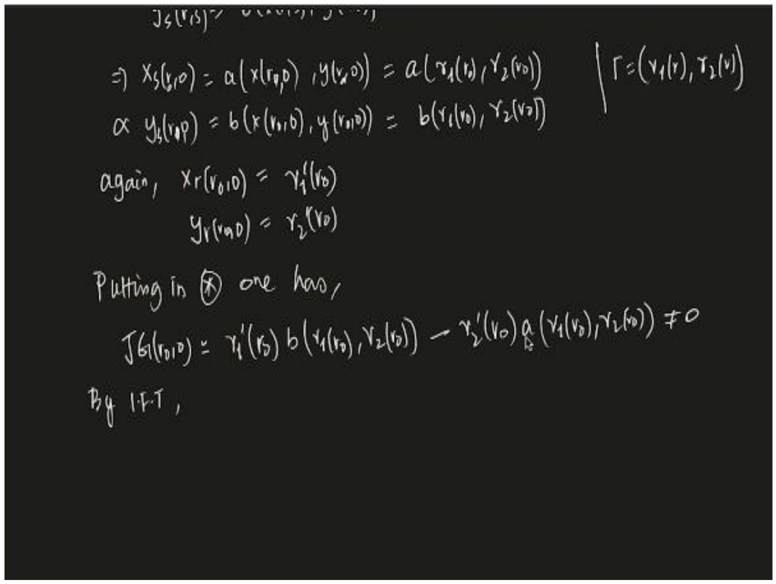

Now, if this given you just putting in star one has G r naught 0 is given by gamma 1 prime r naught, gamma 1 prime r naught times b, b of gamma 1 r naught, gamma 2 r naught minus gamma 2 prime r naught a of gamma 1 r naught, gamma 2 r naught, you see this is the star condition, the Jacobian condition, xr is y1 prime r naught, ys is b of this, that is why I wrote exactly minus again this in calculation, so that is there.

Now, remember see this is the non characteristic condition, and it is given to be nonzero, non characteristics is this and it is given to be nonzero, so by inverse function theorem, so this is nonzero it is given, because gamma is a non characteristics hence, this happened.

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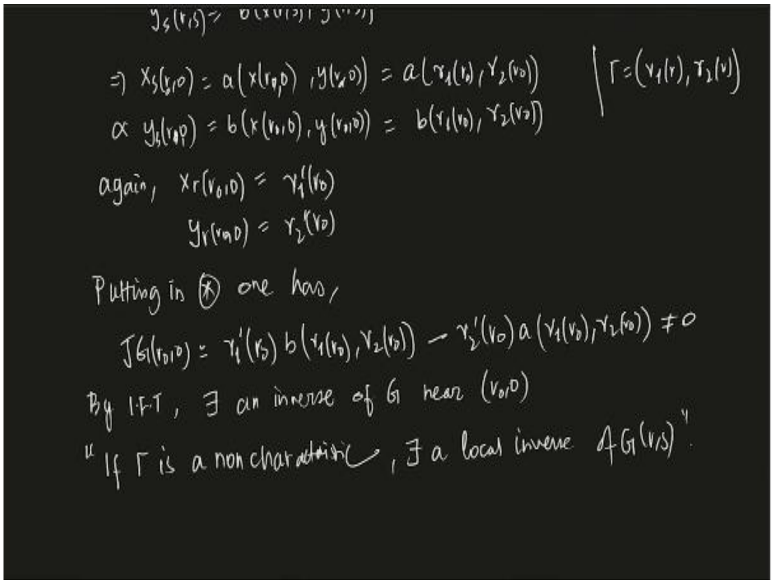

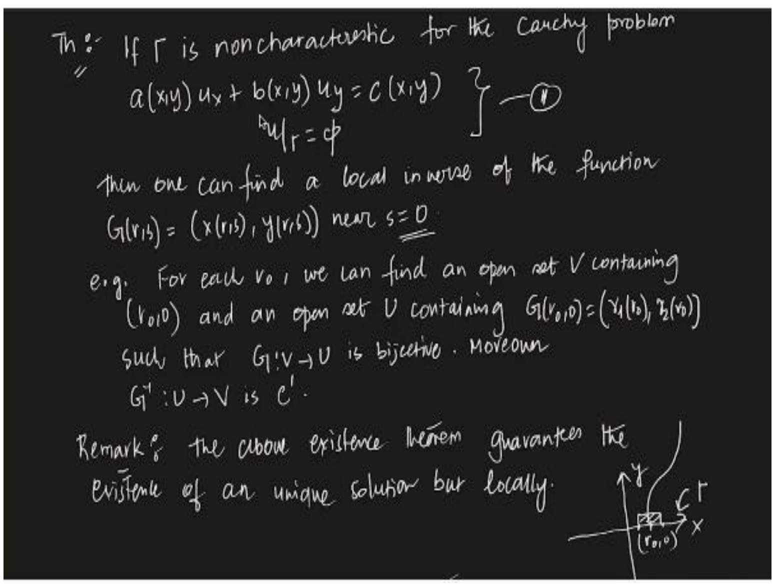

So, by inverse function theorem we can find the inverse, there exist an inverse of G near r naught and 0, because at the point r naught 0, I mean this is, the Jacobian is nonzero so there exists the neighborhood of r naught 0 and G of r naught 0 where G is, I mean G inverse exist and G inverse is C1, so this proves that if gamma is a non characteristic, there exist a local inverse of G rs. And this proves our theorem.

So, let me just end this part by just saying this, see why local inverse is sufficient for this, see if you remember while solving this problem, let us say while solving a problem like this you are getting some characteristic equations for which you want to find x and r, I mean x prime is this, y prime is this, z prime is this and calculate what is x, y and z and that calculation will come in terms of r and s and you understand that it is not always possible to invert those r and s, so it is not always possible to write r and s explicitly in the form of x and y.

And this is where the inverse function theorem comes, so inverse function theorem says that of course, it is possible to invert the theorem, invert r and s in terms of x and y but their inversion is only possible and the unique inversion is only possible in a local neighborhood, in the local neighborhood of r naught 0, so what it means is in a local neighborhood, that initial curve, r naught 0, 0 is s equals to 0, so near the initial curve you can actually say that there is a unique solution, so with this we are going to end this part.