Advanced Partial Differential Equations

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Lecture 33
Shock and Rarefaction Wave: An Example

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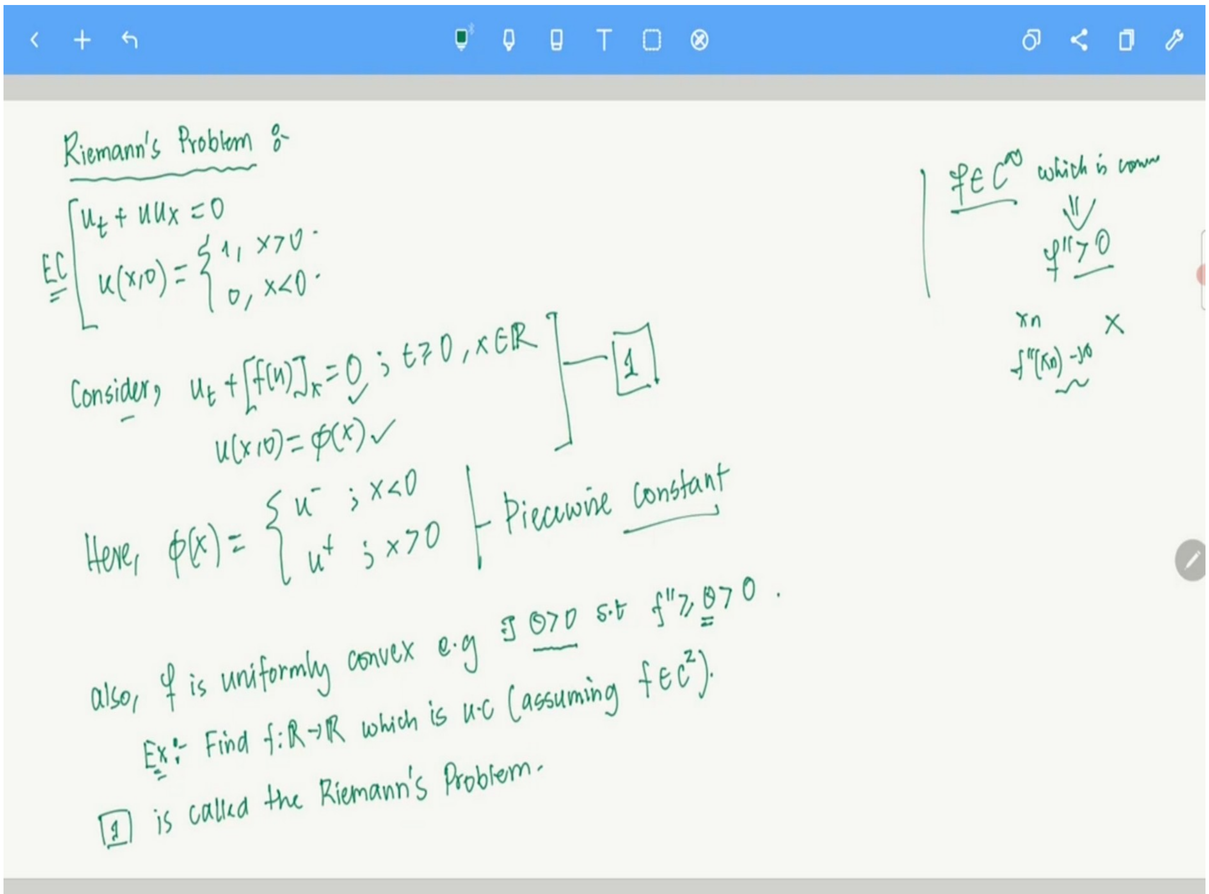

Welcome students, and in this video we are going to talk about Riemann problem. So, in the last video; so this is a continuation of our study of conservation laws.

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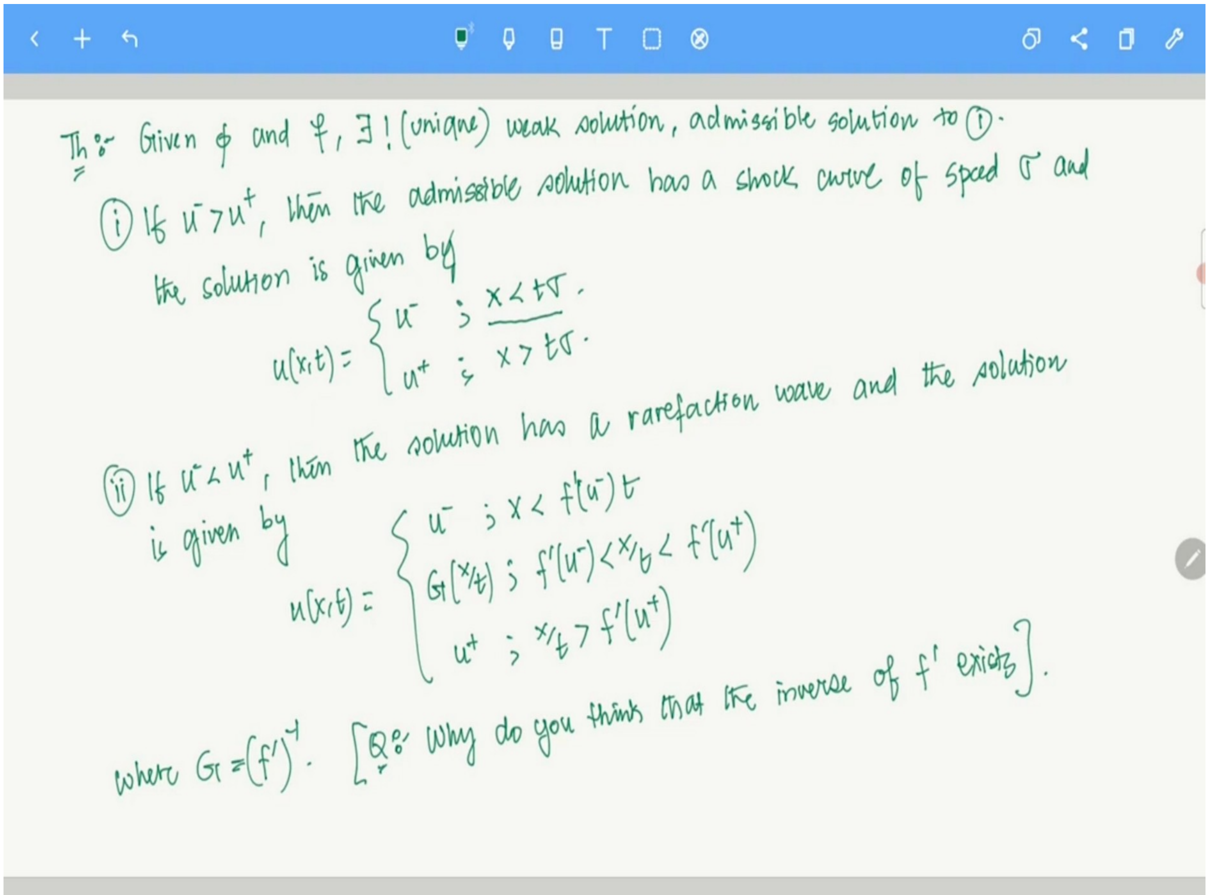

By the way, you see the whole of whatever we did up till here; to solve the second order equations, which we have studied; heat equation, wave equation and Laplace equation. And now this conservation laws, which you are studying; I am doing all of that from Evan’s book.

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Th: Given ϕ and ψ, $\exists\,!$ (unique) weak solution, admissible solution to (1).

(i) If $u^- > u^+$, then the admissible solution has a shock curve of speed σ and the solution is given by

$$u(x,t) = \begin{cases} u^- \, ; & x < t\sigma. \\ u^+ \, ; & x > t\sigma. \end{cases}$$

$$\sigma = \xi'(t)$$

(ii) If $u^- < u^+$, then the solution has a rarefaction wave and the solution is given by

$$u(x,t) = \begin{cases} u^- \, ; & x < f'(u^-)\,t \\ G(x/t) \, ; & f'(u^-) < x/t < f'(u^+) \\ u^+ \, ; & x/t > f'(u^+) \end{cases}$$

where $G = (f')^{-1}$. [Q: Why do you think that the inverse of f' exists].

Riemann's Problem :

$$\underline{\underline{EC}}\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases} \end{cases}$$

Consider, $\left.\begin{array}{l} u_t + [f(u)]_x = 0 \ ; \ t>0, x \in \mathbb{R} \\ u(x,0) = \phi(x) \end{array}\right] - \boxed{1}$

Here, $\phi(x) = \begin{cases} u^- & ; x<0 \\ u^+ & ; x>0 \end{cases}$ — Piecewise constant

also, f is uniformly convex e.g $\exists\ \theta>0$ s.t $f'' \geq \theta > 0$.

Ex:- Find $f: \mathbb{R} \to \mathbb{R}$ which is u.c (assuming $f \in C^2$).

$\boxed{1}$ is called the Riemann's Problem.

$f \in C^\infty$ which is conv

$\Downarrow$

$f'' > 0$

$x_n \quad X$

$f''(x_n) \to 0$

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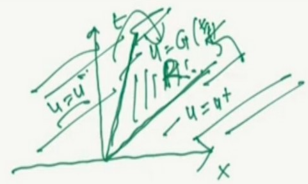
Proof:

$$u(x,t) = \begin{cases} u^- ; & x/t < f'(u^-) \\ G(x/t) ; & f'(u^-) < x/t < f'(u^+) \\ u^+ ; & x/t > f'(u^+) \end{cases} \quad \text{where } G = (f')^{-1}$$



Check: u is a solution in $f'(u^-) < x/t < f'(u^+)$

$$u_t + [f(u)]_x = G'(x/t)(-x/t^2) + f'(G(x/t))\,G'(x/t)(1/t)$$

$$= G'(x/t)(-x/t^2) + G'(x/t)(x/t^2) = 0.$$

$$[f(u)]_x = f'(u)\,u_x$$

∴ u is classical in each of the three regions.

Also, along the curve $x/t = f'(u^-)$, $G(x/t) = (f')^{-1}(x/t) = u^-$ and thus satisfy the entropy condition.

∴ 'u' has curves of discontinuity ∎

Th:- Given ϕ and ψ, $\exists !$ (unique) weak solution, admissible solution to (1).

(i) If $u^- > u^+$, then the admissible solution has a shock curve of speed σ and the solution is given by

$$u(x,t) = \begin{cases} u^- & ; \ x < t\sigma \\ u^+ & ; \ x > t\sigma \end{cases}$$

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(ii) If $u^- < u^+$, then the solution has a rarefaction wave and the solution is given by

$$u(x,t) = \begin{cases} u^- & ; \ x < f'(u^-)t \\ G(x/t) & ; \ f'(u^-) < x/t < f'(u^+) \\ u^+ & ; \ x/t > f'(u^+) \end{cases}$$

where $G = (f')^{-1}$. [Q:- Why do you think that the inverse of f' exists].

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Ex 3: $u_t + u u_x = 0$, $x \in \mathbb{R}, t > 0$

$u(x,0) = \phi(x)$

where, $\phi(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$C(r)$ = Data Curve $= \{(r, 0, \phi(r)) : r \in \mathbb{R}\}$

Char. Curves: $P(s)$

$z(r,s) := u(x(r,s), y(r,s))$

$z'(r,s) = 0$

The projected char. curves are given by

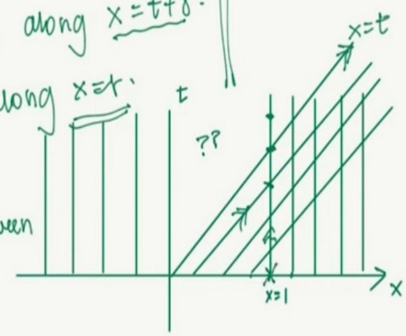
$x(r,t) = \phi(r)t + r$; $r \in \mathbb{R}$.

Case 1: If $r < 0$, then $\phi(r) = 0 \Rightarrow x = r$ and $u = 0$ along $x = r$.

If $0 < r < 1$, $\phi(r) = 1 \Rightarrow x = t + r$ and $u = 1$ along $x = t + r$.

If $r > 1$, then $\phi(r) = 0 \Rightarrow x = r$ and $u = 0$ along $x = r$.

We have shock when the lines $x = t + r$; $0 < r < 1$ and $x = 1$ intersects and we get a rarefaction wave between $x = 0$ and $x = t$.



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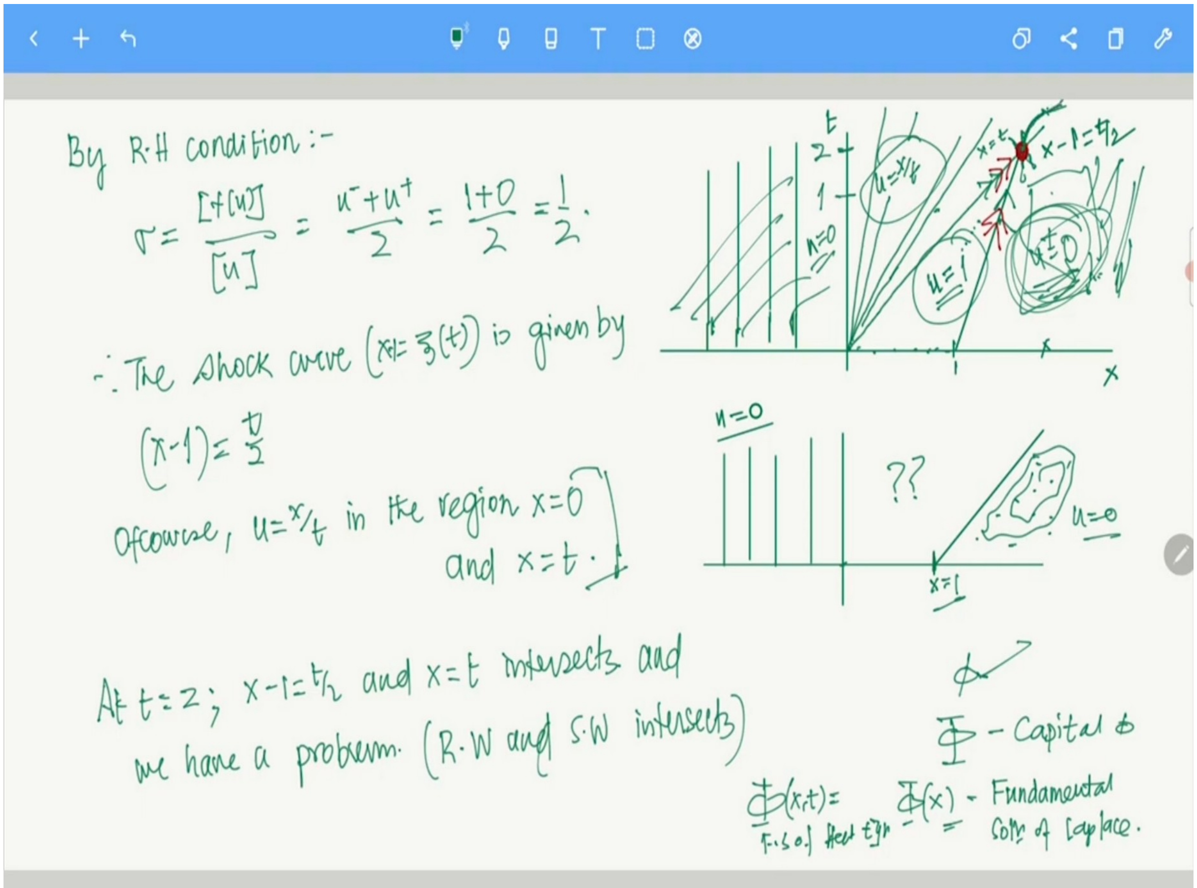

Ex 3: $u_t + u u_x = 0$, $x \in \mathbb{R}, t > 0$

$u(x,0) = \phi(x)$

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$\cancel{f(u)}$ $C(r)$ = Data Curve $= \{(r, 0, \phi(r)) : r \in \mathbb{R}\}$

Char. Curves. $\Gamma(s)$

$f(u) = u^2/2$

$z'(r,s) = 0$

$z(r,s) := u(x(r,s), y(r,s))$

The projected char. curves are given by

$x(r,t) = \phi(r)t + r$; $r \in \mathbb{R}$.

Case 1: If $r < 0$, then $\phi(r) = 0 \Rightarrow x = r$ and $u = 0$ along $x = r$.

If $0 < r < 1$, $\phi(r) = 1 \Rightarrow x = t + r$ and $u = 1$ along $x = t + r$.

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We have shock when the lines $x = t + r$; $0 < r < 1$ and $x = 1$ intersects and we get a rarefaction wave between $x = 0$ and $x = t$

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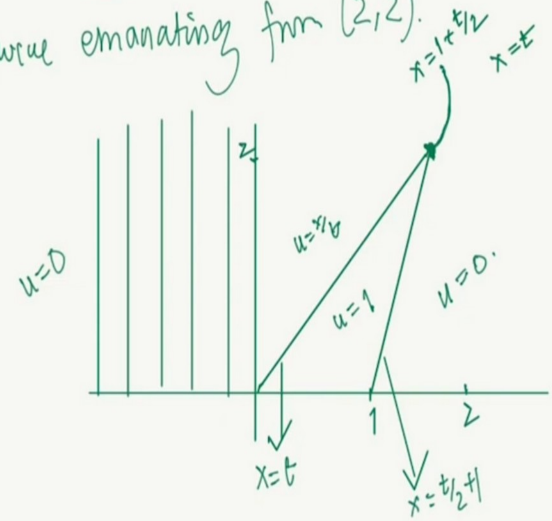
By R.H condition,

$$\sigma = \frac{[f(u)]}{[u]} = \frac{u^- + u^+}{2} = \frac{x}{2t}.$$

$\therefore$ We have a new shock curve emanating from $(2,2)$.

$\therefore$ The curve is given by

$$x(t) = \sqrt{2t}$$



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The weak solutions are given by

$$u(x,t) = \begin{cases} 0, & x<0 \\ x/t, & 0<x<t \\ 1, & t<x<t/2+1 \\ 0, & x>1+t/2. \end{cases} \quad \underline{\text{for } t \leq 2}$$

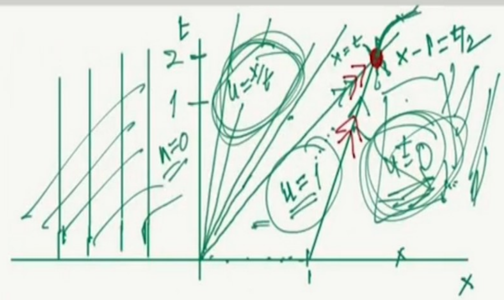
and for $t \geq 2$,

$$u(x,t) = \begin{cases} 0 & \text{if } x<0 \\ x/t & \text{if } 0<x<\sqrt{2t} \\ 0 & \text{if } x>\sqrt{2t}. \end{cases}$$

$\therefore$ 'u' is an admissible weak solution.

By R.H condition :-

$$\sigma = \frac{[f(u)]}{[u]} = \frac{u^- + u^+}{2} = \frac{1+0}{2} = \frac{1}{2}.$$



∴ The shock curve ($x = \xi(t)$) is given by

$$(x-1) = \frac{t}{2}$$

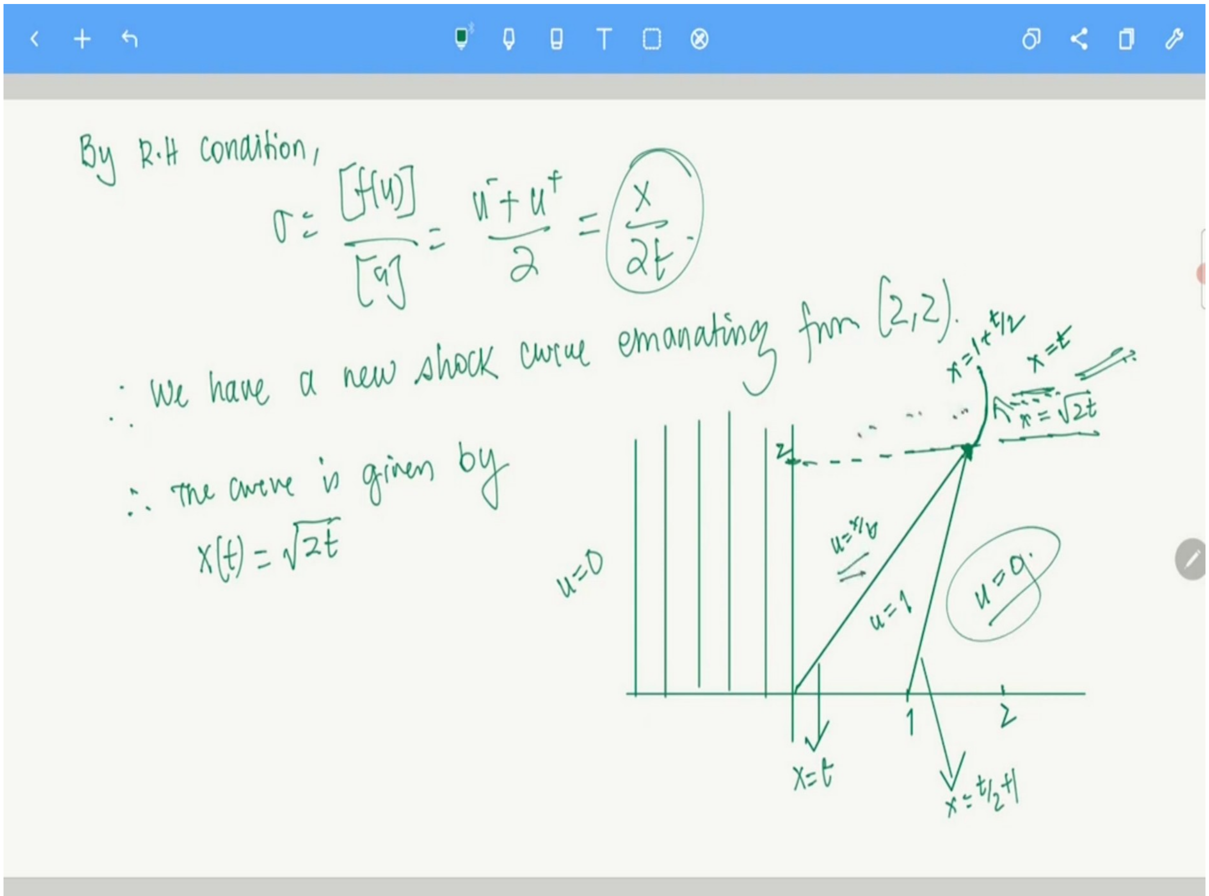
Ofcourse, $u = x/t$ in the region $x=0$ and $x=t$.

At $t=2$; $x-1 = t/2$ and $x=t$ intersects and we have a problem. (R.W and S.W intersects)

ϕ

Φ - Capital ϕ

$\Phi(x,t)$ = F.S.o.f Heat eqn $\qquad$ $\Phi(x)$ - Fundamental soln of Laplace.



I hope this is clear to you; this is the weak solution, which we get. Of course, this is a weak solution, because the Rankine-Hugoniot condition will hold. Thus, we have manipulated our solution like that itself, because there we have derived those things; so that will hold. Then of course, you guys can check that this is also going to, the entropy condition is going to be satisfied. So, this particular solution is a admissible weak solution; hence u is a admissible weak solution. So, if you are not convinced, please check it yourself; but this a admissible weak solution, which we have obtained.

So, with this we are going to end this video.